A Matrix Model for the 2d Black Hole

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Abstract: We construct a large N matrix matrix model describing two-dimensional Euclidean string theory compactified on a circle of radius R and perturbed by an operator creating winding modes (vortices) on the worldsheet. The matrix model is exactly solvable and posesses an integrable structure of the infinite Toda chain hierarchy. We give explicit expressions for its free energy in the sphere- and torus approximation. A conjecture by V. Fateev, A. and Al. Zamolodchikov about the equivalence of the sine-Liouville and $SL(2,\mathbb{R})/U(1)$ conformal field theories implies that for particular values of the parameters (vanishing cosmological constant μ and compactification radius $R = \frac{3}{4}R_{\rm KT}$) the matrix model can be used to study two-dimensional string theory in the Euclidean black hole background to all orders in string perturbation theory.

Keywords: Matrix model, Black hole, Large N, String theory.

1. Introduction

The Euclidean two-dimensional string theory is defined by the world-sheet action

$$\mathcal{S} = \frac{1}{4\pi}\int_\Sigma d^2\sigma[\mathcal{G}_{\mu\nu}(X)\nabla_a X^\mu \nabla^a X^\nu + T(X) + \hat{R}^{(2)}\Phi(X)] \tag{1.1}$$

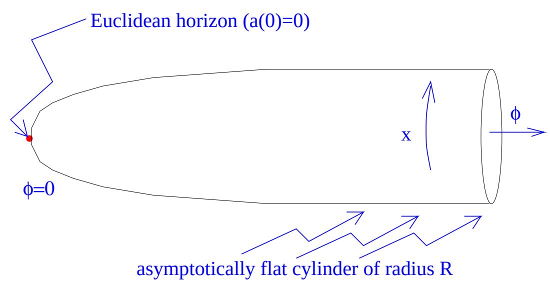
where $X^\mu(\sigma_1,\sigma_2)$ $(\mu = 1,2)$ define the embedding of the world-sheet Σ in a two dimensional spacetime with metric $\mathcal{G}_{\mu\nu}(X)$, ∇_a $(a = 1,2)$ is the covariant derivative on Σ and $\hat{R}^{(2)}$ is the Gaussian curvature on Σ. The target space metric can be considered as a background source. The other background fields are the tachyon T and the dilaton Φ, coupled to the area and the curvature of the world sheet. In the following we will denote $X^1 = x$, $X^2 = \phi$.
The string theory compactified at radius R possesses a classical solution with the geometry of a flat cylinder and an x-independent tachyon condensate,

$$\mathcal{G}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \Phi = -2\phi, \quad T = 2(\phi-\phi_0)\, e^{-2(\phi-\phi_0)} \tag{1.2}$$

as well as a solution

$$\mathcal{G}_{\mu\nu} = \begin{pmatrix} a(\phi) & 0 \\ 0 & 1/a(\phi)\end{pmatrix}, \quad a(\phi) = 1 - e^{-2\phi/R}, \quad \Phi = \Phi_0 - 2\phi, \quad T = 0, \tag{1.3}$$

describing a curved target space (with curvature $\mathcal{R} \sim e^{-2\phi/R}$) having the form of a semi-infinite cigar with asymptotic radius R. The tachyon has zero expectation value and the string coupling constant is determined by the value of the dilaton at the origin.



The metric in the "cigar" background.

The periodic coordinate $x \equiv x + 2\pi R$ in (1.3) is labeling the location around the cigar, while $\phi \geq 0$ is the direction along the cigar, with $\phi = 0$ corresponding to the tip. This geometry can be

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thought of as a Euclidean version of a 1+1 dimensional black hole, with the tip of the cigar representing the Euclidean horizon [1, 2, 3]. The string propagation in the geometry (1.3) is described (in Minkowski space) by $[SL(2,\mathbb{R})]_k/U(1)$ coset CFT on the worldsheet[1], where the level k of the representation of the current algebra is related to the radius R by

$$k = R^2.$$

The mass of the black hole is determined by the value of the dilaton field at the tip of the cigar, $M \propto \exp(-2\Phi_0)$.

A complete solution of the 2d string theory with the flat background (1.2), known also as $c = 1$ string theory, has been obtained by reformulating it as a large N matrix model (see the reviews [5] and [6]). The matrix model has one-dimensional spacetime, the target space of the x-component of the string position field, and therefore is also called Matrix Quantum Mechanics (MQM). The dynamics of the tachyon field is described by the collective field theory of the singlet sector of the MQM [7].

On the other hand, in spite of many attempts (see *e.g.* [5, 8]), the matrix realization of the string theory theory in the nontrivial, "cigar", background (1.3), was not found up to now. Our proposal for such a matrix model [9] is based on an observation by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [12] which we call *the FZZ conjecture*. According to this conjecture the coset CFT describing the string theory on the "cigar" is dual to the Sine-Liouville model, which we describe in section 2. The latter model can be interpreted as a Sine-Gordon theory coupled to quantum gravity, with the cosmological constant tuned to zero.

After being reformulated in such a way, the string theory in the "cigar" background (1.3) can be easily discretized. Indeed, in the T-dual theory, the sine-Gordon operator creates *vortices* on the worldsheet. It is well known [10, 11] how to describe world sheets with vortices by the matrix model: it is sufficient to extend the Hilbert space of MQM to the nonsinglet sectors labeled by the irreducible representations of the $SU(N)$ group.

In this note we show explicitly how to introduce vortices with any vorticity by manipulating the MQM with compact time $\beta = 2\pi R$ and twisted periodic boundary conditions. Our first observation will be that the vortex fugacities t_m are proportional to the moments tr Ω^n of the twisting matrix $\Omega \in SU(N)$. In this way we are able to reformulate the partition function of the string theory with the "cigar" background as the large N limit of a simple matrix integral, depending on the vortex couplings t_m and the cosmological constant μ.

Our second observation is that the theory is exactly solvable and its partition function is a τ-function of the infinite Toda-chain hierarchy. As a consequence, the flows corresponding to the coupling constants t_m are integrable and the partition function for $t_m \neq 0$ can be calculated explicitly by solving the equations of the Toda hierarchy with boundary condition at $t_m = 0$ given by the known partition function of the string theory with flat background.

We are interested in particular in the theory with two couplings, the cosmological constant μ and the lowest vortex coupling $\lambda = \sqrt{t_+ t_-}$, $t_\pm = t_{\pm 1}$, which is the sine-Liouville coupling constant in the T-dual theory. Knowing the exact solution at $\lambda = 0$, we can follow the flow determined by the Toda equation to arrive at the fixed point $\mu = 0$ ($\lambda \to \infty$), described by the "cigar" string theory. As a by-product of our analysis we evaluate the partition function of the two-coupling theory (the sine-Gordon model coupled to gravity). We will present explicit expressions for the partition function for the sphere and the torus. Our result for the sphere reproduces exactly the series expansion obtained by G. Moore [13], while the result for the torus is, to our knowledge, new.

2. FZZ conjecture about the coset/SL duality

The statement of [12] is that the $[SL(2,\mathbb{R})]_k/U(1)$ coset CFT is equivalent to the Sine-Liouville (SL)

[1] The background (1.3) is valid for large level k of the $SL(2,\mathbb{R})$ current algebra. For finite k there are corrections found in [4]

theory

$$L=\frac{1}{4\pi}[(\partial\varphi)^2+(\partial\phi)^2+Q\hat{R}^{(2)}\phi+\lambda e^{b\phi}\cos R\varphi]. \quad (2.1)$$

The matching of the parameters is given by

$$Q^{-1}=-b=\sqrt{k-2}, \quad R=\sqrt{k} \qquad (2.2)$$

and follows from matching of the central charges $(1+6Q^2)+1=\frac{3k}{k-2}-1$
and the requirement that the SL perturbation is marginal,
$\Delta_{SL}=\frac{R^2}{4}-\frac{b(b+2Q)}{4}=1$.

The target space in the coset CFT has the metric of the semi-infinite cigar (1.3). Far from the tip of the cigar. the fields ϕ in the two models can be identified, while the field x of the coset theory should be identified with the T-dual field to the field φ:

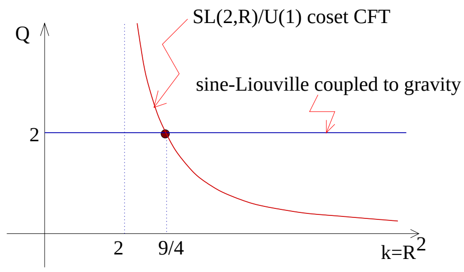
$$x=x_L+x_R, \quad \varphi=x_L-x_R.$$

The nontrivial statement of [12] is that the two CFT's also agree for finite ϕ, where the interactions cannot be neglected. This is far from obvious, since in one case the strong coupling region is eliminated by changing the topology of the cylinder to that of the cigar (1.3), while in the other this is achieved by turning on the potential in (2.1).
The relation between the coset and Sine-Liouville CFT's is a strong-weak coupling duality on the worldsheet. The semi-classical limit of the SL-theory is achieved when $Q\to\infty$ or $k\to 2$ while the classical limit of the coset theory is at $k\to\infty$ or $Q\to 0$.

The operators corresponding to the momentum and winding modes of the cigar are represented in the SL model as combinations of electric (vertex) and magnetic (vortex) operators with discrete spectrum of electric and magnetic charges determined by the compactification radius R. The electric charge (the momentum in the string theory language) is strictly conserved while the magnetic charge (the winding number) is broken. In the cigar CFT the reason for that is that winding number can slip off the tip of the cigar; in (2.1) the interaction breaks this symmetry explicitly.

3. The sine-Gordon model coupled to gravity

The FZZ conjecture relates one-parameter family of coset and the SL theories satisfying the matching conditions (2.2). The criticality condition, $c=26$, determines $k=9/4$ or $R=3/2$. We will however consider in the following a theory with arbitrary compactification radius, satisfying the balance of central charge. The corresponding string theory is well defined for all $R\in(0,2)$, but the equivalence with the coset CFT holds only at the point $R=3/2$. Whether it is possible or not to extend the black hole background interpretation to generic values of R is still an open question.



The SL and the "cigar" string theories are dual only at $k=R^2=9/4$.

Let us now consider a more general worldsheet CFT, the sine-Gordon theory coupled to quantum gravity, having two coupling constants, the Liouville coupling μ playing the role of cosmological constant and the SL (sine-Liouville) coupling λ. Both interactions should correspond to marginal operators and the total central charge should be equal to 26. This fixes all parameters but one, the radius of compactification R. The Lagrangian of the model is

$$L = \frac{1}{4\pi}[(\partial x)^2+(\partial\phi)^2+2\hat{R}\phi+\mu e^{-2\phi} + \lambda e^{(R-2)\phi}\cos R(x_L-x_R)] \qquad (3.1)$$

where $x_{L/R}$ is the left/right component of the compactified boson $x=x_R+x_L$.

The interpretation of this action as a deformation of the $c=1$ "noncritical" string theory is the following: the field x is the periodic time coordinate and the Liouville field ϕ defines the metric on the worldsheet, $ds^2=e^{-2\phi(\sigma)}(d\sigma_1^2+d\sigma_2^2)$.

The constant μ is coupled to the puncture operator and the constant λ is coupled to the vortex and antivortex operators creating discontinuities $\pm 2\pi R$ on the world sheet. The SL-term makes sense only if compactification radius R is smaller than the Kosterlitz-Thouless radius $R_{\rm KT} = 2$, otherwise it would be irrelevant at large scales. The string theory free energy (the partition function of connected surfaces) is a function of μ, λ and the string coupling g_s. The genus expansion of the string theory free energy is

$$\mathcal{F}(g_s, \lambda, \mu) = \sum_{h\geq 0} g_s^{2h-2}\ \mathcal{F}^{(h)}(\lambda, \mu), \qquad (3.2)$$

where $\mathcal{F}^{(h)}(\lambda, \mu)$ is the contribution of the connected worldsheets with genus h. One of the two couplings λ and μ must be nonzero to set the IR scale (the typical area of the worldsheet). Then the other coupling can be considered as a perturbation. There are two perturbative expansions: the expansion in $\lambda^2 = t_+ t_-$ around the Liouville critical point $\lambda = 0$, and the expansion in μ around the sine-Liouville critical point $\mu = 0$.

• *Perturbative expansion at the Liouville critical point.*

Each term of the genus expansion (3.2) is itself a series expansion in the dimensionless parameter

$$z = (R-1)\lambda^2 \mu^{R-2}. \qquad (3.3)$$

The exact form of the expansion for the leading term of (3.2) was obtained by G. Moore in [13]. The partition sum of (3.1) on the sphere can be expanded in the multiple amplitudes of the vortex operators
$\mathcal{V}_{\pm} \sim :e^{\pm iR(x_L - x_R) + (R-2)\phi}:$,
calculated at the Liouville critical point

$$\mathcal{F}^{(0)}(\lambda,\mu) = \langle e^{+t_+\mathcal{V}_+ + t_-\mathcal{V}_-}\rangle_{\rm sphere} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!\ n!} \langle \mathcal{V}_+^n \mathcal{V}_-^n\rangle_{\rm sphere}.$$

The coefficients in this series are $2n$ point functions of the vortex operators on the sphere, which can in principle be computed in the $c = 1$ matrix model.

G. Moore conjectured a general form for these amplitudes based on an extrapolation of the matrix model results. He thus calculated the coefficients in the expansion of the string susceptibility $\chi^{(0)} = \partial_\mu^2 \mathcal{F}^{(0)}$:

$$\chi^{(0)}(\lambda,\mu) = R\left(-\log(\mu a^2) + \textstyle\sum_n C_n \frac{(-z)^n}{n!}\right),$$

namely $C_n = \frac{\Gamma(n(2-R))}{\Gamma(n(1-R)+1)}$.
The series has finite radius of convergence $z_c = \left|(1-R)^{1-R}(2-R)^{R-2}\right|$.

We noticed that this series is the solution of a simple algebraic equation[2]

$$\mu e^{\chi^{(0)}/R} + (R-1)\lambda^2 e^{(2-R)\chi^{(0)}/R} = 1. \qquad (3.4)$$

The perturbative expansion of the partition function on a genus h surface is, according to the DDK-KPZ scaling, of the form

$$\mathcal{F}^{(0)}(\lambda,\mu) = -\frac{1}{2}R\mu^2\log(\mu a^2) + f^{(0)}(z) \qquad (3.5)$$

$$\mathcal{F}^{(1)}(\lambda,\mu) = -\frac{R+\frac{1}{R}}{24}\log(\mu a^2) + f^{(1)}(z), \qquad (3.6)$$

$$\mathcal{F}^{(h\geq 2)}(\lambda,\mu) = \mu^{2-2h} f^{(h)}(z), \qquad (3.7)$$

where $f^{(h)}(z)$ are smooth functions near $z = 0$ and a is the UV cutoff (the elementary length on the worldsheet). $\frac{1}{\mu}$ plays the role of the string coupling constant.

$\star$ *Perturbative expansion at the SL critical point.*

Beyond the convergence radius of the expansion in $\lambda^2 = t_+t_-$, the model should be considered as a perturbation of the SL ($\mu = 0$) model, with perturbation parameter μ. The expansion is actually with respect to the dimensionless strength of the perturbation

$$y = \mu\left[(R-1)\lambda^2\right]^{\frac{1}{R-2}}. \qquad (3.8)$$

The role of the string coupling constant is now played by

$$g_s = \left[(R-1)\lambda^2\right]^{\frac{1}{R-2}}. \qquad (3.9)$$

[2] Note that if we parametrize $\chi^{(0)} = R\zeta/\rho$, $\mu = e^{\frac{v}{\rho}}$, $(R-1)t_+t_- = e^{u\rho}$ $(\rho = \sqrt{2-R})$, then eq. 3.4 takes the form

$$e^{(\zeta+v)/\rho} + e^{\rho(\zeta+u)} = 1$$

which is symmetric with respect to

$$2 - R \leftrightarrow (2-R)^{-1}, \quad u \leftrightarrow v.$$

This symmetry can be used to transform the small λ^2 expansion into a small μ expansion. Note also that the radius $R = 3/2$ is dual to $R = 0$.

The genus h free energy has the form

$$\mathcal{F}^{(0)}(\lambda,\mu) = \frac{1}{g_s^2}\left[A_0 - \frac{1}{2}Ry^2\log\left(\frac{1}{g_s}\right) + f^{(0)}(y)\right], \quad (3.10)$$

$$\mathcal{F}^{(1)}(\lambda,\mu) = C_0\log\left(\frac{1}{g_s}\right) + D_0 + f^{(1)}(y), \quad (3.11)$$

$$\mathcal{F}^{(h\geq 2)}(\lambda,\mu) = g_s^{2h-2} f^{(h)}(y), \quad (3.12)$$

where $f^{(h)}(y)$ can be expanded in Taylor series in y.

◦ *Introducing vortices with arbitrary charge*

We can generalize the two-coupling theory (3.1) by introducing a more general interaction obtained by perturbing with an infinite series of discrete states representing vortex operators of vorticity $m = 1, 2, \ldots$ dressed by the Liouville field[3]

$$L = \frac{1}{4\pi}[(\partial x)^2 + (\partial\phi)^2 + 2\hat{R}\phi + \mu e^{-2\phi} + \sum_{n\neq 0} t_n e^{(|n|R-2)\phi} e^{inR(x_L - x_R)}]. \quad (3.13)$$

The model (3.1) corresponds to the choice $t_n = (\delta_{n,1} + \delta_{n,-1})\ \lambda$.

In what follows we will construct a one-matrix integral, for which the couplings t_n are the coefficient in the matrix potential, and whose saddle point expansion gives the all genus partition functions of the string theory (3.13). This integral will be obtained by evaluating the partition function of the MQM with twisted periodic boundary conditions. In the next section we explain the correspondence between the twisted MQM and the sum over discretized worldsheets containing vortices. We will review some basic facts from the $c = 1$ string theory in order to make the text self-contained. A reader who is not interested in the justification of the matrix/string correspondence can skip this section.

4. A matrix model for the $c = 1$ string theory with vortices

The background (1.2) corresponds to the usual $c = 1$ noncritical string, which can be interpreted as the statistical mechanics of discretized random surfaces. embedded in a compact one-dimensional spacetime $\{x \equiv x + 2\pi R\}$. The second coordinate ϕ is encoded in the intrinsic geometry of the worldsheet determined by the connectivity matrix of the corresponding planar graph. The discretized $c = 1$ noncritical string can be obtained as the collective theory for a more “fundamental” quantum system, a one-dimensional large-N matrix field [14]. More precisely, the compactified $c = 1$ noncritical string theory is contained in the singlet sector of the one-dimensional matrix model (MQM) compactified on a circle of circumference $2\pi R$ [10]. It is necessary to project onto the singlet sector in order to eliminate the vortex excitations, which propagate in the sectors associated with the higher irreducible representations of $SU(N)$ [6, 11].

One way to introduce vortices is therefore to extend the Hilbert space of MQM to the sectors corresponding to all representations of $SU(N)$ and understand the partition function of MQM as a sum over all there sectors, with appropriate weights. But then it is not possible any more to apply the standard methods used to solve the $c = 1$ string theory, because the matrix model is no more described by a system of free fermions. In each sector, the eigenvalues of the random matrix obey a different statistics, which is neither bosonic nor fermionic.

There is however a more efficient and explicit way to introduce vortices in the matrix model, namely to impose *twisted periodic boundary conditions* on the one-dimensional matrix field. We will introduce a matrix source coupled to the vortex excitations by twisting the periodic boundary condition. The twisted partition function of MQM has been considered as an intermediate concept in [11] before projecting onto different representations. In the discussion that follows, the twisted partition function will play a more fundamental role: we will see that the couplings for the vortex operators are actually the moments of the twisting matrix.

4.1 Discretized strings with vortices from the twisted MQM

The partition function of MQM with twisted boundary conditions depends on the unitary matrix Ω

[3] The most general case would correspond to the perturbation of (3.1) by all composite electric-magnetic operators.

and is defined as the functional integral

$$Z_N(\Omega) = \int\limits_{A(2\pi R)=\Omega^\dagger A(0)\Omega} \mathcal{D}A\ e^{-\frac{1}{\hbar}\mathrm{tr}\int_0^{2\pi R}[\frac{1}{2}(\partial_x A)^2+V(A)]dx}. \tag{4.1}$$

where $A = A_i^j(x)$ is a one-dimensional Hermitian $N \times N$ matrix field. The twisted boundary condition means that we identify the value at $x = 0$ of the field A with its value at $x = 2\pi R$, after transforming it by a unitary matrix Ω in the adjoint representation. The simplest choice for the potential from the point of view of the planar diagram expansion is

$$V(A) = \frac{1}{2}A^2 - \frac{1}{3}A^3, \tag{4.2}$$

which we will adopt in the following.

The $1/N$ expansion of the MQM free energy, which we define as the logarithm of the partition function

$$F_N(\Omega) = \log Z_N(\Omega), \tag{4.3}$$

can be expressed in terms of connected planar graphs embedded in the target-space circle of radius R. The planar graph expansion is performed with respect to the trivial classical vacuum $A = 0$. It is easy to see, if we rescale the matrix field as $A \to \kappa A$, with

$$\kappa = \sqrt{\hbar N}, \tag{4.4}$$

that the $1/N$ expansion of the free energy has the form

$$F_N(\Omega) = \sum_{h\geq 0} N^{2-2h} F^{(h)}(\kappa, \Omega), \tag{4.5}$$

$F^{(h)}$ being the contribution of planar graphs with topology of a sphere with h handles. Each planar graph Σ can be considered as a discretized worldsheet, immersed in a one-dimensional spacetime. In the following we will denote the vertices, links and faces of a planar graph by v, ℓ and f, correspondingly. The functional integral over the worldsheet field $X = (x, \phi)$ is discretized as $\int \mathcal{D}x\mathcal{D}\phi \ \to\ \sum_\Sigma \int \prod_v dx_v$.

In order to construct the planar graph expansion, we have to invert the quadratic part of the action in (4.1). The corresponding kernel $G_{ik}^{jl}(x - x')$ satisfies

$$(-\partial_x^2 + x^2)G_{ik}^{jl}(x) = \delta(x)\delta_i^j\delta_k^l,$$

as well as the twisted periodic boundary condition

$$G_{ik}^{jl}(x + 2\pi R) = G_{ik'}^{jl'}(x)\Omega_k^{k'}\Omega^{\dagger l}_{l'}.$$

We write the solution as a series

$$G(x) = \sum_{n=-\infty}^{\infty} e^{-|x+2\pi Rn|}\ \Omega^n \otimes \Omega^{-n} \tag{4.6}$$

and associate with each term in the sum a propagator

$$x \ \underset{k\quad\ n\quad\ l}{\overset{i\qquad\quad j}{=\!=\!=\!=\!=}}\ x' = e^{-|x-x'+2\pi Rn|}\ (\Omega^n)^i_j(\Omega^{-n})^l_k.$$

In this way an additional integer-valued field $n_\ell \equiv n_{<vv'>}$ associated with the links $\ell =< vv' >$ of the graph appears. The nontrivial matrix structure of the propagator leads to a factor $\frac{1}{N}\mathrm{tr}\ \Omega^{m_f}$, associated with each face f of the planar graph, where the integer m_f, which we call vorticity, is the winding number of the boundary ∂f of the face around the target circle[4]. The vorticity is equal to the algebraic sum of n_ℓ along the boundary:

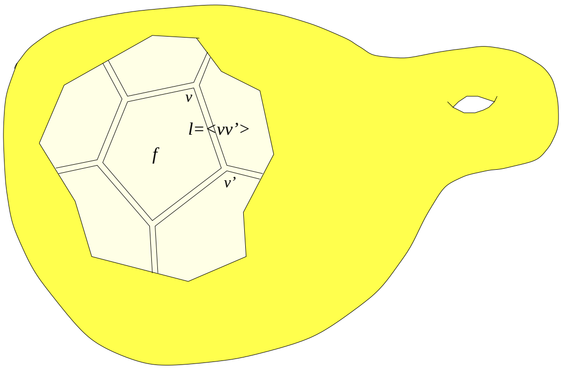
$$m_f = \sum_{\ell\in\partial f} n_\ell. \tag{4.7}$$

The weight of a planar graph with genus h is given by (we take into account the Euler relation $\#f - \frac{1}{2}\#v = 2 - 2h$ for the tri-valent graphs)

$$\hbar^{2h-2}\kappa^{\#\text{faces}} \prod_f \frac{1}{N}\mathrm{tr}\ \Omega^{m_f} \prod_{\ell=<vv>} e^{-|x_{<vv'>}+2\pi Rn_\ell|}$$

We split, following the standard argument due to Berezinski, Kosterlitz and Thouless, the sum over the field n_ℓ into a sum over the vorticity field m_f and the gradient piece $n_v - n_{v'}$. The only effect of the sum over n_v is to extend the integration over x_v, which was originally restricted to the interval $[0, 2\pi R]$, to the whole real axis.

[4]Here we are following an argument suggested by P. Zinn-Justin.



A piece of a discretized worldsheet containing the face f and its boundary ∂f.

Thus the perturbative expansion of the twisted partition function (4.1) can be interpreted as the partition function of the $c = 1$ string theory with vortices, and the fugacities of the vortex operators are proportional to the moments of the twisting matrix. In order to exploit this string/matrix correspondence, it is necessary to express explicitly the partition function (4.1) as a function of the moments $\lambda_n = \text{tr}\ \Omega^n$, *i.e.* $\mathcal{Z}_N[\Omega] = \mathcal{Z}_N[\lambda_{\pm 1}, \lambda_{\pm 2},$ This is not easy, but we can slightly change the problem in order to render it integrable. We will see that the model becomes exactly solvable if we introduce sources t_n coupled to the moments tr Ω^n and then integrate with respect to the twisting matrix Ω. We will also consider the Laplace transformed partition function (the grand canonical ensemble GCE), in which the size of the matrix N is replaced by its conjugated variable, the chemical potential μ_F.

4.2 Integration with respect to the twisting matrix

If we integrate with respect to the twisting matrix, the moments $\lambda_n = \text{tr}\ \Omega^n$ become operators, which create vortices on the worldsheet. We can control the fugacities of the vortices by introducing in the action a set of coupling constants conjugated to these operators. Thus we define the new partition function

$$\mathcal{Z}_N[\vec{t}] = \int [D\Omega]_N \ e^{\sum_{n\neq 0} t_n \text{tr}\Omega^n} Z_N(\Omega) \qquad (4.8)$$

where $\vec{t} \equiv \{t_n\}_{n\neq 0}$ and $[D\Omega]_N$ stands for the the Haar measure on the group $U(N)$ normalized as $\int [D\Omega]_N = 1$.

The evaluation of the integral in Ω in the large N limit can be performed by expanding the free energy $F_N(\Omega) = \log Z_N(\Omega)$ as a sum over connected planar graphs and re-arrange the expansion in the monomials of the moments $\lambda_n = \frac{1}{N}\text{tr}\ \Omega^n$. This series expansion is convergent for sufficiently small κ.

Integrals of the type

$$\int [d\Omega]_N \ \text{tr}\ \Omega^{n_1} ... \text{tr}\ \Omega^{n_k}$$

can be easily evaluated if the sum of all powers is less than N:[5]

$$\int [d\Omega]_N \ \text{tr}\ \Omega^n = N\delta_{n,0}$$
$$\int [d\Omega]_N \ \text{tr}\ \Omega^m \text{tr}\ \Omega^n = |m|\delta_{m+n,0} \qquad (4.9)$$

($m \neq 0$), etc. The generating function for all such integrals is

$$\int [d\Omega]_N \ e^{\sum_{n\neq 0} x_n \text{tr}\ \Omega^n} = e^{\sum_{n>0} n x_n x_{-n}}. \qquad (4.10)$$

The formula (4.10) is valid to all orders in the $1/N$ expansion provided x_n grow less than linearly in N. When $x_n \sim N$, the formula is valid only until some critical value of x_n/N where a Gross-Witten phase transition occurs. The meaning of this formula is the following. If the integrand is a function of the moments that grows slower than exponentially, then the integration with respect to the $U(N)$ Haar measure can be replaced (up to terms $\mathcal{O}(e^{-N})$) by independent gaussian integrations with respect to the moments $\lambda_n = \text{tr}\ \Omega^n$:

$$\int [d\Omega]_N = \prod_{n>0} \int_{-\infty}^{\infty} \frac{d\lambda_n d\lambda_{-n}}{\pi} \ e^{-\frac{1}{n}\lambda_n \lambda_{-n}}. \qquad (4.11)$$

Therefore we can rewrite the integral in (4.8) as a simple Legendre transformation with respect to the moments $\lambda_n = \text{tr}\ \Omega^n$:

$$\mathcal{Z}_N[\ \vec{t}\ ] = \prod_{n\neq 0} \int_{-\infty}^{\infty} \frac{d\lambda_n}{\sqrt{\pi}} \ e^{-\frac{\lambda_n \lambda_{-n}}{2|n|} + \lambda_n t_n} Z_N[\ \vec{\lambda}\ ], \qquad (4.12)$$

[5] The most elegant way to do it is by using the representation of the moments of the unitary matrix bosonic oscillator modes [15]. Replacing Tr $\Omega^n \to \alpha_{-n} + \bar{\alpha}_n$, where $\{\alpha_n\}$ and $\{\bar{\alpha}_n\}$ satisfy the canonical commutation relations $[\alpha_m, \alpha_n] = [\bar{\alpha}_m, \bar{\alpha}_n] = m\delta_{m+n,0}$, integrals of products of traces can be calculated as expectation values with respect to the vacuum satisfying $\alpha_n|0\rangle = \bar{\alpha}_n|0\rangle = 0$ when $n > 0$.

which can be written also as

$$\mathcal{Z}_N[\,\vec{t}\,] = e^{\sum_{n>0} nt_nt_{-n}}\ e^{\hat{T}} Z_N[\,\vec{\lambda}\,]\ e^{-\hat{T}} \quad (4.13)$$

with

$$\hat{T} = \sum_{n\geq 1}\left(n\partial_{\lambda_n}\partial_{\lambda_{-n}} + t_n\partial_{\lambda_{-n}} + t_{-n}\partial_{\lambda_n}\right). \quad (4.14)$$

In the leading order, the two free energies are related by

$$\log \mathcal{Z}_N[\,\vec{t}\,] = \sum_{n\neq 0}\lambda_n t_n - \sum_{n>0}\frac{1}{n}\lambda_n\lambda_{-n} + \log Z_N[\vec{\lambda}]$$

where the r.h.s. is evaluated as the saddle point $\vec{\lambda} = \vec{\lambda}_{\rm s.p.}(\vec{t})$, given by the solution of the equation

$$t_n = \frac{1}{n}\lambda_{-n} - \frac{\partial}{\partial\lambda_n}\log Z_N[\vec{\lambda}]. \quad (4.15)$$

Explicit relations between the observables in the two ensembles can be written for any given genus. For example, the vortex-antivortex correlation functions on the sphere $\mathcal{G}_n[\,\vec{t}\,] = n\frac{\partial^2 \log \mathcal{Z}_N[\,\vec{t}\,]}{\partial t_n \partial t_{-n}}$ and $G_n[\,\vec{\lambda}\,] = n\frac{\partial^2 \log Z_N[\,\vec{\lambda}\,]}{\partial\lambda_n\partial\lambda_{-n}}$ are related by

$$\mathcal{G}_n[\,\vec{t}\,] = (1 - G_n[\,\vec{\lambda}\,])^{-1}. \quad (4.16)$$

The perturbative expansion of the new ensemble involves planar graphs with microscopic "necks" obtained by identifying the boundaries of two faces with equal number of sides but opposite orientations and vorticities and then removing the faces[6]. If the two faces belong to two surfaces of genus h_1 and h_2, the result is a surface with genus h_1+h_2. If the two faces belong to the same surface of genus h, the result is a surface of genus $h+1$. The surfaces contributing to the spherical free energy, $F^{(0)}$, look like trees of spherical bubbles ("cactuces"). Such necks have nontrivial vorticities defined by the number of times they wind around the target circle. The conservation of vorticity allows to express the first derivative in (4.15) as $\frac{\partial}{\partial\lambda_n}\log Z_N[\vec{\lambda}] = \frac{1}{n}G_n[\,\vec{\lambda}\,]\ \lambda_{-n}$ and write it, using (4.16), as

$$\lambda_n = n\mathcal{G}_n[\,\vec{t}\,]\ t_{-n}. \quad (4.17)$$

[6]Such configurations can be also visualized as Euclidean wormholes in the two-dimensional universe.

4.3 The grand canonical ensemble (GCE)

We will also perform the Legendre transformation with respect to the size N of the matrix field

$$\mathcal{Z}[\mu_F, \vec{t}] = \sum_{N=0}^{\infty} e^{2\pi R\mu_F N}\mathcal{Z}_N[\,\vec{t}\,], \quad (4.18)$$

which means that we are considering the grand canonical ensemble in which N becomes an operator coupled to the chemical potential μ_F. In particular, when $\vec{t}=0$, the integral (4.18) gives the standard matrix formulation of the $c=1$ string theory (see, for example, [6]).

Let us see what happens with the planar graphs in the grand canonical ensemble. The relation between the two ensembles can be obtained by performing the quasiclassical expansion around the saddle point $N = N_{\rm s.p.}(\mu_F)$. The saddle point equation for $\kappa = \kappa_{\rm s.p.}(\mu_F)$ can be written, in the genus-zero approximation, as $\mu_F = -\frac{\partial[N^2F^{(0)}(\hbar N,\Omega)]}{2\pi R\ \partial N} = -\frac{1}{2\pi\hbar R}\frac{\partial[\kappa^4F^{(0)}(\kappa)]}{2\kappa\ \partial\kappa}$, where $F^{(0)}(\kappa,\Omega)$ is the leading term in the genus expansion (4.5). This means that the cosmological constant μ_F in the GCE is proportional to the tadpole in the CE, *i.e.* the contribution of the spherical surfaces with one pinned point. In the planar limit the surfaces in the grand canonical ensemble are trees (cactuces) made of the spherical surfaces of the original model. The expansion around the saddle point in the sum over N is organized by a set of Feynman rules with propagator $-\frac{1}{\partial_N^2(N^2F^{(0)})}$, tadpole $\mu_F = \partial_N(N^2F^{(0)})$, and vertices $\partial_N^n(N^2F^{(h)})$ with $n+2h\geq 2$ representing the contribution in the CE of the surfaces of genus h with n punctures. The propagator can be visualized as a microscopic tube connecting two punctures and the diagrammatical expansion can be thus as the sum over surfaces with microscopic wormholes.

We have seen that the integration in Ω and the sum over the size N have similar effects: they introduce contact interactions due to microscopic tubes, connecting two surfaces or different points of the same surface in the original string theory. However, as we shall discuss in the next section, only the the microscopic tubes with zero vorticity will survive in the large N limit.

4.4 The scaling limit

It is relatively easy to calculate the functional integral with respect to the matrix field $A(x)$ at fixed Ω, once we are interested only in the scaling limit. In this section we will try to give an intuitive picture of the scaling limit and extract the relevant piece of the the integrand of (4.1) before performing the integral. We will consider in detail the geometrical meaning if the scaling limit in terms of planar graphs, since such a discussion cannot be easily found in the published literature. Let us first consider the scaling limit of the theory with no vortices.

∘ *The scaling limit in the singlet sector of MQM: the canonical ensemble*

Consider first the ensemble with N fixed, which we call canonical (CE), saving the word grand canonical (GCE) for the ensemble with fixed μ. In the ensemble with N fixed, the scaling limit is achieved in the vicinity of the critical value $\kappa = \kappa_c$ of the coupling (4.4), for which the sum of planar graphs diverges. The size of a typical planar graph grows as

$$\#\text{faces} \sim (\kappa_c - \kappa)^{-1}.$$

We assume that each link has length a and take the limit $a \to 0$ so that the area

$$\mathcal{A} = a^2(\#\text{faces})$$

of the discretized worldsheet remains fixed. The corresponding constant is then defined as

$$\Delta = \frac{\kappa_c - \kappa}{a^2\ \kappa_c}. \qquad (4.19)$$

The scaling limit is approached by letting $a \to 0$ so that Δ remains finite. Then the area of the typical planar graph will be of order $1/\Delta$.

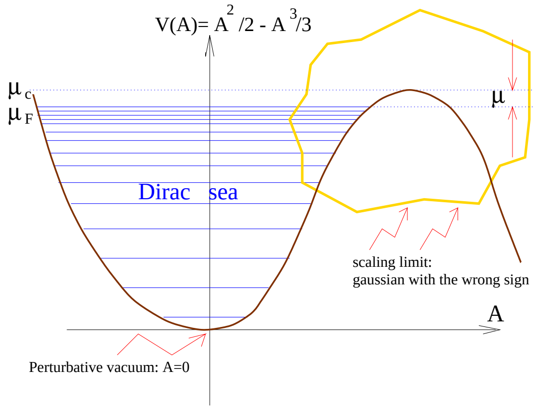
The integration with respect to Ω projects on the singlet representation of MQM, in which the wave function $\Psi(a_1, ..., a_n)$ should be totally antisymmetric with respect to permutations of the eigenvalues $A_1, ..., A_N$ of the matrix A. Then the matrix model is reduced to a system of N noninteracting one-dimensional fermions in a common potential $V(A)$. The scaling behavior is reached when the fermi level becomes close to the top of the matrix potential $V(A) = -\frac{1}{2}A + A^3/3$. The piece of the potential relevant to the scaling limit is Gaussian (with the wrong sign). The nongaussian part of the potential plays the role of a cutoff wall placed at distance of $\sim 1/\sqrt{\hbar}$ from the top.

Since we are going to pass to the GCE, we return to the original parameter $\hbar$ and denote its critical value by $\hbar_c$. Then $\kappa_c^2 = \hbar_c N$. It will be convenient to denote $\hbar_c$ as

$$\hbar_c = \frac{1}{N_c}$$

and choose the cutoff as $a^2 = \frac{1}{N_c}$. Then the renormalized cosmological coupling is simply

$$\Delta = N_c - N.$$



The Dirac sea near the top of the potential.

Once the energy levels ϵ_n of the Hamiltonian $\mathcal{H} = -\frac{1}{2}\partial_A^2 + V(A)$ are known, the partition function is given by

$$Z_N = \sum_{n=1}^{N} e^{-\frac{1}{\hbar}2\pi R\epsilon_n}.$$

The quasiclassical fermionic levels are obtained by the Bohr-Sommerfeld quantization condition $\oint p_n(A)dA = 2\pi n\hbar$, with $p_n = \sqrt{2[\epsilon_n - U(A)]}$. Then the leading term of the free energy in the scaling limit is (see, for example, [6])

$$\mathcal{F}^{(0)}(\Delta) = \frac{R}{2g_s^2}\frac{\Delta^2}{\log(a^2\Delta)}, \qquad (4.20)$$

where g_s is some finite constant. In the following we will rescale the cutoff a so that $g_s = 1$. Eq. 4.20) means that the number of genus zero discretized surfaces of area A grows as

$$e^{N_c A}A^{-3}(\log A)^{-2}.$$

$\circ$ *The scaling limit in the singlet sector of MQM: the grand canonical ensemble*

In the grand canonical ensemble (4.18), the partition function becomes

$$\mathcal{Z}_N(\mu_F) = \sum_{N=1}^{\infty} e^{\frac{1}{\hbar}2\pi R(\mu_F - \epsilon_N)}.$$

The cosmological constant is now

$$\mu = \mu_c - \mu_F$$

where $\mu_c = 1/6$ is the energy at the top of the potential. In our normalization, the role of string coupling constant is played by $1/\mu$: the contribution of the surfaces of genus $h > 1$ is given by the coefficient in front of the term μ^{2-2h}, which is the statement of the double scaling limit in this case. The genus expansion in the scaling limit is (see eqs. (3.30) and (9.12) in [6]):

$$\mathcal{F}(\mu) = -\frac{R}{2}\,\mu^2 \log(\mu a) - \frac{R+\frac{1}{R}}{24}\log(\mu a) + R\sum_{h=2}^{\infty}\mu^{2-2h)} f^{(h)} \qquad (4.21)$$

where
$f^{(k)} = (2k-3)!\; 2^{-2k}\sum_{n=0}^{k}\left(\frac{1}{R}\right)^{2n}(2^{2(k-n)}-2)$ $(2^{2n}-2)|B_{2(k-n)}B_{2n}|([2(k-n)]![2n]!)^{-1}$
and B_m are the Bernoulli numbers.

Eq. (4.21) means that in the grand canonical ensemble the contribution of the genus zero surfaces with area $\mathcal{A}$ grows as

$$e^{\mu_c A}A^{-3},$$

i.e. there are no logarithmic scaling violations[7]. Note that the genus zero free energies in the two ensembles have opposite signs. This fits to the general observation by I. Klebanov and collaborators [17, 18] that surfaces with critical self-touching interaction (microscopic necks) are described by the non-convenient branch of the Liouville exponent. As a consequence, at the point when the self-touching interaction becomes critical, the string susceptibility exponent $\gamma_{\rm str}$ jumps to a positive value

$$\gamma_{\rm str} \;\to\; \frac{\gamma_{\rm str}}{\gamma_{\rm str}-1}.$$

In the limit $\gamma_{\rm str} \to 0$ relevant for our discussion, this means that the genus zero free energy changes its sign:

$$F^{(0)} \sim \frac{\Delta^{2-\gamma_{\rm str}}-\Delta^2}{\gamma_{\rm str}} \;\to\; -F^{(0)}.$$

The cosmological constants in the two ensembles are related to the value of the Liouville field at the origin in the flat background (1.2) as

$$\Delta = 2\phi_0 e^{2\phi_0}, \quad \mu = e^{2\phi_0}.$$

In this way the CE of fermions (MQM with N fixed) describes a gaussian field coupled to a Liouville theory with potential $\sim 2\phi e^{-2\phi}$, while the GCE (μ fixed) describes a theory with potential $\sim e^{-2\phi}$.

$\circ$ *The scaling limit in presence of vortices*

The scaling behavior of the vortex couplings $\lambda_n = {\rm tr}\ \Omega^n$ is dictated, according to the KPZ-DDK scaling arguments, by the dimension of the m-vortex operators:

$$\lambda_n = \hat{\lambda}_n\ \Delta^{1-\frac{1}{2}|m|R} \qquad (4.22)$$

with $\hat{\lambda}_n$ being dimensionless constants.

Generically the wormholes are not critical, and the original partition function (4.1) and its Legendre transform (4.8) will have the same scaling limit, up to a finite rescaling of the coupling constants. This change is not dramatic unless the the matrix potential are specially tuned so that the relation (4.16) becomes singular, i.e. $G_n[\,\vec{\lambda}\,] = 1$ for some n. At such critical point, if we assume that it exists, the scaling dimension of the vortex operators $V_{\pm n}$ will change from $\frac{|n|R}{2}$ to $-\frac{|n|R}{2}$. We will assume that no such critical point is nearby. This is certainly true when the couplings t_n are small, because otherwise the wormholes along the handles on a higher genus surface would alter its critical behavior, which is not the case in the $c = 1$ string theory.

We also assume that vortices do not change the UV behavior of the worldsheet theory and therefore the continuum limit of our generalized theory is also described by the inverse gaussian potential. One can come to this conclusion after the analysis of the twisted matrix model made in [11]. We thus rescale $A \to \frac{A-1}{\sqrt{\hbar}}$ and retain

[7] The grand canonical ensemble is in fact the simplest realization the "modified $c = 1$ matrix model" considered in [16, 17].

only the gaussian part of the potential partition function (4.1):

$$Z_N(\Omega) = \int_{A(2\pi R)=\Omega^\dagger A(0)\Omega} \mathcal{D}A\, e^{-\frac{1}{2}\mathrm{tr}\int_0^{2\pi R}[(\partial_x A)^2 - A^2]dx}. \quad (4.23)$$

If we integrate first in Ω and then in A, the result will depend logarithmically on the cutoff a, which was absorbed in the definition of the functional measure. However, the cutoff is not necessary if we perform first the integration in X and then in Ω, as suggested in [11]. The integral in Ω can be regularized by adding a small imaginary part to R and consider the integration with respect of the eigenvalues of Ω as a contour integration. In the case $\vec{t} = 0$, such analytical regularization reproduces the result 4.21) for the free energy [11], up to the logarithmic dependence of the UV cutoff a, which is a consequence of our analytical regularization.

5. Integrability of the scaling limit

It has been already pointed out that at the self-dual radius, the integrable structure of the $c = 1$ string theory is described by the Toda lattice hierarchy [20, 21, 22, 23] We will see that this is the case for any compactification radius. For that we will need to write explicitly the GCE partition function in terms of the eigenvalues $z_1, ...z_N$ of the twisting matrix Ω.

5.1 The GCE partition function as a Fredholm determinant

Let us evaluate the integral (4.23). First we integrate with fixed boundary conditions $A = A(0)$ and $A' = A(2\pi R)$, which gives the propagation kernel of the inverse oscillator:

$$\left\langle A \left| \left(\frac{-\partial_A^2 - A^2}{2}\right)^{-1} \right| A' \right\rangle = \frac{\exp\left(i\frac{(Aq^{1/2} - A'q^{-1/2})^2}{q - q^{-1}}\right)}{[-i\pi(q - q^{-1})]^{\frac{N^2}{2}}}.$$

Here and below we use the shorthand notation

$$q = e^{2\pi i R}. \quad (5.1)$$

Then we impose the twisted boundary condition

$$A' = \Omega^{-1}A\Omega$$

and integrate with respect to the initial value A of the matrix field. The result depends only on the eigenvalues z_j of Ω:

$$\mathcal{Z}_N(\Omega) = \prod_{j,j'=1}^{N} \frac{1}{|z_j q^{1/2} - z_{j'} q^{-1/2}|}. \quad (5.2)$$

The same holds for the integral (4.8) and we can replace the integration measure $[d\Omega]_N$ by its diagonal part

$$\int [D\Omega]_N = \frac{1}{N!}\prod_{k=1}^{N} \oint \frac{dz_k}{2\pi i z_k} \prod_{j<k} |z_j - z_k|^2. \quad (5.3)$$

As the integration goes along the unit circle, the absolute value can be dropped out. This allows us to consider the integral as a multiple contour integral, in which case we can regularize it by adding a small imaginary part to $R = \frac{1}{2i\pi}\log q$.

It will be convenient to rescale the couplings t_n in the definition (4.18) of the partition function $\mathcal{Z}_N[\,\vec{t}\,]$ as $t_n \to 2it_n \sin(n\pi R)$. Using the Cauchy identity, we represent the canonical partition function as

$$\mathcal{Z}_N[\vec{t}] = \oint \prod_{k=1}^{N} \oint \frac{dz_k}{2\pi i z_k} \det_{jk} \frac{e^{u(z_j q^{1/2}) - u(z_k q^{-1/2})}}{z_j q^{1/2} - z_k q^{-1/2}}$$

where

$$u(z) = \sum_{n\neq 0} t_n z^n. \quad (5.4)$$

The GCE partition function

$$\mathcal{Z}[\mu, \vec{t}\,] = \sum_{N=0}^{\infty} e^{-2\pi\mu R N} \mathcal{Z}_N[\,\vec{t}\,]$$

can be represented as a Fredholm determinant

$$\mathcal{Z}[\mu, \vec{t}] = \mathrm{Det}(1 + e^{-\mu 2\pi R}\hat{K}), \quad (5.5)$$

where the operator $\hat{K}$ is defined as

$$(\hat{K}f)(z) = \oint \frac{dz}{2\pi i} \frac{e^{u(z_j q^{1/2}) - u(z_k q^{-1/2})}}{q^{1/2}z - q^{-1/2}z'} f(z').$$

Such Fredholm determinants have been studied by many authors (see for example [24, 25, 26, 27]), and it is well known that they contain the integrable structure of the Toda chain hierarchy [30].

5.2 Toda equation and KPZ-DDK scaling

It can be shown following the same arguments as in [27], that our Fredholm determinant with the normalization of the coupling constants we adopted, is almost a τ function of the Toda chain hierarchy with $t_0 = -i\mu$:

$$\tau_l[-i\mu, \vec{t}] = e^{-\sum_{n\geq 1} n t_n t_{-n}} \mathcal{Z}[\mu, \vec{t}]. \qquad (5.6)$$

The matrix model in the scaling limit is therefore characterized by an infinite set of commuting renormalization flows associated with the couplings t_n and its partition function satisfies the PDE of the Toda hierarchy. The dispersionless limit of the Toda hierarchy describes the genus zero approximation of the string theory given by the leading term at $\mu \to \infty$.

In the following we will concentrate on the lowest equation of the hierarchy, the Toda lattice equation

$$\tau(t_0)\partial_+\partial_- \log\tau(t_0) + \frac{\tau(t_0+1)\tau(t_0-1)}{\tau(t_0)\tau(t_0)} = 0. \qquad (5.7)$$

It is easy to see that due to the symmetry of the measure, the τ-function depends only on the product $\lambda^2 = t_+t_-$. In terms of the grand canonical free energy $\mathcal{F}(\mu,\lambda) = \log\mathcal{Z}(\mu,\lambda)$, the Toda equation reads

$$\partial_+\partial_- F(\lambda,\mu) + e^{F(\lambda,\mu+i)+\mathcal{F}(\lambda,\mu-i)-2\mathcal{F}(\lambda,\mu)} = 1. \qquad (5.8)$$

Since we are interested in the dispersionless limit $\mu \to \infty$, we write this equation in a differential-operator form

$$\partial_+\partial_-\mathcal{F}(\lambda,\mu) + e^{-4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\lambda,\mu)} = 1 \qquad (5.9)$$

and expand the equation in a series in $1/\mu$. This equation is of second order in λ and has unique solution satisfying the boundary conditions $\partial_\lambda\mathcal{F}(\lambda,\mu)|_{\lambda=0} = 0$ and $\mathcal{F}(0,\mu) = \mathcal{F}(\mu)$, with $\mathcal{F}(\mu)$ given by (4.21).

It is remarkable that equation (5.9) is is compatible with the KPZ-DDK scaling, i.e. with the expansions (3.5)-(3.7) and (5.17)-(3.12), if we take $a = 1$. Let us remind that the definition of the τ-function implies that the integration with respect to the eigenvalues z_i is considered as a contour integration, after adding a small imaginary part to the compactification interval. With such analytic regularization the partition function does not depend on the cutoff.

Since we are mainly interested in the critical point $y \to 0$ or $\lambda \to \infty$ with μ finite, we will use the expansion (5.17)-(3.12). Than the Toda chain equation (5.8) gives a nonlinear ODE for $f^{(0)}(y)$ and a triangular system of linear second order ODE for the functions $f^{(h\geq 1)}(y)$.

The boundary condition (4.21) defines the asymptotics of solution at $y \to \infty$. It can be satisfied only if we choose $a = 1$. Indeed, the free energy calculated from the τ-function does not depend on the UV cutoff a. The integrals over the eigenvalues of the twisting matrix entering in the definition of the τ-function should be understood as contour integrals, otherwise the integrable structure will be destroyed. No trace of a cutoff is left in such an analytic regularization.

Substituting (5.17)-(3.12 in (5.9), we get for the universal part of the free energy

$$f(g_s, y) = \sum_{h=0}^{\infty} g_s^{2h-2} f^{(0)}(y) \qquad (5.10)$$

the equation

$$(1-\omega^2)g_s^2(y\partial_y + g_s\partial g_s)^2 f + 4e^{4\sin^2(\frac{g_s}{2}y\partial_y)f} = 0 \qquad (5.11)$$

where we denoted

$$\omega = \frac{R}{2-R}. \qquad (5.12)$$

It is evident that this equation can be developped in a Taylor series in g_s^2.

5.3 Partition functions on the sphere and on the torus

Genus zero

In the leading order in g_s, eq. (5.11) reduces to a non-linear ODE for $f^{(0)}(y)$

$$(1-\omega^2)(y\partial_y - 2)^2 f^{(0)} + 4e^{\partial_y^2 f^{(0)}} = 0. \qquad (5.13)$$

Its solution is given, in terms of the universal part of the susceptibility

$$X_0 = \partial_y^2 f^{(0)} \qquad (5.14)$$

by

$$y = e^{-\frac{1}{R}X_0} - e^{\frac{1-R}{R}X_0}, \qquad (5.15)$$

which is equivalent to (3.4), with $\chi^{(0)} = X_0 + R\log g_s$. Integrating twice in μ we get for the partition function itself

$$\mathcal{F}^{(0)}(\lambda,\mu) = \lambda^2 + \tfrac{1}{2}\tfrac{1}{g_s^2}y^2\left(R\log g_s + X_0(y)\right) + \tfrac{1}{g_s^2}R\left(\tfrac{3}{4}\tfrac{1}{R-1}e^{-2\frac{R-1}{R}X_0(y)} + \tfrac{3}{4}e^{-\frac{2}{R}X_0(y)}\right) - \tfrac{1}{g_s^2}\tfrac{R^2-R+1}{R-1}e^{-X_0}, \tag{5.16}$$

with $X_0(y)$ defined by (5.15), and the string coupling is related to λ by

$$g_s = \left[(R-1)\lambda^2\right]^{\frac{1}{R-2}}.$$

In the limit $y \to 0$, or $\mu \to \infty$ with λ fixed, we reproduce from (5.16) the known asymptotics of the $c = 1$ string theory unperturbed by vortices:

$$\mathcal{F}^{(0)}(\lambda,\mu) = -\frac{1}{2}R\mu^2\log\mu + \lambda^2(1-\mu^R) + ...$$

For R finite, the term λ^2 in the free energy can be dropped, but when $R \to 0$, this term assures that the the free energy also vanishes. The free energy calculated from the τ-function does not depend on the UV cutoff, therefore we have to add by hand the term $\mu^2\log a$.

In the "black hole" limit $y \to \infty$ or $\lambda \to \infty$ with μ fixed, we obtain the asymptotics

$$\mathcal{F}^{(0)}(\lambda,\mu) \sim A\lambda^{4/(2-R)}, \tag{5.17}$$

with $A = -\frac{1}{4}(2-R)^2(R-1)^{R/(2-R)}$.

For the black hole case ($R = \frac{3}{2}$) we get

$$F_0 \sim \lambda^8. \tag{5.18}$$

Genus one

The linear in g_s^2 term of (5.11) gives the linear equation

$$(1-\omega^2)(y\partial y)^2 f^{(1)} + 4e^{X_0}\partial_y^2\left(f^{(1)} - \frac{1}{12}X_0\right) = 0. \tag{5.19}$$

The solution whose large y asymptotics is compatible with (4.21), is

$$f^{(1)} = \frac{R+R^{-1}}{24R}X_0 - \frac{1}{24}\ln\left(1-(R-1)e^{\frac{2-R}{R}X_0}\right).,$$

and the value of the constant C_0 in (3.11) is $-\frac{1}{24}\frac{R+1}{R}$. Therefore

$$\mathcal{F}^{(1)} \sim -\frac{R+R^{-1}}{12(2-R)}\ln\lambda, \qquad \lambda \to \infty. \tag{5.20}$$

5.4 Nonperturvative ambiguities

It is easy to see that the terms

$$\Delta\mathcal{F} = \sum_{n\geq 1}(C_n\log\lambda + D_n)e^{-2n\pi\mu}$$

can be always added to $\mathcal{F}(\mu,\lambda)$ and it will still satisfy the eq. (5.8). As usual, the string loop expansion is defined up to these exponentially small terms $\sim e^{-2\pi\mu}$ which are negligible in the string perturbation theory. This nonperturbative ambiguity is related to the $\sim (2h)!$ divergence of the coefficients f_h in the genus expansion. It is possible in principle to fix them directly from the matrix integral, as it is the case in the known large N matrix models. At the $c = 1$ critical point they are not essential, but they can become important at the "cigar" critical point, where μ is kept finite. These terms in the free energy deserve further study, because they should describe the nonperturbative states at the Euclidean horizon (tip of the cigar), where vortices are created and annihilated.

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