Dynamical Quantum groups and Applications

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Abstract: We give an introduction to the theory of dynamical quantum groups and precise its relation to the harmonic analysis on non-compact quantum groups through the example of the quantum Lorentz-Group.

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Representation of non compact quantum groups is of central importance in different theoretical physics systems. We can at least give three examples of this fact:

1. Chern-Simons theory with non compact group G. This is particularly important in view of its applications to quantum gravity in 2+1 dimensions where the group G is equal to $SO(3,1)$ $(\Lambda > 0)$, $ISO(2,1)$ $(\Lambda = 0)$ or $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, $(\Lambda < 0)$ depending on the sign of the cosmological constant Λ.
2. Discretization of Lorentzian gravity in the spirit of Ponzano-Regge.
3. Liouville theory in the weak and probably strong coupling regime.

In these three cases, there is an associated "non compact" quantum group, which is a star Hopf algebra, with a category of unitary representations. The computation of physical quantities in these three systems amounts to compute explicitely, or to have a good understanding of the

- coupling of two unitary representations (i.e decomposition in irreducible representations of the tensor product of two unitary representations). This is what physicists call the computation of Clebsch-Gordan coefficients or $3J$.
- recoupling of three unitary representations. This is what physicists call the computation of Racah-Wigner coefficients or $6J$.

We are not only interested in the exact value of these coefficients but also in the relations they satisfy.

When the group is $SU(2)$, this is already a not completely trivial task. Of central importance is to understand the important link between hypergeometric functions and Clebsch-Gordan and Racah coefficients. Indeed $3J$ are expressed in terms of ${}_3F_2$ and $6J$ in terms of ${}_4F_3$ [1].As a result one can obtain non trivial results in the theory of hypergeometric functions using representation theory and vice-versa. Of course all this can be generalized to $U_q(su(2))$ with relatively minor modifications, i.e replacing hypergeometric functions by basic hypergeometric functions. For an introduction to the theory of hypergeometric function of one variable and its q-analog, called basic hypergeometric function, a very good reference is [2].

When the group is non compact and of rank 1, i.e $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, this is a much harder problem, because it is not just purely algebraic, but also functional analysis techniques have to be added. In this case, even up to now, very little is known for example of the Racah coefficients of unitary representations of $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$.

When the group is compact and of rank higher than 1, then this problem is not fully solved, because multiplicities occur when tensoring two irreducible representations.

The theory of dynamical quantum groups has proven to be a very effective tool to obtain results in the theory of Racah coefficients, particularly in the case where the Lie algebra is of rank greater than one and even in the classical case. The aim of this note is to introduce the reader to the basic ideas of the theory, and to emphasize its future relevant role for the study of Clebsch-Gordan and Racah coefficients for non compact quantum groups.

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1. Fusion matrices

The fundamental article on this topic is [3], a survey article is [4]. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. A choice of a polarisation on the roots gives a decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and induces a partial order $<$ on the vector space of weights $\mathfrak{h}^\star$. We denote by $U_q(\mathfrak{g})$ the standard quantization of the universal envelopping algebra of $\mathfrak{g}$. It is a Hopf algebra and denote by ϵ the counit. Let λ be a weight and denote by M_λ the Verma module of $U_q(\mathfrak{g})$, with m_λ highest weight vector, i.e $hm_\lambda = \lambda(h)m_\lambda, em_\lambda = 0, \forall e \in U_q(n^+) \cap ker\epsilon$. Let V be a finite dimensional $U_q(\mathfrak{g})$ module, it is necessarily $\mathfrak{h}$ semi-simple and we denote by $V[\nu]$ the subspace of weight ν for the action of $\mathfrak{h}$. Let Φ be an element of $Hom_{U_q(\mathfrak{g})}(M_\lambda, M_\mu \otimes V)$, we can define an element $< \Phi > \in V[\lambda - \mu]$ as follows: $\Phi(m_\lambda) = m_\mu \otimes < \Phi > + \sum_i x_i \otimes v_i$ where x_i is of weight strictly less than μ. A fundamental result [3] is: if M_μ is irreducible then the map $<>$: $Hom_{U_q(\mathfrak{g})}(M_\lambda, M_\mu \otimes V) \to V[\lambda - \mu]$ is an isomorphism. In particular M_μ is irreducible if and only if the Shapovalov form on M_μ is non degenerate. As a result if μ is not located on the zeroes of the Shapovalov determinant, which are a countable family of hyperplanes, M_μ is irreducible. In this case we will say that μ is generic.

Assume that all the weights of the Verma modules appearing in the sequel are generic. Let v in V an homogeneous element of weight $\underline{v}$, we denote by $\Phi^v_\lambda \in Hom_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\underline{v}} \otimes V)$, the unique element such that $< \Phi^v_\lambda >= v$. As a result, if λ is generic, if V and W are two finite dimensional modules and if $v \in V[\underline{v}], w \in V[\underline{w}]$, we can define the intertwiner $(\Phi^v_{\lambda-\underline{w}} \otimes \mathrm{id}_W)\Phi^w_\lambda$: $M_\lambda \to M_{\lambda-\underline{v}-\underline{w}} \otimes V \otimes W$. Let us define the *fusion operator* $J_{VW}(\lambda) : V \otimes W \to V \otimes W$ by $J_{V,W}(\lambda)(v \otimes w) =< (\Phi^v_{\lambda-\underline{w}} \otimes \mathrm{id}_W)\Phi^w_\lambda >$. The following shifted 2-cocycle condition is then a direct consequence of the definition:

$$\begin{aligned} & J_{U\otimes V,W}(\lambda)(J_{U,V}(\lambda - h^{(3)}) \otimes 1) = \\ = \; & J_{U,V\otimes W}(\lambda)(1 \otimes J_{VW}(\lambda)). \end{aligned} \tag{1.1}$$

Note that there exists a universal element $J(\lambda) \in U_q(\mathfrak{g})^{\otimes 2}$ such that $J_{V,W}(\lambda)$ is the action of $J(\lambda)$ on $V \otimes W$, as a result the 2-cocycle equation can be also written as:

$$\Delta_{12}(J(\lambda))J_{12}(\lambda - h^{(3)}) = \Delta_{23}(J(\lambda))J_{23}(\lambda). \tag{1.2}$$

Let define $F(\lambda) = J(-\lambda)$, from this we can define $R(\lambda) = F_{21}(\lambda)^{-1}R_{12}F_{12}(\lambda)$ which satisfies the *quantum dynamical Yang-Baxter Equation* also called Gervais-Neveu-Felder Equation:

$$\begin{aligned} & R_{12}(\lambda + h^{(3)})R_{13}(\lambda)R_{23}(\lambda + h^{(1)}) = \\ = \; & R_{23}(\lambda)R_{13}(\lambda + h^{(2)})R_{12}(\lambda). \end{aligned} \tag{1.3}$$

"Dynamical Quantum groups" are the algebraic objects which are associated to the solution of this equation [6]. The reader is invited to read the work of P.Etingof, A. Varchenko on the study of the algebra of interwiners and on the numerous results on representation theory that can be obtained [3, 8].

One can wonder if there are other way to compute $F(\lambda)$ apart from the definition: a central result is the linear equation satisfied by $F(\lambda)$ which has been first discovered in [10]. We will denote by ω the restriction of the Killing form to $\mathfrak{h}\times\mathfrak{h}$. This form identifies $\mathfrak{h}$ and $\mathfrak{h}^*$, we will denote by h_λ the element of $\mathfrak{h}$ associated to the weight λ via this identification. $F(\lambda)$ is the unique solution, in $U_q(b^+) \otimes U_q(b^-)$, of the linear equation:

$$F(\lambda)(1 \otimes B(\lambda)) = R^{(0)-1}(1 \otimes B(\lambda))F(\lambda) \tag{1.4}$$

with $B(\lambda) = q^{h_\lambda + \omega^{ij} h_i h_j}, R = q^{\omega^{ij} h_i \otimes h_j} R^{(0)}$.

When λ is non generic, for example when λ is a dominant integral weight, this theory can be still applied when λ is sufficiently far from the wall of the Weyl chamber. ([7] for details)

Indeed let λ be an integral dominant weight and denote by V_λ the finite dimensional simple module of highest weight vector v_λ. To any $\phi \in Hom(V_\lambda, V_\mu \otimes V_\nu)$ we can still define $< \phi >$ by $\phi(v_\lambda) = v_\mu \otimes < \phi > + \sum_i v_i \otimes w_i$ with v_i of weight strictly less than μ. If $\lambda, \mu, \nu \in P^+$, $<>$: $Hom(V_\lambda, V_\mu \otimes V_\nu) \to V_\nu[\lambda - \mu]$ is always injective, and is also surjective if (for fixed ν) μ (and therefore λ) is sufficiently far from the wall of the Weyl chamber. We can therefore still define operators $J_{VW}(\lambda)$ in this situation, and they satisfy the 2-cocycle equation. In the case of $U_q(sl(2))$ the construction of [5] is very similar, the only difference amounts to the fact that the normalization of the intertwiners ϕ^v_λ are chosen differently. A very important relation, first described in [5], is the relation between Racah Coefficients and matrix elements of the matrix $F(\lambda)$ in the representation $V_{j_1} \otimes V_{j_2}$. More precisely we have the relation:

$$F^{j_1 j_2}(\lambda)_{\sigma_1\sigma_2,\sigma_1'\sigma_2'} = \sum_{j_{12}} \begin{pmatrix} j_1 & j_2 & j_{12} \\ \sigma_1 & \sigma_2 & \sigma_1+\sigma_2 \end{pmatrix}$$
$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j(\lambda) & j(\lambda)+\sigma_1'+\sigma_2' & j(\lambda)+\sigma_2' \end{Bmatrix}. \quad (1.5)$$

where we have used the conventions of [3] and $2j(\lambda)+1 = \lambda \in \mathbb{Z}^+$. This results shows that the matrix elements of $F(\lambda)$ can be expressed in terms of $3J$ of $U_q(su(2))$ and $6J$ of $U_q(su(2))$. From this relation, it is trivial to show that the pentagonal equation on $6J$ of $U_q(su(2))$ is equivalent to the 2-cocycle identity on $F(\lambda)$. Note that because $F(\lambda)$ can be defined for generic $\lambda \in \mathbb{C}$, the previous relation still hold exactly true with $6J$ coefficients of $U_q(su(2))$ prolongated in the complex variable λ. We call these coefficients $6J(1)$ to recall that they have been extended in 1 complex parameter.

2. Harmonic Analysis on $U_q(sl(2,\mathbb{C})_\mathbb{R})$

It is impossible in such a small amount of space to fully describe the content of the papers [11][12]. We will give here a brief review of the theorems which have been proven in these works. $U_q(su(2))$, $q = e^{-\hbar} \in]0,1[$ is a star Hopf algebra whose quantum double $D(U_q(su(2))$ can be taken as the definition of the quantum envelopping algebra $U_q(sl(2,\mathbb{C})_\mathbb{R})$ of the realification of the Lie algebra $sl(2,\mathbb{C})$. In [13] a definition of the spaces of compact supported continuous functions on $SL(2,\mathbb{C})$ is given, it is a *multiplier C^* algebra*. In [14] the classification of irreducible unitary representations is given and is completely parallel to the classification of unitary representations of the complex group $SL(2,\mathbb{C})$ (up to tensoring with a one dimensional representation which has no classical analog). Infinite irreducible unitary representations are classified by two parameters (m,ρ) and we obtain:

- Principal representations $\Pi(m,\rho)$, $m \in \frac{1}{2}\mathbb{Z}^+$, $\rho \in]-\frac{\pi}{\hbar}, \frac{\pi}{\hbar}[$, acting on $V_{(m,\rho)}$
- Complementary representations $\Pi^c(0,\rho)$, $\rho \in]0,1[$.

In [11] we have proved a Plancherel theorem, i.e we have shown that:

$$L^2(SL_q(2,\mathbb{C})_\mathbb{R}) = \\ = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}^+} \int^{\oplus} V_{(m,\rho)} \otimes V^*_{(m,\rho)} dP(m,\rho) \quad (2.1)$$

with $dP(m,\rho) = [m+i\rho]_q[m-i\rho]_q$, where as usual we denote $[z]_q = \frac{q^z - q^{-z}}{q-q^{-1}}$. In order to prove this theorem, we have shown that the matrix elements of $\Pi(m,\rho)$, acting on any element of $U_q(sl(2,\mathbb{C})_\mathbb{R})$ can be expressed in terms of prolongation of 6J of $U_q(su(2))$ in one complex spin, i.e $m + i\rho$. Plancherel theorem amounts to integral identities on these prolongation of $6J$ symbols.

It would be particularly interesting to generalize these results to the case of the quantization of any semi-simple complex Lie group.

In [12] we have shown the following theorem:

$$\Pi(m_1,\rho_1) \otimes \Pi(m_2,\rho_2) = \\ = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}^+, m+m_1+m_2\in\mathbb{Z}} \int^{\oplus} \Pi(m,\rho) d\rho, \quad (2.2)$$

and we have computed exactly the Clebsch-Gordan coefficients of this decomposition. We have shown that these $3J$ of the principal representation can be expressed naturally in terms of both prolongation of $6J$ of $U_q(su(2))$ in one complex variable, called 6J(1), and prolongation of $6J$ of $U_q(su(2))$ in three complex parameters, namely $m_1+i\rho_1, m_2+i\rho_2, m+i\rho$, that we called $6J(3)$.

We have also shown that these Clebsch-Gordan coefficients can be expressed in terms of Racah polynomials and Askey-Wilson polynomials. The orthogonality relations on the Clebsch-Gordan coefficients of the principal representations translate to orthogonality relations mixing Racah polynomials and Askey-Wilson polynomials. The simplest of these relations being the orthogonality relation of Askey-Wilson polynomial for the Askey-Wilson measure.

These $6J(3)$ have a natural representation theory interpretation[1] and can be computed using the universal $F(\lambda)$. More precisely, there exists a family of infinite dimensional irreducible modules, denoted $G_{\nu,\mu}$ which basis is (v_β) with $\beta \in \nu+\mathbb{Z}$. The action of the standard generators h, e, f of $U_q(sl(2))$ reads:

$$hv_\beta = 2\beta v_\beta,$$
$$ev_\beta = [\mu-\beta]_q v_{\beta+1}, fv_\beta = [\mu+\beta]_q v_{\beta-1},$$

the value of the Casimir element in this representation is $[\mu]_q[\mu+1]_q$. The $6J(3)$ are obtained by representing $F(\lambda)$ in the representation $V_I \otimes G_{\nu,\mu}$, with $2\lambda+1 = m+i\rho, 2\nu+1 = m_2+i\rho_2, 2\mu+1 = m_1+i\rho_1-m-i\rho$.

It remains to compute exactly Racah coefficients for recoupling of principal representations. We expect that these coefficients can be expressed in terms of non terminating ${}_8\phi_7$ basic hypergeometric functions.

We are presently applying these mathematical tools to the two previously stated physical problems:

- we can modify the construction of the representation of the moduli algebra in the compact case [15] to apply it to the quantum Lorentz group case. We will therefore obtain non trivial result in 2+1 quantum gravity with positive cosmological constant.
- the exact expression of Racah coefficients of principal representations of the Lorentz and of the quantum Lorentz groups are the very beginning step for the definition of Lorentzian quantum gravity models in 3+1 dimensions. [16]. The use of dynamical quantum groups, through the use of the Fusion matrices, should simplify the computation of these coefficients.

[1] I thank P.Etingof for explaining this to me.

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