Dual Baxter equations and quantum algebraic geometry.

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ABSTRACT: According to Mumford the applicability of algebraic geometry to integrable models is due to their relation to affine Jacobi variety. We give elementary explanation of this fact based on the method of separation of variables. Further we explain the quantum version of this construction making stress on Baxter equation and duality.

1. Introduction.

The importance of methods of algebraic geometry for the theory of classical integrable models is well-known. The solutions of integrable equations are written in terms of Riemann theta-function. The question usually asked is where is the analogue of the Riemann theta-function in the quantum case. The answer to this exactly question is not known, but the quantum analogue of classical methods of algebraic geometry can be explained in wider context.

This wider point of view on the classical integrable models was nicely explained by Mumford [1]. First, we have to admit that all the exact solutions that we know are obtained for algebraically integrable model. The Liouville theorem tells that the phase space of any integrable model is foliated into tori everyone of these tori represents the level of integrals of motion. In the case of algebraic integrable model the tori in question happen to be real sections of Jacobi variety of Riemann surfaces, the moduli of these Riemann surfaces are defined by values of the integrals of motion.

Obviously, the complexification must be important in this situation. Usually, it can be shown that the complexified phase-space allows embedding into affine complex space. Under this embedding the complexified Liouville torus happens to be non-compact affine Jacobi variety. Affine Jacobi variety is obtained from the Jacobi variety by removing the theta-divisor: the sub-variety on which the theta-function vanishes. This is how the theta-functions enter the game.

Let me explain what is going on in some details. Typically the phase space of an algebraically integrable model is defined as follows. There is a $N\times N$ matrix $\boldsymbol{m}(z)$ which depends polynomially on the spectral parameter z. The coefficients of decomposition of the matrix elements are considered as coordinates in an affine space V. Consider the characteristic polynomial $f(z,w)=\det(\boldsymbol{m}(z)-wI)$. all the coefficients of these polynomial are integrals of motion. The level of these integrals of motion is affine Jacobi variety of the algebraic curve $f(z,w)=0$.

This construction allows quantum deformation as we shall explain in this talk. We shall consider the case 2×2 matrix $\boldsymbol{m}(z)$ when the spectral curve is hyper-elliptic. We shall show that certain important properties of the affine ring (the classical algebra of observables) are preserved. Finally, we shall see that in the quantum case new phenomenon of duality appears.

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2. Algebraic model of affine Jacobi variety.

Consider the matrix

$$\boldsymbol{m}(z) \equiv \begin{pmatrix} \boldsymbol{a}(z) & \boldsymbol{b}(z) \\ \boldsymbol{c}(z) & \boldsymbol{d}(z) \end{pmatrix}$$

which depends polynomially on the parameter z. We require that the matrix elements are of the form:

$$\begin{aligned} \boldsymbol{a}(z) &= z^{g+1} + \boldsymbol{a}_1 z^g + \cdots + \boldsymbol{a}_{g+1}, \\ \boldsymbol{b}(z) &= z^g + \boldsymbol{b}_1 z^{g-1} + \cdots + \boldsymbol{b}_g, \\ \boldsymbol{c}(z) &= \boldsymbol{c}_2 z^g + \boldsymbol{c}_3 z^{g-1} \cdots + \boldsymbol{c}_{g+2}, \\ \boldsymbol{d}(z) &= \boldsymbol{d}_2 z^{g-1} + \boldsymbol{d}_3 z^{g-2} \cdots + \boldsymbol{d}_{g+1} \end{aligned}$$

Let us impose further the equation:

$$\boldsymbol{a}(z)\boldsymbol{d}(z) - \boldsymbol{c}(z)\boldsymbol{b}(z) = 1$$

which defines a quadric $\mathcal{M}$ in the space $\mathbb{C}^{4g+2}$ with coordinates

$$\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{g+1}, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_g, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_{g+2}, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_{g+1}$$

The algebra of functions on $\mathcal{M}$ is denoted by $\mathcal{A}$. Consider the g-dimensional section of $\mathcal{M}$

$$\boldsymbol{t}(z) \equiv \boldsymbol{a}(z) + \boldsymbol{d}(z) = t(z)$$

where the coefficients of $t(z)$ are fixed. These g-dimensional variety coincides with affine Jacobi variety $\mathcal{J}(t)$ of the hyper-elliptic curve $X(t)$:

$$w^2 - t(z)w + 1 = 0 \tag{2.1}$$

This construction is explained in details in [1].

Equivalent description of affine Jacobi variety is as follows:

$$\mathcal{J}(t) = X(t)^{\times g}/S_g - D$$

where the divisor D is:

$$D = \{(p_1, \cdots, p_g) |\, p_i = \sigma(p_j)\}$$

By σ we denote the hyper-elliptic involution:

$$\sigma(z, w) = (z, t(z) - w)$$

Equivalence of the two description is established as follows. Consider zeros of the polynomial $\boldsymbol{b}(z)$:

$$\boldsymbol{b}(z) = \prod_{j=1}^{g} (z - \boldsymbol{z}_j) \tag{2.2}$$

Obviously these zeros together with the variables

$$\boldsymbol{w}_j = \boldsymbol{d}(\boldsymbol{z}_j)$$

satisfy equation of the curve (2.1). So we construct from the matrix $\boldsymbol{m}(z)$ the divisor $p = (p_1, \cdots, p_g)$ where $p_j = \{\boldsymbol{z}_j, \boldsymbol{w}_j\}$. Oppositely, suppose we are given the divisor p. Then $\boldsymbol{b}(z)$ is constructed trivially,

$$\boldsymbol{d}(z) = \sum_{j=1}^{g} \prod_{k \neq j} \left(\frac{z - \boldsymbol{z}_k}{\boldsymbol{z}_j - \boldsymbol{z}_k} \right) \boldsymbol{w}_j \tag{2.3}$$

$\boldsymbol{a}(z)$ and $\boldsymbol{c}(z)$ can be found from trace and determinant. It is important to notice that the expression (2.3) has singularities on the divisor D.

Let us return to the algebra of function $\mathcal{A}$. The dynamical structure defined by integrable model allows to define vector fields D_j, $j = 1, \cdots, g$:

$$[D_j, D_k] = 0, \qquad D_j \boldsymbol{t}_k = 0$$

The algebra $\mathcal{A}$ possesses the important property: $\forall \boldsymbol{x} \in \mathcal{A}$ we have

$$\boldsymbol{x} = \sum_{\alpha} P_\alpha \left(D_1 \cdots D_g \right) h_\alpha$$

The coefficients of P_α depend on $\boldsymbol{t}_j$,

$$h_\alpha \, d\zeta_1 \wedge \cdots \wedge d\zeta_g$$

represent $H^g(\mathcal{J}(t))$. Thus $\mathcal{A}$ is generated by action of vector-fields D_j from finite number of functions. This property is important for quantization.

In the paper [2] it is conjectured that

$$h_\alpha = h_\alpha(\boldsymbol{b}_1 \cdots, \boldsymbol{b}_g)$$

The algebra $\mathcal{A}$ can be identified with the algebra of functions of certain integrable model. $\mathcal{A}$ is Poisson algebra. Thus the Poisson structure can be introduced. The coefficients of $\boldsymbol{t}(z)$ are commuting Hamiltonians:

$$\{\boldsymbol{t}_j, \boldsymbol{t}_k\} = 0, \qquad D_j \boldsymbol{x} = \{\boldsymbol{t}_j, \boldsymbol{x}\}$$

The introduction of the Poisson structure is the first step to quantization.

3. Algebra A(q).

Introduce the parameter of deformation:

$$q = e^{i\gamma}$$

The deformed algebra $\mathcal{A}(q)$ is defined by quadratic commutation relations:

$$\begin{aligned} &r_{21}(z_1, z_2)\, \boldsymbol{m}_1(z_1)\, k_{12}(z_1)\, s_{12}\, \boldsymbol{m}_2(z_2)\, k_{21}(z_2) = \\ &= \boldsymbol{m}_2(z_2)\, k_{21}(z_2)\, s_{21}\, \boldsymbol{m}_1(z_1)\, k_{12}(z_1)\, r_{12}(z_1, z_2) \end{aligned} \tag{3.1}$$

This equation is written in $\mathbb{C}^2 \otimes \mathbb{C}^2$, $a_1 = a \otimes I$, $a_2 = I \otimes 2$, $a_{21} = Pa_{12}P$ The $\mathbb{C}$-number matrices r, k, s are given by:

$$\begin{aligned} &r_{12}(z_1, z_2) = \frac{z_1 - qz_2}{1-q}\,(I \otimes I) + \\ &+ \frac{z_1 + qz_2}{1+q}\,(\sigma^3 \otimes \sigma^3) + \\ &+ 2\,(z_1\sigma^- \otimes \sigma^+ + z_2\sigma^+ \otimes \sigma^-), \\ &k_{12}(z) = I \otimes (I - \sigma^3) + \\ &+ \left(q^{-\sigma^3} + z(q^2-1)\sigma^-\right) \otimes (I + \sigma^3), \\ &s_{12} = I \otimes I - (q - q^{-1})\sigma^- \otimes \sigma^+ \end{aligned}$$

The relation of this algebra to usual r-matrix algebra is explained in [3]. The quantum determinant

$$q\boldsymbol{d}(z)\boldsymbol{t}(zq^{-2}) - q^2\boldsymbol{d}(z)\boldsymbol{d}(zq^{-2}) - q\boldsymbol{b}(z)\boldsymbol{c}(zq^{-2}) = 1$$

belongs to the center. One finds the commutative family created by

$$\boldsymbol{t}(z) = q\boldsymbol{a}(z) + q^2\boldsymbol{d}(z) - z(q^2-1)\boldsymbol{b}(z)$$

We accept the following

Conjecture 1. Every $x \in \mathcal{A}(q)$ can be presented as

$$\boldsymbol{x} = p_L(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)h_\alpha(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g)p_R(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)$$

with finite number of functions h_α playing the role of quantum cohomologies.

4. Realization of A(q).

I would like to describe a realization of the algebra $\mathcal{A}(q)$ which explains the relation to integrable models. Consider the self-adjoin operators x_j and y which satisfy the commutation relations:

$$\begin{aligned} &x_k x_l = q^2 x_l x_k \qquad k < l, \\ &y x_k = q x_k y \qquad \forall k \end{aligned}$$

The Hamiltonian of the system:

$$\boldsymbol{h} = q^{-1}\sum_{k=1}^{2g+2} x_k x_{k-1}^{-1}$$

where

$$x_{2g+3} \equiv qyx_1$$

Consider the algebra A generated by u and v satisfying the commutation relations:

$$uv = qvu$$

These operators are realized in $L_2(\mathbb{R})$ as follows:

$$v = e^{\varphi}, \qquad u = e^{i\gamma\frac{d}{d\varphi}}$$

The algebra $A^{\otimes(2g+2)}$ is generated by u_j, v_j with $j = 1, \cdots, 2g+2$.
We have the following representation of x_j, y:

$$x_k = v_k \prod_{j=1}^{k-1} u_j^{-2}, \quad y = \prod_{j=1}^{2g+2} u_j$$

in the space $\mathfrak{H} = (L_2(\mathbb{R}))^{\otimes(2g+2)}$.

Define so called monodromy matrix:

$$\widetilde{\boldsymbol{m}}(z) = \begin{pmatrix} \widetilde{\boldsymbol{a}}(z) & \widetilde{\boldsymbol{b}}(z) \\ \widetilde{\boldsymbol{c}}(z) & \widetilde{\boldsymbol{d}}(z) \end{pmatrix} = l_{2g+2}(z)\, \cdots\, l_1(z)$$

where the l-operators are

$$l(z) = \frac{1}{\sqrt{z}}\begin{pmatrix} zu & -qvu \\ zv^{-1}u^{-1} & 0 \end{pmatrix}$$

The modified monodromy matrix

$$\begin{aligned} \boldsymbol{m}(z) \equiv &\begin{pmatrix} \boldsymbol{a}(z) & \boldsymbol{b}(z) \\ \boldsymbol{c}(z) & \boldsymbol{d}(z) \end{pmatrix} = \\ = &\begin{pmatrix} \widetilde{\boldsymbol{a}}_0\widetilde{\boldsymbol{b}}_0^{-1} & 0 \\ -\widetilde{\boldsymbol{d}}_1\widetilde{\boldsymbol{b}}_0^{-1} & 1 \end{pmatrix} \widetilde{\boldsymbol{m}}(z) \begin{pmatrix} \widetilde{\boldsymbol{b}}_0\widetilde{\boldsymbol{a}}_0^{-1} & 0 \\ q\widetilde{\boldsymbol{d}}_1\widetilde{\boldsymbol{a}}_0^{-1} & 1 \end{pmatrix} \end{aligned}$$

($\widetilde{\boldsymbol{a}}_0, \widetilde{\boldsymbol{b}}_0, \widetilde{\boldsymbol{d}}_1$ are leading coefficients of corresponding polynomials) satisfies the commutation relations (3.1), and their matrix elements have correct polynomial structure. Thus we are dealing with a realization of the algebra $\mathcal{A}$. The Hamiltonian $\boldsymbol{h}$ is one of coefficients of $\boldsymbol{t}(z)$.

5. Diagonilization of Hamiltonians.

The next step of our construction is diagonalization of the Hamiltonian $\boldsymbol{h}$ in the space $\mathfrak{H}$. We find the Baxter's operator $\mathcal{Q}(\zeta)$ which depends on the logarithmic spectral parameter $\zeta = \frac{1}{2}\log(z)$. The Baxter's operator commutes with Hamiltonians:

$$[\mathcal{Q}(\zeta_1), \boldsymbol{t}(z_2)] = 0$$

It satisfies the equation:

$$(-1)^{g+1}\boldsymbol{t}(z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta + i\gamma) + \mathcal{Q}(\zeta - i\gamma) \tag{5.1}$$

Baxter's operator is constructed as the trace:

$$\mathcal{Q}(\zeta) = \mathrm{tr}_a \left(\mathcal{L}_{a2q+2}(\zeta) \ \cdots \ \mathcal{L}_{a1}(\zeta)\right)$$

Where the universal $\mathcal{L}$-operator acting in $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ is defined as follows. Consider

$$v = e^{\varphi}, \quad u = e^{i\gamma\frac{d}{d\varphi}}, \quad w = e^{\psi} = uvu$$

It is convenient to describe the universal $\mathcal{L}$-operator by its matrix elements in mixed representation:

$$\begin{aligned}&\langle\, \varphi' \mid \otimes \langle\, \psi' \mid \ \mathcal{L}(\zeta) \ \mid \psi \,\rangle \otimes \ \mid \varphi \,\rangle = \\ &= \delta(\varphi - \varphi')\delta(\psi - \psi')\lambda(\zeta \mid \varphi - \psi')\end{aligned}$$

where

$$\lambda(\zeta \mid \psi) = e^{-\frac{1}{2i\gamma}\zeta\psi + \frac{\pi+\gamma}{\gamma}(\psi - \zeta)} \ \Phi(\psi - \zeta),$$

The important function $\Phi(\varphi)$ is given by:

$$\Phi(\varphi) = \exp\left(\int\limits_{\mathbb{R}+i0} \frac{e^{ik\varphi}}{4\sinh\gamma k \ \sinh\pi k}\frac{dk}{k}\right)$$

It satisfies the functional equation:

$$\frac{\Phi(\varphi + i\gamma)}{\Phi(\varphi - i\gamma)} = \frac{1}{1 + e^{\varphi}}$$

This is the only property of $\Phi(\varphi)$ important for construction the universal $\mathcal{L}$-operator.

Here we realize the extraordinary property of our integrable model which is the duality. Notice that the function $\Phi(\varphi)$ satisfies the dual functional equation:

$$\frac{\Phi(\varphi + i\pi)}{\Phi(\varphi - i\pi)} = \frac{1}{1 + e^{\frac{\pi}{\gamma}\varphi}}$$

Duality implies existence of $\boldsymbol{T}(Z)$ which satisfies the equation:

$$(-1)^{g+1}\boldsymbol{T}(Z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta + \pi i) + \mathcal{Q}(\zeta - \pi i) \tag{5.2}$$

where $Z = e^{\frac{2\pi}{\gamma}\zeta}$. At the same time $\boldsymbol{T}(Z)$ defines the Hamiltonians of the dual model:

$$\boldsymbol{T}(Z) = \mathrm{tr}(\widetilde{\boldsymbol{M}}(Z))$$

with

$$\widetilde{\boldsymbol{M}}(Z) = L_{2g+2}(Z) \ \cdots \ L_1(Z)$$

$$L(Z) = \frac{1}{\sqrt{Z}}\begin{pmatrix} ZU^{-1} & -QVU \\ ZV^{-1}U^{-1} & 0 \end{pmatrix}$$

The dual operators

$$U = e^{\frac{\pi}{\gamma}\varphi}, \qquad V = e^{\pi i\frac{d}{d\varphi}}$$

satisfy the commutation relations

$$UV = QVU$$

with dual

$$Q = e^{i\frac{\pi^2}{\gamma}}$$

It is easy to see:

$$[\, \boldsymbol{t}(z), \boldsymbol{T}(Z)\,] = 0$$

The equations (5.1,5.2) hold for the eigen-vectors of commuting operators $\boldsymbol{t}(z)$, $\boldsymbol{T}(Z)$ and $\mathcal{Q}(\zeta)$. The main conjecture concerning these dual integrable models is that their spectrum is defined by different solutions of the equations (5.1,5.2) with $\mathcal{Q}(\zeta)$ being an entire function of its argument with certain asymptotic at the infinity [3]. The above construction proves, in particular, existence of solutions to these equations.

6. Separation of variables.

The last important ingredient of our construction is the method of separation of variables. This method was developed in the quantum case by Sklyanin [4].

According to the commutation relations (3.1) we have the commutative family:

$$[\, \boldsymbol{b}(z), \boldsymbol{b}(z')\,] = 0$$

Introduce the operator-valued zeros $\boldsymbol{z}_j$:

$$\boldsymbol{b}(z)=\prod(z-\boldsymbol{z}_j)$$

Introduce further the operators

$$\boldsymbol{w}_j=(-1)^{g+1}q\,\boldsymbol{d}(\overleftarrow{\boldsymbol{z}}_j)$$

where $\overleftarrow{\boldsymbol{z}}_j$ means that $\boldsymbol{z}_j$ is substituted from the left into the polynomial $\boldsymbol{d}$ whose coefficients are operators. One finds the commutation relations

$$\boldsymbol{z}_j\boldsymbol{w}_k=\boldsymbol{w}_k\boldsymbol{z}_j,\quad j\neq k;\qquad \boldsymbol{z}_j\boldsymbol{w}_j=q^2\boldsymbol{w}_j\boldsymbol{z}_j$$

The operators $\boldsymbol{z}_j$, $\boldsymbol{w}_j$ satisfy equation of "Quantum hyper-elliptic curve":

$$\boldsymbol{w}_j^2-\boldsymbol{w}_j\boldsymbol{t}(\overleftarrow{\boldsymbol{z}}_j)+1=0$$

Introduce the operators:

$$\boldsymbol{\zeta}_j=\tfrac{1}{2}\log(\boldsymbol{z}_j)$$

Following [4] we find that the eigen-function in $\boldsymbol{\zeta}$-representation factorize:

$$\langle\,\zeta_1,\cdots,\zeta_g\mid t_1,\cdots,t_g\,\rangle=\mathcal{Q}(\zeta_1)\cdots\mathcal{Q}(\zeta_g)$$

where $\mathcal{Q}(z)$ satisfies the equation:

$$\mathcal{Q}(\zeta+i\gamma)+\mathcal{Q}(\zeta-i\gamma)=(-1)^{g+1}t(z)\mathcal{Q}(\zeta)$$

This is nothing but equation on the eigen-values of Baxter's operator $\boldsymbol{\mathcal{Q}}(z)$.

Consider $\boldsymbol{\mathcal{X}}\in\mathcal{A}(q)\otimes\mathcal{A}(Q)$, following the Conjecture 1 it can be presented as

$$\begin{aligned}\boldsymbol{\mathcal{X}}&=\boldsymbol{x}\boldsymbol{X}=\\&=p_L(\boldsymbol{t}_1,\cdots,\boldsymbol{t}_g)P_L(\boldsymbol{T}_1,\cdots,\boldsymbol{T}_g)\\&\times g(\boldsymbol{b}_1,\cdots,\boldsymbol{b}_g)G(\boldsymbol{B}_1,\cdots,\boldsymbol{B}_g)\\&\times P_R(\boldsymbol{T}_1,\cdots,\boldsymbol{T}_g)p_R(\boldsymbol{t}_1,\cdots,\boldsymbol{t}_g)\end{aligned}\tag{6.1}$$

Introduce the notation:

$$\begin{aligned}h(\boldsymbol{z}_1,\cdots,\boldsymbol{z}_g)&=\prod\boldsymbol{z}_i\prod_{i<j}(\boldsymbol{z}_i-\boldsymbol{z}_j)\\&\times g(\boldsymbol{b}_1(\boldsymbol{z}_1,\cdots,\boldsymbol{z}_g),\cdots,\boldsymbol{b}_g(\boldsymbol{z}_1,\cdots,\boldsymbol{z}_g))\end{aligned}$$

the polynomial $h(\boldsymbol{z}_1,\cdots,\boldsymbol{z}_g)$ is anti-symmetric. Following [4, 3] one introduces the weight of integration in the space of functions of ζ_j consistent with the scalar product in $\mathfrak{H}$. For the matrix elements we find:

$$\begin{aligned}&\langle\,t_1,\cdots t_g\;;\;T_1,\cdots,T_g\mid\boldsymbol{\mathcal{X}}\mid t_1',\cdots t_g'\;;\;T_1',\cdots,T_g'\,\rangle=\\&=p_L(t_1,\cdots,t_g)p_R(t_1',\cdots,t_g')\\&\times P_L(T_1,\cdots,T_g)P_R(T_1',\cdots,T_g')\\&\times\int_{-\infty}^{\infty}d\zeta_1\cdots\int_{-\infty}^{\infty}d\zeta_g\prod_{j=1}^{g}\mathcal{Q}(\zeta_j)\mathcal{Q}'(\zeta_j)\\&\times h(z_1,\cdots,z_g)H(Z_1,\cdots,Z_g)\end{aligned}\tag{6.2}$$

Obviously, the only non-trivial piece of this formula is the matrix element of gG given by the integral. This explains importance of the representation (6.1). We consider h as "differential forms". Actually we need to calculate the integrals only for "cohomologies" identifying the "forms" whose integrals vanish. The relation to the classical case is obvious. On the "differential forms" define the $\wedge$-product:

$$\begin{aligned}&(h\wedge h')(z_1,\cdots,z_{k+l})=\\&=\frac{1}{k!\,l!}\sum_{\pi\in S_{k+l}}(-1)^{\pi}\,h(z_{\pi(1)},\cdots,z_{\pi(k)})\\&\times h'(z_{\pi(k+1)},\cdots,z_{\pi(k+l)})\end{aligned}$$

Let $\mathcal{V}^1$ be the space of polynomials of one variable z (of degree ≥ 1) with coefficients polynomial in t_j, t_j', and

$$\mathcal{V}^k=\wedge^k\mathcal{V}^1$$

Certain basis can be defined in $\mathcal{V}^1$ [3] for which the following properties are satisfied.

$$s_k,\quad -g\leq k\leq\infty,\quad \deg(s_k)=k+g+1$$

We shall denote that the "form" vanishes under the integral by $\simeq$. We have

1. For $k\geq g+1$ we have:

$$s_k\wedge\mathcal{V}^{g-1}\simeq 0$$

2. Consider $c\in\mathcal{V}^2$ defined as

$$c=\sum_{j=1}^{g}s_j\wedge s_{-j}$$

we have

$$c\wedge\mathcal{V}^{g-2}\simeq 0$$

3. Consider $d \in \mathcal{V}^1$ defined as

$$d = (t_j - t_j')s_{-j}$$

we have

$$d \wedge \mathcal{V}^{k-1} \simeq 0$$

The description of "cohomologies" following from these "exact forms" is a deformation of classical cohomologies of affine Jacobi variety [2].

However, the most interesting feature of our construction is its duality. One introduces the dual objects: $\mathcal{V}_1$ is the space of polynomials of Z (of degree ≥ 1), $\mathcal{V}_k = \wedge^k \mathcal{V}_1$, and

$$S_k, \quad C, \quad F$$

There are two classical limits: one of them is usual: $\gamma \to 0$, another is dual: $\gamma \to \infty$. Computing the asymtotics of integrals in these limits one describes the duality as follows:

$$s_{j_1} \wedge \cdots \wedge s_{j_g} = \begin{cases} \text{form: } \gamma \to 0 \\ \text{cycle: } \gamma \to \infty \end{cases}$$

$$S_{j_1} \wedge \cdots \wedge S_{j_g} = \begin{cases} \text{cycle: } \gamma \to 0 \\ \text{form: } \gamma \to \infty \end{cases}$$

Thus the objects that define cycles and forms in the classical limit are represented on quantum level as absolutely similar objects related by week-strong duality $\gamma \to \frac{\pi^2}{\gamma}$.

References

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