BCFT: from the boundary to the bulk

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ABSTRACT: The study of boundary conditions in rational conformal field theories is not only physically important. It also reveals a lot on the structure of the theory "in the bulk". The same graphs classify both the torus and the cylinder partition functions and provide data on their hidden "quantum symmetry". The Ocneanu triangular cells – the $3j$-symbols of these symmetries, admit various interpretations and make a link between different problems.

KEYWORDS: conformal field theory, boundary, graphs, weak Hopf algebra.

1. Introduction

Ten years after the work of Cardy [1], Boundary Conformal Field Theory (BCFT) is experiencing a renewal of interest. This is motivated by applications to string and brane theory, and to problems of statistical mechanics and condensed matter. In these situations, it may be important to have an a priori knowledge of the possible boundary conditions compatible with conformal invariance, and to master the algebra of boundary fields etc. There is, however, another reason to be interested in BCFT: as we want to explain in this note, there is much to learn on the general structure and the quantum symmetries of a CFT from the study of its properties in the presence of a boundary. After a brief review of notations and general aspects of rational CFT, we shall discuss how boundary conditions may be systematically classified in terms of non-negative integer valued matrix representations of the fusion algebra, or equivalently in terms of graphs generalising the ADE Dynkin diagrams (see [2] for more details). We shall see how the "cells" – a concept introduced by Ocneanu and associated with these graphs – determine many properties of the BCFT or of the associated lattice models and, in particular, are at the heart of the quantum symmetry of the CFT described by an algebraic structure called weak C^*-Hopf algebra. A more detailed presentation will appear in [3].

2. General set-up

A rational conformal field theory (RCFT) is generally described by data of different nature:
• **Chiral data:** Chiral data specify the chiral algebra $\mathfrak{A}$, e.g., the Virasoro algebra itself, or a $\mathcal{W}$ algebra, a current algebra $\widehat{g}$ etc, and its finite set $\mathcal{I}$ of irreducible representations, $\mathcal{V}_i\,,\ i\in\mathcal{I}$; notations are such that $i=1$ labels the identity (vacuum) representation and i^* the conjugate of $\mathcal{V}_i$. The fusion rule multiplicities ${N_{ij}}^k$, $\mathcal{V}_i\star\mathcal{V}_j=\oplus_k {N_{ij}}^k\,\mathcal{V}_k$, are assumed to be given by the Verlinde formula,

$$ {N_{ij}}^k=\sum_{\ell\in I}\frac{S_{il}\,S_{jl}\,(S_{kl})^*}{S_{1l}}\,, \qquad (2.1)$$

*permanent address: Institute for Nuclear Research and Nuclear Energy, 72 Tzarigradsko Chaussee, 1784 Sofia, Bulgaria; petkova@inrne.bas.bg with S the symmetric, unitary matrix of the modular transformations of the characters $\chi_i(\tau) = \mathrm{tr}\, e^{2i\pi\tau(L_0-c/24)}$, $\chi_i(\tau) = \sum_{j\in\mathcal{I}} S_{ij}\chi_j(-1/\tau)$. The nonnegative integers ${N_{ij}}^k$ give the dimensions of linear spaces of chiral vertex operators (CVO)

$$\phi^k_{ij;t}(z):\ \mathcal{V}_i\otimes\mathcal{V}_j\to\mathcal{V}_k\,,\quad z\in\mathbb{C}\,, \qquad (2.2)$$

with a finite basis label $t=1,2,\dots {N_{ij}}^k$. The chiral data finally include the knowledge of the duality matrices: the genus 0 fusing F and braiding $B(\pm)$ matrices, and the matrix $S(j)$, which gives the modular transformation of 1-point conformal block $\langle\phi^i_{ji}\rangle$ on the torus. These matrices satisfy a set of consistency relations: pentagon, hexagon and torus identities [4].

A typical example is provided by the chiral algebra $\mathfrak{A}=\widehat{sl}(2)_k$, the affine algebra at level k, for which $\mathcal{I}=\{1,2,\cdots,k+1\}$, $S_{ij}=\sqrt{2/(k+2)}\,\sin(\pi ij/(k+2))$, and the F are, up to a gauge transformation, the quantum $6j$ symbols [5].

Those are the basic ingredients of algebraic nature in the construction of a RCFT. In the following, they are supposed to be known.

• **Spectral and OPE data:** The physical spectrum of a RCFT *in the bulk*, i.e. on a closed Riemann surface, is described by irreducible representations of *two* copies of the chiral algebra. Thus the Hilbert space is decomposed according to

$$\mathcal{H}_P=\oplus_{j,\bar{j}\in\mathcal{I}} Z_{j\,\bar{j}}\,\mathcal{V}_j\otimes\bar{\mathcal{V}}_{\bar{j}}\,. \qquad (2.3)$$

The integer multiplicities $Z_{j\,\bar{j}}$ (with $Z_{11}=1$, expressing the unicity of the vacuum) are conveniently encoded in the modular invariant torus partition function

$$Z(\tau)=\sum_{j,\bar{j}\in\mathcal{I}} Z_{j\bar{j}}\ \chi_j(\tau)\,\chi_{\bar{j}}(\tau)^*\,. \qquad (2.4)$$

The particular case where $Z_{ij}=\delta_{ij}$ will be referred to as the diagonal theory, the others as "non-diagonal".

On the other hand, the (primary) physical fields and their correlators factorise

$$\Phi_{(i,\bar{i})}(z,\bar{z})=\sum_{j,\bar{j},k,\bar{k},t,\bar{t}} d^{(k,\bar{k});t,\bar{t}}_{(i,\bar{i})(j,\bar{j})}\,\phi^k_{i,j;t}(z)\otimes\phi^{\bar{k}}_{\bar{i},\bar{j};\bar{t}}(\bar{z})\,. \qquad (2.5)$$

The expansion coefficients $d^{(k,\bar{k});t,\bar{t}}_{(i,\bar{i})(j,\bar{j})}$ determine, up to the normalisation of the chiral blocks, the coefficients of the short distance operator product expansion (OPE), i.e., these are the *relative* OPE coefficients of the non-diagonal model if d are chosen trivial for the diagonal model of same central charge. These numbers are constrained by the requirement of locality of the physical correlators, which makes use of the braiding matrices $B(\pm)$. The resulting set of coupled quadratic equations has been fully solved only in the $sl(2)$ cases (see [6, 7] and further references therein).

Starting with the ADE classification of the $sl(2)$ modular invariants [8], it has been gradually realised that behind these data there are hidden **graphs** G. A systematic study of the graphs generalising the $sl(2)$ ADE Dynkin diagrams was initiated in [9, 10]. It was empirically observed that these graphs are encoding a non trivial information on some spectral and OPE data, more precisely on those pertaining to the *spin-zero* fields of the theory. These graphs G_i, which share the same set of vertices $\mathcal{V}$, are labelled by an index $i\in\mathcal{I}$ –but it is sufficient to restrict i to a generating subset (generating in the sense of fusion)– and their adjacency matrices are commuting and simultaneously diagonalizable in an orthonormal basis $\{\psi\}$:

• the diagonal part of the physical spectrum, labelled by the so-called *"exponents"*:

$$\mathcal{E}=\{j\in\mathcal{I}|j=\bar{j},Z_{jj}\neq 0\}\,, \qquad (2.6)$$

counted with the multiplicity Z_{jj}, is in one-to-one correspondence with the spectrum of eigenvalues of G_i, of the form S_{ij}/S_{1j}, $j\in\mathcal{E}$.

• in the $sl(2)$ (ADE) cases the structure constants ${M_{ij}}^k$ of the so-called Pasquier algebra [11] can be identified with the (relative) OPE coefficients of the scalar fields [7]

$$d^{(k,k)}_{(i,i)(j,j)}={M_{ij}}^k:=\sum_{a\in\mathcal{V}}\frac{\psi^i_a\,\psi^j_a\,\psi^{k\,*}_a}{\psi^1_a}\,. \qquad (2.7)$$

The summation in (2.7) runs over the set $\mathcal{V}$ of vertices of the Dynkin diagram.

The purpose of this note is to show that the study of the CFT on a half-plane or a cylinder (i.e. on a manifold with boundary), [1], provides an alternative (chiral) approach in which these graphs and the related algebraic structures become manifest.

3. Graphs and conformal boundary conditions

We may summarize the results of [2] as follows: boundary conditions that respect the chiral algebra $\mathfrak{A}$ are described by a set of commuting, non-negative integer valued matrices $\{n_i = n_{ia}{}^b\}\,, i \in \mathcal{I}\,,\ a,b \in \mathcal{V}\,,\ |\mathcal{E}| = |\mathcal{V}|$, s.t. $n_1 = I,\ n_i^T = n_{i^*}$, realising a representation of the Verlinde fusion algebra

$$n_i\, n_j = \sum_{k\in\mathcal{I}} N_{ij}{}^k\, n_k\,. \qquad (3.1)$$

The matrices n_i thus admit a spectral decomposition

$$n_{ia}{}^b = \sum_{j\in\mathcal{E}} \frac{S_{ij}}{S_{1j}}\, \psi_a^j\, \psi_b^{j*}\,, \qquad (3.2)$$

where ψ_a^j are unitary matrices of dimension $|\mathcal{E}| = |\mathcal{V}|$ and the sum runs over the set $\{(j,\alpha)\,, j \in \mathcal{I}\,, \alpha = 1,2,\ldots, Z_{jj}\}$, for simplicity of notation identified with the set of exponents $\mathcal{E}$. This comes about as follows: On the upper half-plane parametrised by a coordinate $z \in \mathbb{H}^+$, (resp. on a finite-width strip $w = \frac{L}{\pi}\log z$), with boundary conditions b and a imposed on the negative and positive real axes, (resp. on the two sides of the strip), only one copy of the chiral algebra acts: only real analytic coordinate transformations, $\epsilon(z) = \bar\epsilon(\bar z)$ for real $z = \bar z$, are allowed. Thus the Hilbert space of the theory $\mathcal{H}_{ba}$ splits into a linear sum of representations

$$\mathcal{H}_{ba} = \oplus_{i\in\mathcal{I}} n_{ib}{}^a\, \mathcal{V}_i\,. \qquad (3.3)$$

On a finite segment of the strip of length T with periodic boundary conditions $w \sim w + T$, i.e. on a cylinder, the partition functions $Z_{b|a}$ is thus a linear form in the characters

$$Z_{b|a}(\tau) = \sum_{i\in\mathcal{I}} n_{ib}{}^a\, \chi_i(\tau)\,, \quad \tau = \frac{iT}{2L}\,. \qquad (3.4)$$

The multiplicities $n_{ia}{}^b$ are constrained by the Cardy consistency condition [1] which expresses that $Z_{b|a}$ can also be evaluated in a dual way: by mapping the cylinder to an annulus region in the full plane, one computes $Z_{b|a}$ as the matrix element of the evolution operator between boundary states $|a\rangle$ and $\langle b|$. The latter are decomposed on a standard basis of complete and orthonormal "Ishibashi states" $|j\rangle\rangle$ labelled by an exponent $j \in \mathcal{E}$, according to $|a\rangle = \sum_{j\in\mathcal{E}} \frac{\psi_a{}^j}{\sqrt{S_{1j}}}\, |j\rangle\rangle$. This gives

$$\begin{aligned} Z_{b|a}(\tau) &= \langle b| e^{\frac{-\pi i}{\tau}(L_0 + \overline{L}_0 - \frac{c}{12})} |a\rangle \\ &= \sum_{j\in\mathcal{E}} \frac{\psi_a{}^j \psi_b{}^{j*}}{S_{1j}} \chi_j\Big(\frac{-1}{\tau}\Big)\,. \end{aligned} \qquad (3.5)$$

Comparing the resulting two expressions for $Z_{b|a}$ yields the spectral decomposition (3.2). See [2] for a discussion of the assumptions and a more detailed derivation.

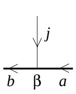
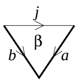
Each set of matrices $\{n_i\}$ solving (3.1), (3.2) may be regarded as the adjacency matrices of a collection of graphs G_i, thus explaining the occurence of graphs with the special spectral properties noticed above. Solving in general the system (3.1) provides a classification of boundary conditions, labelled by the vertices of the graphs G_i and specified by the spectrum (3.3). The case $n_i = N_i$ provides the regular representation with spectrum $\mathcal{E} = \mathcal{I}$ corresponding to the diagonal case; then (3.1) reduces to (2.1) with $\psi = S$ and the boundary states are labelled by the same indices $i \in \mathcal{I}$ as the representations and primary fields [1].

- In $\widehat{sl}(2)$ RCFT (the WZNW models and the Virasoro minimal models) this reduces to the classification of the symmetric, irreducible, non-negative integer valued matrices of spectrum $|\gamma^j| = |\frac{S_{2j}}{S_{1j}}| < 2$. This is well known to lead to an ADE classification, after the spurious "tadpole" graphs $A_{2n}/\mathbb{Z}_2$ have been discarded, on the basis that their spectrum does not match any known modular invariant.

- A new situation arises in the $sl(3)$ case, where there is no known a priori classification of (oriented) graphs with the required spectral properties. Solutions may be found [10] and comparison with the complete list of modular invariants [12] enables one to discard some spurious solutions and leads to a list of graphs, see [2]. One also finds that in a few cases, there are isospectral graphs, i.e. more than one solution with a given spectrum, indicating that there are several choices of boundary conditions (and in fact of OPE structure constants, see [2]), for a given spectrum of spin-zero fields in a RCFT.

4. Ocneanu quantum graph symmetry

According to Cardy the boundary conditions (a,b) are created by insertions of fields on the boundary (*boundary fields*), ${}^a\Psi^b_{j;\beta}(x)$, $\beta = 1,2,\dots n_{jb}{}^a$, $x \in \mathbb{R}$. Exploiting some ideas of Ocneanu [13, 14] one can interpret these fields, extended to the full plane, as generalised CVO.

Given a solution of the system of equations (3.1), consider for each $j \in \mathcal{I}$ an auxiliary Hilbert space $V^j \cong \mathbb{C}^{m_j}$ of dimension $m_j = \sum_{a,b} n_{ja}{}^b$ with basis states $|e^{j,\beta}_{ba}\rangle$, $\beta = 1,2,\dots,n_{ja}{}^b$, depicted as  or . A scalar product in $\oplus_{j\in\mathcal{I}} V^j$ is defined as

$$\langle e^{j,\beta}_{ab}|e^{j',\beta'}_{a'b'}\rangle \;=\; \delta_{bb'}\delta_{aa'}\,\delta_{jj'}\,\delta_{\beta'\beta}\,\sqrt{\frac{P_a\,P_b}{d_j}}\,,$$

$$d_j := \frac{S_{j1}}{S_{11}}\,,\ P_a := \frac{\psi^1_a}{\psi^1_1}\,. \qquad (4.1)$$

Restricting to a subspace $V^i \otimes_h V^j$ of $V^i \otimes V^j$, with coinciding intermediate labels, one defines a decomposition $V^i \otimes_h V^j \cong \oplus_k N_{ij}{}^k\, V^k$, or, explicitly,

$$|e^{i,\eta}_{ab}\rangle \otimes_h |e^{j,\zeta}_{bc}\rangle \;=\; \sum_{k\in\mathcal{I}} \sum_{\gamma=1}^{n_{kc}{}^a} \sum_{t=1}^{N_{ij}{}^k} {}^{(1)}F_{bk}\begin{bmatrix} i & j\\ a & c\end{bmatrix}^{\gamma\, t}_{\eta\,\zeta}$$

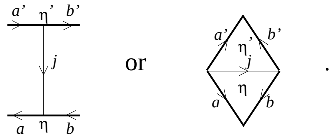
$$\times\sqrt{P_b}\;\left(\frac{d_k}{d_i d_j}\right)^{\frac14}\,|e^{k,\gamma}_{ac}(ij;t)\rangle\,. \qquad (4.2)$$

The counting of states in both sides is consistent due to (3.1). The Clebsch-Gordan coefficients ${}^{(1)}F \in \mathbb{C}$ (**"3j- symbols"**) satisfy unitarity conditions (completeness and orthogonality) and a pentagon identity, reflecting the associativity of the product (4.2), written schematically as

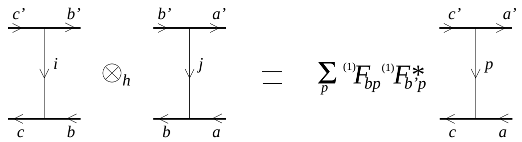
$$F\ {}^{(1)}F\ {}^{(1)}F\ = {}^{(1)}F\ {}^{(1)}F\,, \qquad (4.3)$$

where F are the $6j$-symbols. In the diagonal case F and ${}^{(1)}F$ are identified, and (4.3) becomes the conventional pentagon identity satisfied by the fusing matrices of the corresponding RCFT.

The space $\mathcal{A} = \oplus_{j\in\mathcal{I}}\,\mathrm{End}\,(V^j)$ is a matrix algebra $\oplus_{j\in\mathcal{I}} M_{m_j}$. A matrix unit basis $\{e^{(ab)\,,(a'b')}_{j;\eta\,,\eta'} = |e^{j,\eta}_{ab}\rangle\langle\, e^{j,\eta'}_{a'b'}|\,, j \in \mathcal{I}\}$ in $\mathcal{A}$ is identified with a product of states in $V^j \otimes V^{j^*}$, and depicted as 4-point blocks or double triangles



Along with the matrix ("vertical") product, a second "horizontal" product (or, alternatively, a coassociative coproduct), is defined on $\mathcal{A}$ via the $3j$-symbols ${}^{(1)}F$



The algebra generated by these objects is the Ocneanu **double triangle algebra** (DTA) [13], further studied in the subfactor theory, see [15, 16] and references therein. Equipped with a counit and an antipode satisfying a weakened version of the Hopf algebra axioms ($\triangle(1_v) \neq 1_v \otimes 1_v$, etc.), it was considered in [17] as an example of a *weak C^*- Hopf algebra* (WHA) (see also the recent paper [18]). This Ocneanu "graph quantum algebra", associated to any solution $\{n_i\,, i \in \mathcal{I}\}$ of (3.1), together with its dual algebra $\hat{\mathcal{A}}$ structure (see below), appears as the quantum symmetry of the CFT model, either diagonal or non-diagonal.

5. Generalised CVO (GCVO)

We define operators $\oplus_{j\in\mathcal{I}}\, \mathcal{V}_j\otimes V_j \to \oplus_{k\in\mathcal{I}}\, \mathcal{V}_k\otimes V_k$ associated with the basis states in the tensor product (4.2),

$${}^a\Psi^c_{i,\beta;I}(z) \;=\; \sum_{j,k,t} \phi^k_{ij;t,I}(z)\otimes \sum_{b,\alpha,\gamma} {}^{(1)}F_{ck}\begin{bmatrix} i\ j\\ a\ b\end{bmatrix}^{\alpha\, t}_{\beta\,\gamma}$$

$$\times\sqrt{\frac{d_j}{P_c\,P_b}}\;|e^{k,\alpha}_{ab}\rangle\langle\, e^{j,\gamma}_{cb}|\,, \qquad (5.1)$$

covariant under the action of $\mathcal{A}$; here I labels descendent states in $\mathcal{V}_i$. Their correlators, obtained by projecting on a state $|0\rangle\otimes|e^1_{aa}\rangle$ in the vacuum space $\mathcal{V}_1 \otimes V^1$, as well as their fusing and braiding properties, are inherited from those of the conventional CVO. In contrast with previous approaches introducing quantum symmetry covariant operators, e.g., [19, 20], note that in the diagonal case the $6j$-symbols F of the related quantum group (instead of its $3j$-symbols) appear in the r.h.s. of (5.1). For real z the operators Ψ of (5.1) represent (after an appropriate choice of normalisation) the boundary fields, and their short distance $x_{12} \approx 0$ expansion is recovered, with the $3j$-symbols ${}^{(1)}F$ serving as *boundary fields OPE coefficients.* The recoupling equation (4.3) expresses then the associativity of the boundary fields OPE [2] and is equivalent to the Lewellen boundary fields sewing identity [22].

In general the chiral operators (5.1) have nontrivial braiding with a new braiding matrix $\hat{B}(\epsilon)$, with $4+2$ indices of two types,

$$ {}^{a}\Psi^{b}_{j,\alpha}(z_1)\, {}^{b}\Psi^{c}_{k,\gamma}(z_2) = \sum_{d,\alpha',\gamma'} \hat{B}_{bd} \begin{bmatrix} j & k \\ a & c \end{bmatrix}^{\alpha'\,\gamma'}_{\alpha\,\gamma} (\epsilon)\ {}^{a}\Psi^{d}_{k,\alpha'}(z_2)\, {}^{d}\Psi^{c}_{j,\gamma'}(z_1)\,, \tag{5.2}$$

$z_{12} \notin \mathbb{R}_-$, $\epsilon = \mathrm{sign}\,(\mathrm{Im}\, z_{12})$. The matrices $\hat{B}$ satisfy various relations:

- Inversion (unitarity) relation

$$\hat{B}^{12}(\epsilon)\, \hat{B}^{21}(-\epsilon) = 1\,. \tag{5.3}$$

- The braid group ("Yang-Baxter") relation

$$\hat{B}^{12}\, \hat{B}^{23}\, \hat{B}^{12} = \hat{B}^{23}\, \hat{B}^{12}\, \hat{B}^{23}\,. \tag{5.4}$$

- The braiding–fusing (pentagon) identity

$$\hat{B}\ {}^{(1)}F = {}^{(1)}F\ \hat{B}\ \hat{B}\,, \tag{5.5}$$

which turns into a recursion relation for $\hat{B}$, given the fundamental $3j$-symbols,

- Intertwining relation

$${}^{(1)}F\, {}^{(1)}F\ B = \hat{B}\ {}^{(1)}F\, {}^{(1)}F\,, \tag{5.6}$$

which implies a bilinear representation of $\hat{B}$ in terms of the $3j$-symbols

$$\hat{B}_{bd} \begin{bmatrix} j & k \\ a & c \end{bmatrix}^{\alpha'\gamma'}_{\alpha\gamma}(\epsilon) = \sum_{i\,,\delta\,,t} {}^{(1)}F_{bi} \begin{bmatrix} j & k \\ a & c \end{bmatrix}^{\delta\, t}_{\alpha\gamma} e^{-i\pi\epsilon\, \triangle^{i}_{jk}}\ {}^{(1)}F^{*}_{di} \begin{bmatrix} k & j \\ a & c \end{bmatrix}^{\delta\, t}_{\alpha'\gamma'}\,, \tag{5.7}$$

where $\triangle^{i}_{jk}$ is the combination of conformal weights $\triangle_j + \triangle_k - \triangle_i$.

6. Connection to lattice models

Some of these identities have been already encountered in critical lattice models, the ADE Pasquier models and their $\widehat{sl}(n)_{h-n}$ generalisations, in which the degrees of freedom may be regarded as vertices of the previous graphs [23, 10].

In the limit $u \to -i\epsilon\infty$, ($\epsilon = \pm 1$), of the spectral parameter u, the face Boltzmann weights $W(u)$ satisfy the same equation (5.4) as the braiding matrix $\hat{B}$ of (5.2). Indeed, denoting the representations of $sl(n)$ by their Young tableau, and with $q = e^{\frac{2\pi i}{h}}$,

$$\hat{B}_{bd} \begin{bmatrix} \square\,\square \\ a\ \ c \end{bmatrix}^{\gamma'\,\alpha'}_{\gamma\,\alpha}(\epsilon) = 2i\, q^{-\epsilon\frac{1}{2n}} \lim_{u\to -i\epsilon\infty} e^{-i\pi\epsilon u}\, W_{bd} \begin{pmatrix} a \\ c \end{pmatrix}^{\gamma'\,\alpha'}_{\gamma\,\alpha} (u)\,,$$

$$W_{bd}(u) = \sin(\frac{\pi}{h} - u)\delta_{bd} + \sin(u)\, [2]_q\, U_{bd}\,, \quad U^2 = U\,. \tag{6.1}$$

Using (5.7), the Hecke algebra generators U are expressed via the $3j$-symbols, recovering an Ansatz of Ocneanu [14]

$$U_{bd} = \sum_{\beta} {}^{(1)}F_{b\,\boxminus} \begin{bmatrix} \square\,\square \\ a\ \ c \end{bmatrix}^{\beta\ 1}_{\gamma\,\alpha}\ {}^{(1)}F^{*}_{d\,\boxminus} \begin{bmatrix} \square\,\square \\ a\ \ c \end{bmatrix}^{\beta\ 1}_{\gamma'\,\alpha'}\,. \tag{6.2}$$

Moreover, equation (5.6) shows that we can identify the $3j$- symbols ${}^{(1)}F$ with the *"intertwining cells"* studied in these models, see e.g. [10]; then lattice results provide solutions for particular ${}^{(1)}F$, namely, those for which one of the labels in $\mathcal{I}$ is equal to $\square$.

For $sl(2)$: $\boxminus\ = 1$, $[2]_q\ U_{bd} = \delta_{ac}\frac{\sqrt{P_b\, P_d}}{P_a}$, while for $sl(3)$: $\boxminus\ = \square^*$, and these cells ${}^{(1)}F_{b\,\square^*}$ exist for all graphs listed in [2] but one, according to the recent work of Ocneanu, [14]. (See also [21], where the existence of the Boltzmann weights (cells ${}^{(1)}F$) has been proved in the subfactor approach and where the relations (5.3) – (5.7) also appear, for a subclass of graphs corresponding to conformal embeddings.)

7. Bulk fields

For $(j,\bar{j})$ in the physical spectrum (2.3), i.e., $Z_{j\bar{j}} \neq 0$, and $z \in \mathbb{H}^+$, define (upper) half-plane bulk fields by two-point compositions of GCVO (5.1)

$$\Phi^{H}_{(j,\bar{j})}(z,\bar{z}) = \sum_{\substack{a,b\,, \\ \beta'\,,\beta}} C^{a,b,a;\beta,\beta'}_{(j,\bar{j})}\ {}^{a}\Psi^{b}_{j,\beta}(z)\ {}^{b}\Psi^{a}_{\bar{j}^*,\beta'}(\bar{z})\,. \tag{7.1}$$

From this operator representation, which can be also rewritten in terms of the conventional CVO, one recovers the bulk field small distance $z-\bar{z}=2i\,y\approx 0$ expansion, in the leading order

$$\Phi^H_{(j,\bar{j})}(z,\bar{z}) = \sum_{p,a,\alpha,t} R^{(j,\bar{j};t)}_{(a;\alpha)}(p)\,\langle p|\phi^p_{j\bar{j}^*;t}(2iy)|\bar{j}^*\rangle^{\,a}\,\Psi^a_{p,\alpha}(x)+\dots\,, \tag{7.2}$$

with $R=\sum {}^{(1)}F\;C$ – the bulk-boundary reflection coefficients of Cardy–Lewellen [22]. Given the chiral representation (7.1) all the bulk-boundary sewing relations of [22] are derived from the duality identitities of the conventional CVO.

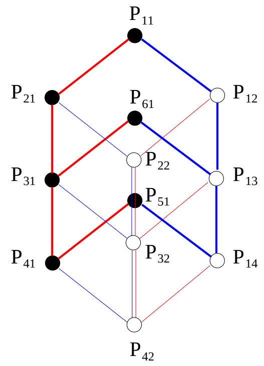
• Diagonal case: the two basic bulk-boundary Lewellen equations are not independent and are equivalent to the torus duality identity of Moore-Seiberg with $R(p)\sim S(p)$ [2].

8. The dual DTA and the physical spectrum

The dual (in the algebraic sense) of Ocneanu DTA leads to the consideration of new graphs $\tilde{G}$, [13, 14], (see also [16, 15]), with a set of vertices $x\in\tilde{\mathcal{V}}=\{1,2,\dots,\sum_{i,j}(Z_{ij})^2\}$. These graphs are described by a set of nonnegative integer valued matrices $\big(\tilde{V}_{ij}\big)_x^{\;y}$, forming a representation of the product of two Verlinde algebras, thus generalising (3.1)

$$\tilde{V}_{i_1j_1}\tilde{V}_{i_2j_2}=\sum_{i_3,j_3} N_{i_1i_2}{}^{i_3}N_{j_1j_2}{}^{j_3}\;\tilde{V}_{i_3j_3} \tag{8.1}$$

with the additional condition that $\big(\tilde{V}_{ij^*}\big)_1^{\;1}=Z_{ij}$, the matrix of (2.4). The eigenvalues of $\tilde{V}_{ij}$ are described by this matrix Z, i.e., are labelled by the set $\tilde{\mathcal{E}}=\{(\ell,\bar{\ell}),\ell,\bar{\ell}\in\mathcal{I}$, taken with multiplicity $(Z_{\ell\bar{\ell}})^2\}$, $|\tilde{\mathcal{E}}|=|\tilde{\mathcal{V}}|$, and are of the form $S_{i\ell}S_{j\bar{\ell}}/S_{1\ell}S_{1\bar{\ell}}$. For example in the E_6 case, this graph has 12 vertices that we label by a pair $x=(a,b)$, $a=1,2,\cdots,6$, $b=1,2$ of two vertices of the ordinary E_6 diagram; the new graph is generated by $\tilde{V}_{2,1}$ and $\tilde{V}_{1,2}$ and the corresponding edges are depicted in red and blue respectively on the following figure due to Ocneanu



On that figure $(P_{ab})_{ij}:=\sum_{c=1,5,6}n_{ia}{}^c n_{jb}{}^c$, expressed in terms of n_i of (3.1), give the matrices $(\tilde{V}_{ij})_1{}^x$, generalising a well known formula for $Z=P_{11}$ [24, 25].

This Ocneanu graph $\tilde{G}$ gives rise to a (noncommutative in general) fusion algebra with structure constants $\tilde{N}_{xy}{}^z$; the nodes of the graph can be equivalently associated with the matrices $\tilde{N}_x$. This algebra also admits a representation by matrices $(\tilde{n}_x)_a^b=\tilde{n}_{xa}{}^b$ of size $|\mathcal{V}|$ with nonnegative integer entries

$$\sum_{b\in\mathcal{V}}\tilde{n}_{xa}{}^b\,\tilde{n}_{yb}{}^c=\sum_{z\in\tilde{\mathcal{V}}}\tilde{N}_{xy}{}^z\,\tilde{n}_{za}{}^c\,. \tag{8.2}$$

These integers describe the dimensions $\tilde{m}_x=\sum_{a,b}\tilde{n}_{xa}{}^b$ of the dual spaces $\hat{V}_x$, spanned by states (dual triangles) $|E^b_{ax;\alpha}\rangle$, $\alpha=1,2,\dots,\tilde{n}^b_{ax}$. The "dual double triangles" span a basis of $\hat{\mathcal{A}}$, with vertical and horizontal products reversed. The two basis sets are related by a "fusing" matrix, ${}^{(2)}F$, satisfying the pentagon identity

$${}^{(2)}F\;{}^{(2)}F={}^{(2)}F\;{}^{(1)}F\;{}^{(1)}F^*\,. \tag{8.3}$$

More pentagon relations involve the dual $3j$- and $6j$-symbols; the full set of such relations is named the "Big Pentagon identity" in [17]. In the diagonal cases, all these F matrices coincide. The problem is to extend this general construction beyond the diagonal case.

Example: $sl(2)$ D_r series, $r=\frac{h}{2}+1$ odd. Here $\tilde{\mathcal{V}}=\{x\equiv k=1,2,\dots,h{-}1\}$, $\tilde{\mathcal{E}}=\{(l,\zeta(l))\,|\,l=1,2,\dots,h-1\}$, where $\zeta(l)=h-l$ for l even, $\zeta(l)=l$ for l odd. Then $\tilde{V}_{ij}=N_i\,N_{\zeta(j)}$ is diagonalised by S and $\tilde{G}=A_{h-1}$, $\tilde{N}_k=N_k$, $\tilde{n}_k=n_k$. Hence $\mathcal{A}$ is self-dual, with both the $3j$- and $6j$-symbols being self-dual. The $3j$-symbols ${}^{(1)}F$ were found in [26] and it remains to determine the ${}^{(2)}F$ to have a non-diagonal example of WHA.

While $\mathcal{A}$ gives meaning to the generalised CVO, the CFT interpretation of the dual $\hat{\mathcal{A}}$ is less clear. The representations of the $\tilde{N}$-algebra carry the labels $(j, \bar{j})$ of the physical fields with $Z_{(j,\bar{j})} \neq 0$. In the cases $Z_{ij} = 0, 1$, when this algebra is commutative with only 1-dimensional representations, one can consider its dual algebra, a generalised Pasquier algebra, with structure constants

$$\tilde{M}_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k})} = \sum_{x \in \tilde{\mathcal{V}}} \frac{\Psi_x^{(i,\bar{i})}}{\Psi_x^{(1,1)}} \Psi_x^{(j,\bar{j})} \Psi_x^{(k,\bar{k})*}, \qquad (8.4)$$

where $\Psi_x^{(i,\bar{i})}$ is a unitary matrix diagonalising $\tilde{V}_{ij}$ and $\tilde{N}_x$. In all $sl(2)$ cases of this type, i.e., A, D_{odd}, E_6, E_7, E_8, one finds [3] a relation generalising (1.4),

$$\tilde{M}_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k})} = \left(d_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k})} \right)^2, \qquad (8.5)$$

involving the relative OPE coefficients of fields of non-zero spin. For E_6, E_8, one has $\tilde{M}_{(i,\bar{i})(j,\bar{j})}^{(k,\bar{k})} = M_{ij}^{k} M_{\bar{i}\bar{j}}^{\bar{k}}$, in agreement with the known factorisation of the OPE coefficients in these cases. The problem remains to recover (8.5) directly in the field theory framework as it has been achieved in the boundary CFT [27, 2] for its scalar counterpart (2.7).

One step towards the CFT interpretation of the dual DTA structures is the observation that the system of equations (8.1) and the spectral decomposition of the matrices $\tilde{V}_{ij}$ can be recovered, generalising the derivation of (3.1) and (3.2), by considering a partition function $\tilde{Z}_{x|y}(\tau)$ defined on a double cylinder and bilinear in the characters. It is interesting to note that the simplest of these partition functions, namely the one corresponding to the diagonal case $Z_{ij} = \delta_{ij}$ for which $\tilde{V}_{ij} = N_i N_j$, has been independently obtained (for fixed $x = y = 1$), looking at tensor products of BCFT [28].

Acknowledgements:
Our warmest thanks to Adrian Ocneanu for his generous explanations of his recent work prior to publication. We also thank Paul Pearce and Peter Vecsernyes for useful discussions and Andreas Recknagel and Gerard Watts for communicating to us the unpublished result [28]. V.B. P. acknowledges partial support of the Bulgarian National Research Foundation (Contract Φ-643). J.-B. Z. acknowledges partial support of the EU Training and Mobility of Researchers Program (Contract FMRX-CT96-0012).

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