PROCEEDINGS

Renormalisation group flows of boundary theories

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ABSTRACT: We review recent developments in the theory of renormalisation group flows in minimal models with boundaries. Among these, we discuss in particular the perturbative calculations of Recknagel et al, not only as a tool to predict the IR endpoints of certain flows, but also as a motivation for considering the particular limiting case of $c = 1$. By treating this limit, we are able to investigate a wide class of perturbations by considering them as deformations away from the $c = 1$ point. We also present the truncated conformal space approach as a tool for investigating the space of RG flows and checking particular predictions.

1. Introduction

In this talk we consider two dimensional theories defined on manifolds with boundaries. This allows the possibility for renormalisation group (RG) flows generated by boundary fields and flows between different boundary conditions. The discussion of RG flows of boundary theories can be usefully divided into

1. Purely boundary flows

2. Combined bulk and boundary flows

In this talk we shall consider only the first class of flows. Some examples of combined bulk and boundary flows are considered in the talk by R. Tateo [20]. Before starting our discussion of boundary flows, it is useful to recall the well-known results for purely bulk perturbations.
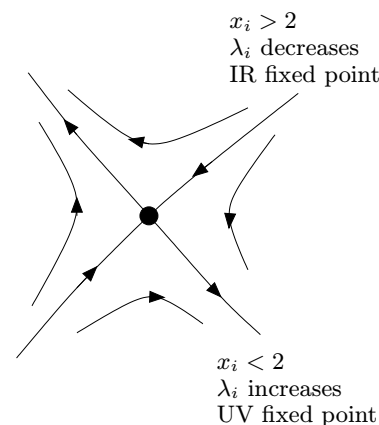
2. The bulk case

In two-dimensions, the field theory at an RG fixed point is a conformal field theory (cft). Hence, near such a fixed point, the (bulk) action takes the form

$$S_{\text{bulk}} = S_0 + \sum_i \lambda_i \int \varphi_i(z, \bar{z}) \, \mathrm{d}^2 z \,, \qquad (2.1)$$

where S_0 is the fixed point cft action and $\varphi_i(z, \bar{z})$ are conformal scaling fields of scaling dimension x_i, with corresponding couplings λ_i. Under scale transformations, the couplings have β–functions

$$\dot{\lambda}_i \equiv \beta_i = (2 - x_i)\lambda_i + \dots \,, \qquad (2.2)$$

so that λ_i increases or decreases as $x_i < 2$ or $x_i > 2$ respectively, as in figure 1. (The case $x_i = 2$ needs more careful analysis to decide the nature of the corresponding RG flow, if any) Order is brought

$x_i > 2$
λ_i decreases
IR fixed point

$x_i < 2$
λ_i increases
UV fixed point

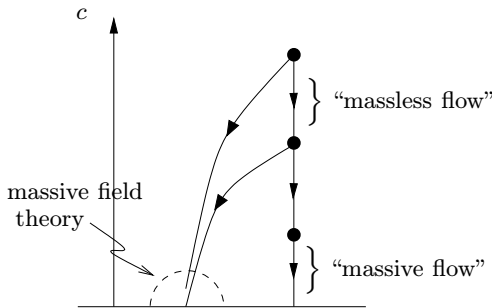**Figure 1:**
The RG flows in the neighbourhood of a fixed point

to the space of RG flows by Zamolodchikov's c–theorem [22]. This states that for unitary theories there is a function C which is decreasing along RG flows and which is equal at a fixed

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point to the conformal central charge of the corresponding cft. In terms of the trace of the stress-energy tensor one has

$$\begin{cases} \dot{C} = -\frac{3}{4} \langle \, \Theta(1)\,\Theta(0) \, \rangle \\ C\big|_{\Theta=0} = c \, , \quad \text{central charge} \end{cases}$$

As a result, one knows that the bulk RG flows always cause c to decrease and one has a schematic picture as in figure 2. Flows which have non-trivial cfts as both UV and IR fixed points are known as 'massless flows', while those which end at a fixed point with $c=0$ are known as 'massive flows'. In the latter case these define massive scattering theories in the neighbourhood of the IR fixed point.

Figure 2:
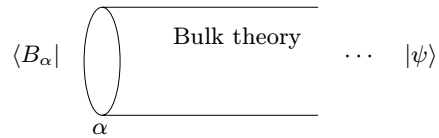Schematic picture of the space of bulk RG flows

    Given this picture, the next most important fact is that for many cases one has a classification of the conformal field theories, for example for unitary minimal models with $c<1$ one has the 'ADE' classification of Cappelli et al [6] (for these and other results on the minimal models see e.g. [9]). From this result one knows the possible UV and IR endpoints, and all the possible relevant ($x_i < 2$) operators and one has then 'simply' to join up the cfts by flows.

    Finally, one has the Landau-Ginzburg model for the theories with a given UV fixed point [22] in which the space of relevant perturbations are described by an action for scalar field(s) with a particular class of polynomial potentials. This picture gives one a heuristic understanding of the space of RG flows, and the vacuum structure of the IR fixed points; for a detailed treatment of the tricritical Ising model using this method, see [15].

    Let's now turn to the case of models with boundaries and see how much can be said there.

3. The boundary case

If the space on which our theory is defined has boundaries, then we must consider the possible boundary conditions that can be put on these boundaries and the space of fields which can exist on corresponding boundary conditions. For a semi-infinite cylinder, a boundary condition α on the end of the cylinder defines a 'boundary state' $\langle B_\alpha |$. One can take the inner product of such a state with a bulk states $| \psi \rangle$ (as in figure 3), but the boundary state is not actually a normalisable state in the bulk Hilbert space.

Figure 3:
Bulk and boundary states associated to a cylinder

    If the bulk theory is at an RG fixed point, it is not necessarily the case that the boundary is invariant under scale transformations. Those boundary conditions which are invariant are called 'conformal boundary conditions' and Cardy gave the conditions for a boundary state to correspond to a conformal boundary condition in [7]. For each bulk CFT there is a set of possible conformal boundary conditions $\{ B_\alpha \}$, and associated to each conformal boundary condition there is a set of conformal scaling fields which can exist on that boundary.

    Consequently, near a fixed point the action for the theory on a space M with boundary ∂M takes the form

$$S = S_{\text{bulk}} + \sum_i \mu_i \oint_{\partial M} \phi_i(l) \, \mathrm{d}l \, , \qquad (3.1)$$

where S_{bulk} is the action (2.1) and ϕ_i are boundary scaling fields of dimension h_i.

    For each CFT and boundary condition B_α we have to determine the possible RG trajectories generated by both bulk and boundary perturbations, and the IR bulk theory and its boundary condition B' which is quite a challenge.

There is one important simplification that we can make, and that is to set all the couplings λ_i in the bulk action (2.1) to zero. If that is the case, then (to all orders in perturbation theory) the bulk theory remains conformally invariant, and so the cft of the IR fixed point will be the same as that at the UV fixed point. Consequently, the only effect of the RG is to flow in the space of possible boundary conditions, in which the fixed points are the conformal boundary conditions.

We can calculate the β function for the coupling μ_i to find

$$\dot{\mu}_i = (1 - h_i)\mu_i \; + \; \ldots \; . \qquad (3.2)$$

Again we have a flow *out* of a UV fixed point for $h_i \leq 1$ and a flow *into* an IR fixed point for $h_i \geq 1$. Hence to understand the space of flows starting at a given boundary condition, we only need to consider the perturbations by operators with weight $h_i < 1$, which (for a rational cft, and in particular for a minimal model) will be a finite dimensional space of perturbations.

How do we bring order to this space of flows? The first requirement is to a have a quantitative estimate of the number of degrees of freedom associated to a boundary condition (which should then decrease along RG flows).

In [2], Affleck and Ludwig defined the 'generalised ground state degeneracy' g_α associated to a conformal boundary condition B_α as follows. Consider the partition function of a cft on a cylinder of length R and circumference L and with boundary conditions α and β at the two ends, as in figure 4.
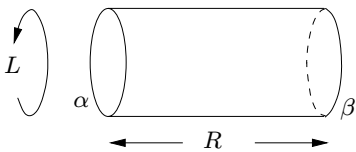

Figure 4: The cylinder partition function

In terms of the Hamiltonian $H_{\alpha\beta}(R)$ propagating around the cylinder, this partition functions has a canonical normalisation

$$Z_{\alpha\beta} \; = \; \text{Tr} \, e^{-L \, H_{\alpha\beta}(R)} \; . \qquad (3.3)$$

In the limit $R \to \infty$, one finds

$$Z_{\alpha\beta} \; \sim \; g_\alpha \, g_\beta \, e^{-R \, E_0(L)} \; + \; \ldots \; , \qquad (3.4)$$

where $E_0(L)$ is the energy of the ground state $|\Omega\rangle$ of the Hamiltonian propagating along the cylinder, and where the constants g_α, g_β are given by

$$g_\alpha = \langle B_\alpha | \Omega \rangle \; . \qquad (3.5)$$

These constants are the generalised ground state degeneracies associated to the conformal boundary conditions[1].

For a perturbed theory with a non-trivial scale-dependence, it is not so easy to define such ground state degeneracies. If we try to adapt the previous method, instead of (3.4), we find

$$Z_{\alpha\beta} \sim \mathcal{G}_\alpha(L) \, \mathcal{G}_\beta(L) \, e^{-R \, E_0(L)} \; + \; \ldots \; , \quad (3.6)$$

where the functions $\mathcal{G}_\alpha(L)$ generically behave for large L as

$$\log \mathcal{G}_\alpha(L) = -L \, f_\alpha^B \; + \; g_\alpha(L) \; . \qquad (3.7)$$

Here f_α^B is the boundary free energy per unit length, and it is the function $g_\alpha(L)$ which interpolates g_{UV} and g_{IR}. If we try to calculate $\mathcal{G}_\alpha(L)$ or $g_\alpha(L)$, we find different behaviour for $h < 1/2$ and $h > 1/2$. (The special case $h = 1/2$ is again different).

For $h < 1/2$, f_α^B is physical and non-perturbative. This means that it is not possible to calculate $g_\alpha(L)$ in perturbation theory.

For $h > 1/2$, f_α^B is non-physical and divergent. Surprisingly, this is a better situation than the previous one, as it is possible to remove the divergent terms proportional to L and calculate $g_\alpha(\mu_R)$ as a perturbation expansion in the renormalised coupling μ_R, as shown by Affleck and Ludwig in [3].

In this second case, Affleck and Ludwig show that g_α defined via (3.7) satisfies

$$\dot{g}_\alpha < 0 \qquad (3.8)$$

to leading order in perturbation theory; nevertheless this result has gone by the name of the "g–theorem", namely that for a perturbation of a unitary theory by purely boundary fields, the UV and IR boundary conditions satisfy $g_{\text{IR}} < g_{\text{UV}}$.

[1]As pointed out by A. Cappelli and W. Nahm in the talk, care must be taken when dealing with irrational theories for which it may be the case that only *ratios* of the constants g_α can be defined sensibly

The upshot of this conjecture is that we can order our flows by the value of g.

Hence, in almost all respects the quantity g_α behaves for boundaries as the central charge c does for bulk theories – except that there is no non-perturbative proof of the corresponding "g–theorem". (For a recent development see [14])

While this result was known for a long time, as were the values of g_α for a large class of boundary conditions, it was not until very recently that there was a complete classification of the possible boundary conditions of the unitary minimal models. This was found by Behrend et al. and discussed in a series of papers [5, 4]. They classified the boundary conditions which admit only a single scalar ($h = 0$) boundary field – equivalently, those for which boundary correlation functions which obey the cluster property – the so–called 'Cardy' boundary conditions.

For the 'A' type 'diagonal' modular invariant theories, they found that the elementary (Cardy type) boundary conditions are in 1–1 correspondence with the Virasoro highest weight representations
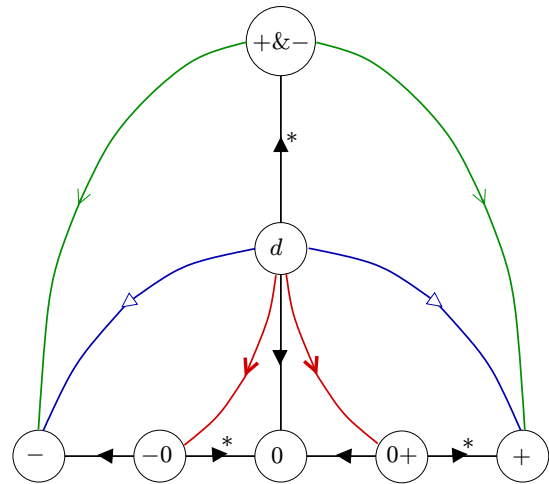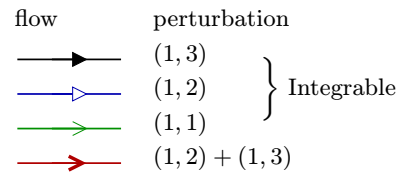
$$B_\alpha \;\leftrightarrow\; h_\alpha \;. \qquad (3.9)$$

As an example, consider the tricritical Ising model, which is the 'A' type unitary minimal model $M_{4,5}$. This cft has $c = 7/10$ and six Virasoro representations of interest; consequently there are six conformal boundary conditions. It can be thought of as the continuum limit of a critical lattice model on each site of which there can either be a spin in state down ($-$) or state up ($+$), or the site can be empty (0). The six Cardy boundary conditions can be labelled by the possible values of the sites on the boundary and associated to the six representations of the Virasoro algebra as in table 1.

In [1], the boundary condition '0' was identified as the free boundary condition and 'd' as an additional 'degenerate' boundary condition; however in view of the fact that 'd' has the maximal number of boundary degrees of freedom, it seems natural to consider it as the free (or possibly freest) boundary condition.

The space of boundary RG flows in the tricritical Ising model has been given recently by Affleck in [1], which we reproduce in figure 5.

$h_{11} = 0$	$h_{14} = 3/2$	$h_{21} = 3/16$
$-$	$+$	0
$g = 0.5127$		$g = 0.725$
$h_{12} = 1/10$	$h_{13} = 3/5$	$h_{22} = 3/80$
-0	$0+$	'd'
$g = 0.8296$		$g = 1.173$

Table 1: The conformal boundary conditions of the tricritical Ising model

Figure 5: The boundary RG flows of the tricritical Ising model

A few comments are in order on the label 'integrable' applied to the flows generated by the fields $\phi_{(1,2)}$ and $\phi_{(1,3)}$. As discussed in [11], one can show show the conservation *to first order in perturbation theory* of non-trivial conserved quantities for perturbations by boundary fields for which a similar argument shows the corresponding bulk perturbation to be integrable. To be explicit, for a generic minimal model this comprises perturbations by fields of type $(1,2)$, $(1,3)$ and $(1,5)$, and their duals $(2,1)$, $(3,1)$ and $(5,1)$. Further evidence for the integrability of the $(1,3)$ perturbation on boundary conditions of type $B_{(1,s)}$

is given by the existence of exact TBA equations which are supposed to describe these flows [17]. For the perturbation by $\phi_{(1,2)} \equiv \phi_{(3,3)}$ the obvious line crossings in figure 9 are usually taken as a clear indication of integrability of the RG flow.

What tools are there which would enable us to map out such a picture? Here we list three:

1. If $y \equiv 1 - h$ is small, then we can use perturbation theory as outlined in [2]. In this way one can calculate Δg along a flow.

2. If the perturbation is integrable, one may be able to use exact integral equation (TBA or NLIE) techniques. In this way one may be able to calculate g, or some or all of the spectrum, along the flow.

3. For any perturbation one can use the Truncated conformal space approach (TCSA). In principle, this can be used to calculate any quantity which can be defined in terms of the perturbed conformal field theory construction. However, one has little control over the accuracy of the results and ranges of parameters for which the results are reliable.

We now consider these in more detail.

4. Perturbation theory

Let us consider a perturbation by a single relevant ($h < 1$) boundary field ϕ. The first condition we must check is that no new perturbations are generated as counterterms to remove divergences in the perturbation expansion in μ. No new counterterms are needed if the operator product expansion of this field with itself takes the form

$$\phi(x)\,\phi(x') = \frac{1}{(x-x')^{2h}} \;+\; \frac{C\,\phi(x')}{(x-x')^h} \;+\; \dots\,, \\ x > x'\,, \qquad (4.1)$$

where the right hand side contains no further contributions from fields of weight h' with $h' \leq 1 - 2y$. Under this condition, Affleck and Ludwig found the β–function to be (in a particular regularisation scheme)

$$\dot{\mu} = y\,\mu \;-\; C\,\mu^2 \;+\; \dots\,, \qquad (4.2)$$

where it is assumed that μ is $O(y)$ and terms of order y^3 have been dropped; they also found the change in g along the flow to be

$$\log\left(\frac{g(\mu)}{g_{\mathrm{UV}}}\right) = -\pi^2 y \mu^2 \;+\; \frac{2\pi^2}{3}C\mu^3 \;+\; O(y^4)\,. \tag{4.3}$$

In figure 6 we plot the β–function and the RG flows, and find a perturbative fixed point at $\mu_R^* = y/C$. This gives

$$\log\left(\frac{g_{\mathrm{IR}}}{g_{\mathrm{UV}}}\right) = -\frac{\pi^2}{3}\,\frac{y^3}{C^2} \;+\; O(y^4)\,. \qquad (4.4)$$
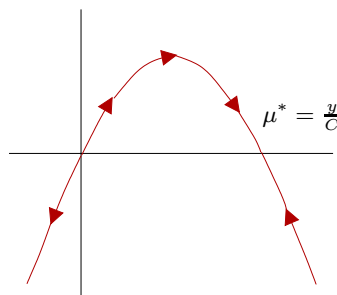

Figure 6: The β–function and RG flows

The technical restriction on the ope (4.1) means that for unitary minimal models one can only consider perturbations by fields of type $(1,3)$. For a long time, the obstacle to using (4.4) to analyse the space of RG flows was the absence of any formula for the structure constant C appearing in the ope (4.1). The equations that the structure constants must satisfy were known since 1991 from the work of Cardy and Lewellen [8], but it was not until 1999 that this obstacle was removed when Runkel found the boundary structure constants for 'A' and 'D' type models in [19]. In retrospect, the solution for the A-models is quite simple to understand, but the general situation is still rather intriguing, as is described in the talk by J.-B. Zuber [23].

Runkel's results for the A-series were applied to Affleck and Ludwig's calculation by Recknagel et al in [18]. The calculation of the ratio (4.4) is itself not hard, but the analysis of the possible boundary conditions consistent with this ratio is rather involved, and has a surprising result. They found the result that the value of g_{IR} at the perturbative IR fixed point of the RG

5

flow generated by the field $\phi_{(1,3)}$ away from the (r,s) boundary is given by the superposition of $\min(r,s)$ elementary boundary conditions

$$B_{(r,s)} \ + \ \phi_{(1,3)} \to \bigoplus_{t=1}^{\min(r,s)} B_{(r+s+1-2t,1)} \ . \quad (4.5)$$

Note that each of the end-point boundary conditions is *by itself* stable, but that the superposition is not itself stable against perturbations by scalar operators, as described by Affleck [1].

Given the surprising nature of their results, it is desirable to have an independent check, and one way to check these results is by using TCSA, which we describe next.

5. The Truncated Conformal Space Approach

This method was introduced by Yurov and Al. Zamolodchikov in 1990 in [21] as a numerical method to study bulk perturbations of conformal field theories. It was first applied to the boundary situation by Dorey et al in [10]. The idea is to construct the Hamiltonian of the model on a strip of width R, restrict it to a finite-dimensional subspace, and diagonalise it numerically.

We consider a strip of width R with action (3.1) with the boundary perturbation restricted to one edge of the strip. If we map the strip to the upper-half-plane, the Hamiltonian can be expressed in terms of the operators acting on the upper-half-plane Hilbert space as

$$H = \frac{\pi}{R}\Big((L_0 - \frac{c}{24}) + \lambda \left(\frac{R}{\pi}\right)^{2-x} \int \varphi(e^{i\theta}, e^{-i\theta}) \, \mathrm{d}\theta$$
$$+ \mu \left(\frac{R}{\pi}\right)^{1-h} \phi(1) \ \Big) , \quad (5.1)$$

where the combinations $\lambda \left(\frac{R}{\pi}\right)^{2-x}$ and $\mu \left(\frac{R}{\pi}\right)^{1-h}$ are dimensionless.

The truncation of the Hilbert space to a finite dimensional space can be easily achieved by discarding all states for which the eigenvalue of L_0 is greater than some cut-off N, and the matrix elements of (5.1) can (often) be calculated exactly on this space. Given the truncated space, one can then simply find the eigenstates and eigenvalues of the Hamiltonian restricted to this space

numerically. For moderate values of R (including the vicinity of the IR fixed point, with luck) this can give reasonably accurate numerical results for many quantities. (Again see [20] for examples).

To check the predictions of Recknagel et al. in the case of the tricritical Ising model it turns out to be sufficient to examine the spectra of the truncated Hamiltonian. For purely boundary perturbations one expects TCSA to give the following results for the spectra:

For small R, the UV behaviour

$$E_i \ \sim \ \frac{\pi}{R} \, (h_i^{(\mathrm{UV})} - \frac{c}{24}) + \dots \ .$$

For R in the 'scaling region', the IR behaviour

$$E_i \ \sim \ f_B \ + \ \frac{\pi}{R} \, (h_i^{(\mathrm{IR})} - \frac{c}{24}) + \dots \ .$$

For large R, the truncation-dominated regime.
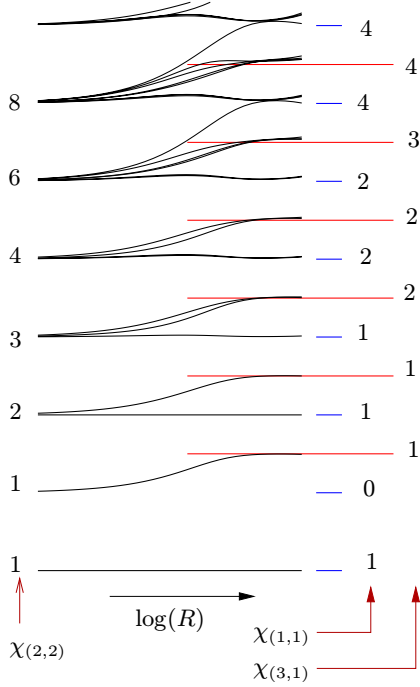
$$E_i \ \propto \ R^{1-h} \ .$$

It is important to note that there is no guarantee that the value of R for which the scaling region becomes apparent is less than the value at which truncation errors start to dominate. At the moment there is little analytic control over the errors in the TCSA and it is a matter of luck whether the scaling region is accessible in any particular model.

Furthermore the TCSA results are expressed in terms of the bare coupling constant μ, and certain quantities may become divergent as the cutoff N is removed. For example, for the models analysed by Recknagel et al, h is close to 1 and so f_B is a divergent quantity. This means that the individual eigenvalues of the TCSA Hamiltonian depend strongly on N and it is only the energy differences which are physical.

As an example we give the results for the spectra of the TCSA applied to the tricritical Ising model with the particular case of the strip with boundary conditions $(1,1)$ and $(2,2)$, with the $(2,2)$ boundary perturbed by the field $\phi_{(1,3)}$ in the direction of the perturbative fixed point. In figure 7 we plot the normalised energy levels

$$\mathcal{E}_i = 2\frac{E_i - E_0}{E_2 - E_0} \ \text{ vs. } \ \log R \ ,$$

so that the second excited state always has normalised $\mathcal{E}_2 = 2$. The numbers shown are the multiplicities of the corresponding UV or IR levels.

Figure 7: The boundary $B_{(2,2)}$ perturbed by $\phi_{(1,3)}$

Note that truncation errors are already affecting the higher levels for large R; these errors decrease on increasing the truncation level.

It is one of Cardy's results [7] that the states in the Hilbert space of the strip with boundary conditions $(1,1)$ and (r,s) are encoded by the character $\chi_{(r,s)}$. This means that we can read off the boundary content of the strip by simply counting the degeneracies of states.

For example, the UV fixed point is the boundary condition $B_{(2,2)}$ which has singular vectors at levels 4 and 6 (amongst others) which would have enabled us to identify it through

$$
\begin{aligned}
& q^{c/24-h_{2,2}} \times \chi_{(2,2)}(q) \\
& = 1 + q + 2q^2 + 3q^4 + 4q^5 + 6q^6 + 8q^8 + \dots \\
& = (1 - q^4 - q^6 + \dots)\, \varphi(q)\,,
\end{aligned} \tag{5.2}
$$

where the partition generating function φ is

$$
\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}\,. \tag{5.3}
$$

Counting the degeneracies of the states in the IR we see that indeed they are grouped into the

characters $\chi_{(1,1)}$ and $\chi_{(3,1)}$, in agreement with the prediction of Recknagel et al. that

$$
B_{(2,2)} + \phi_{(1,3)} \to B_{(1,1)} \& B_{(3,1)}\,. \tag{5.4}
$$

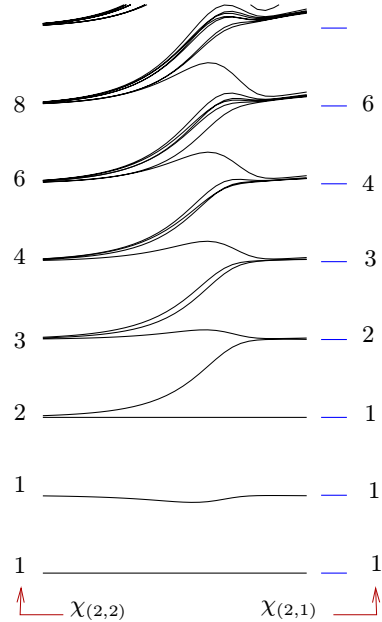As a second check, we can also read off the first gap, which in the IR should be

$$
\mathcal{E}_1^{(\mathrm{IR})} = h_{(3,1)} - h_{(1,1)} = 3/2 \tag{5.5}
$$

We have indicated this on figure 7 and the TCSA data agrees very well with this prediction.

Using the results of Recknagel et al it is possible to fill in all the flows in figure 5 marked with an asterisk, and two others by Z_2 symmetry.

This leaves unchecked the flows away from the $(2,2)$ boundary condition generated by $\phi_{(1,2)}$, the flow generated by $\phi_{(1,3)}$ in the direction of the non-perturbative fixed point, and the two-dimensional space of flows generated by linear combinations of $\phi_{(1,2)}$ and $\phi_{(1,3)}$. In these cases we are left with the TCSA as the only viable tool at the moment.

To complete the picture of the $\phi_{(1,3)}$ flows, in figure 8 the spectra $\mathcal{E}_i$ for the perturbation of the boundary condition $B_{(2,2)}$ by $\phi_{(1,3)}$ in the opposite direction to that in figure 7. It is easy
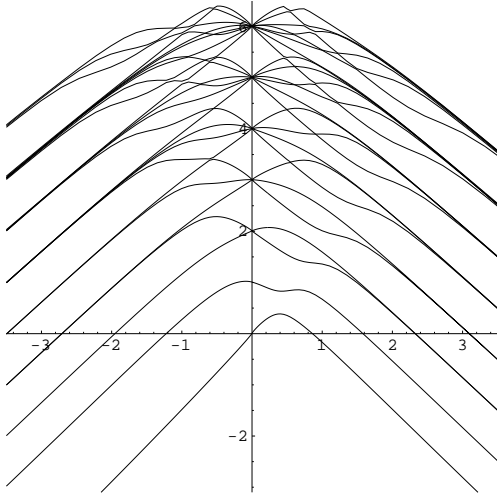

Figure 8: The boundary $B_{(2,2)}$ perturbed by $\phi_{(1,3)}$

to check that this is given by

$$
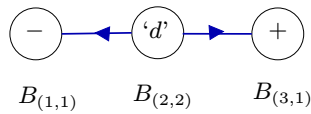B_{(2,2)} - \phi_{(1,3)} \to B_{(2,1)}\,, \tag{5.6}
$$

an example of one of the conjectures given in [18].

The other simple flows away from $B_{(2,2)}$ in the tricritical Ising model are generated by $\phi_{3,3} \equiv \phi_{(1,2)}$. In this case, $h_{33} = h_{12} = 1/10 < 1/2$ so that the constant f_B is finite and physical (being the expectation value of the perturbing field) and so we can plot the whole spectrum. In figure (9) we plot the eigenvalues of the dimensionless operator $(R/\pi)H$ against the dimensionless variable $\kappa = \mu R^{1-h_{33}}$.

Figure 9: $M_{4,5} : B_{(22)} \pm \phi_{33}$.
The first 25 eigenvalues of $(R/\pi)H(\kappa)$ plotted vs. κ.

Here again by simply counting the states we can identify the end points clearly as $B_{(3,1)}$ in the positive direction and $B_{(1,1)}$ in the negative direction. This is in agreement with the picture in figure 5, where we have
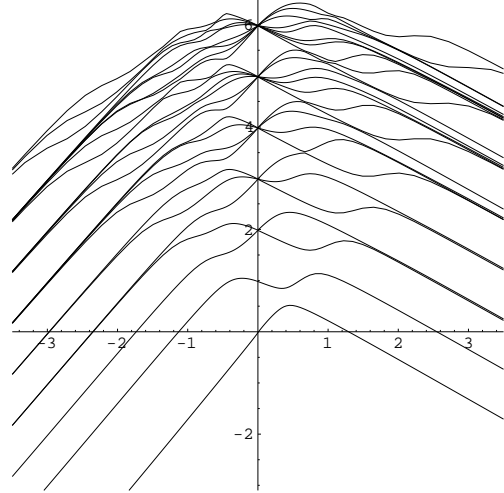

$\quad B_{(1,1)} \qquad B_{(2,2)} \qquad B_{(3,1)}$

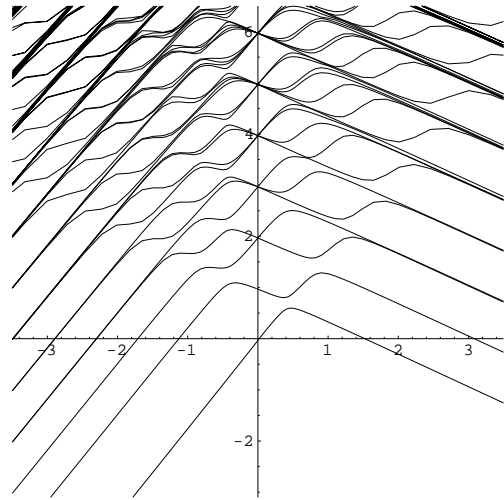**Figure 10:** $\phi_{(33)}$ boundary RG flows

It is also evident from the graph that there are numerous line crossings, which are usually taken to be an indication of integrability. Since the perturbing field in this case $\phi_{(33)} \equiv \phi_{(12)}$, this is presumably related in some manner to the $a_2^{(2)}$ boundary affine Toda theory, in the same manner that $\phi_{(1,3)}$ perturbations are related to the boundary sine-Gordon theory.

Using the TCSA we can just as easily examine other models. For unitary minimal mod-

els the relevant fields are $\phi_{(rr)}$ and $\phi_{(r,r+2)}$. As an example, in figures 11 and 12 we consider for the unitary minimal models $M_{6,7}$ and $M_{10,11}$ the same perturbation $B_{(2,2)} \pm \phi_{(3,3)}$ as in figure 9.
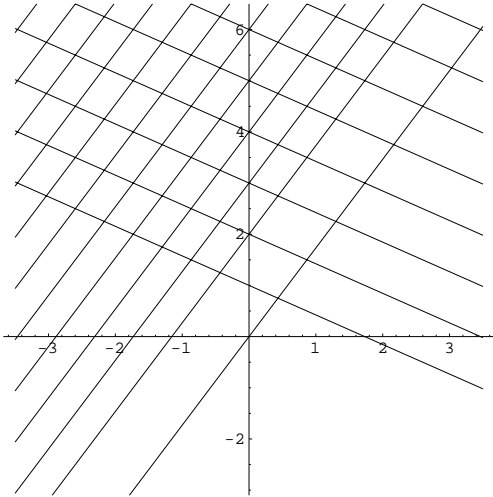

Figure 11: $M_{6,7} : B_{(22)} \pm \phi_{33}$.
The first 25 eigenvalues of $(R/\pi)H(\kappa)$ plotted vs. κ.

Figure 12: $M_{10,11} : B_{(22)} \pm \phi_{33}$.
The first 25 eigenvalues of $(R/\pi)H(\kappa)$ plotted vs. κ.

It is apparent that these are deformations of figure 9, and that the IR endpoints are unchanged from $B_{(1,1)}$ and $B_{(3,1)}$, but that the line crossings so obvious there are missing here. However, if we continue all the way to the limiting point of the unitary minimal models at $c = 1$, in figure 13 we find something quite striking.

Figure 13: $M_{\infty,\infty+1} : B_{(22)} \pm \phi_{33}$.
The eigenvalues of $(R/\pi)H(\kappa)$ plotted against κ.

How can we understand that the flows all become straight lines?
The key to understanding these RG flows is the idea that

> "perturbative flows for $c < 1$ become identities at $c = 1$"

This is quite natural given the β–function for the $\phi_{(1,3)}$ boundary perturbations of the unitary minimal model $M_{p,p+1}$. In these models

$$c = 1 - \frac{6}{p(p+1)} \,, \quad h_{13} = 1 - \frac{2}{p+1} \,, \quad y = \frac{2}{p+1} \,.$$

The limit $c = 1$ is given by taking $p \to \infty$, and the perturbative fixed point is at $\mu^* = y/C \to 0$. Hence as we approach $c = 1$, the perturbative IR fixed point gets closer and closer to the UV fixed point, until at $c = 1$ they have identical properties, and indeed they then represent the same boundary condition. However, for $c < 1$ the UV fixed point is $B_{(2,2)}$ and the IR fixed point the superposition $B_{(1,1)} + B_{(3,1)}$. Hence at $c = 1$ these must be equivalent descriptions of the same boundary condition.

We need to be somewhat careful when we come to check this idea. The "$c = 1$" boundary conditions are defined as limits of the $c < 1$ boundary conditions such that the space of states has a smooth limit, i.e. given any truncation level N, there is some $p(N)$ such that for $p > p(N)$ the space of states in $M_{p,p+1}$ up to level N has

constant dimension. This requirement is already non-trivial for $N = 1$, as we shall see now.

One of the fundamental results in boundary conformal field theory is that the space of boundary fields on a boundary condition is in 1–1 correspondence with the space of states in the Hilbert space on the upper half plane associated to that boundary condition; furthermore, for the 'A' type models considered here, the boundary conditions B_α are in 1–1 correspondence with the Virasoro representations (α) and the space of primary fields interpolating two boundary conditions is given by the fusion of the corresponding representations. In particular, the space of primary fields on a given boundary B_α is given by the fusion product of the representation (α) with itself. Hence the primary fields on the boundary condition $B_{(2,2)}$ are given by the fusion

$$(22) \times (22) = (11) + (13) + (31) + (33) \,. \quad (5.7)$$

The weights (and their $c \to 1$ limits) of these 4 primary fields are

$$\begin{aligned} h_{11} &= 0 & &\to 0 \\ h_{13} &= 1 - 2/p + \ldots & &\to 1 \\ h_{31} &= 1 + 2/p & &\to 1 \\ h_{33} &= 1/(2p^2) + \ldots & &\to 0 \end{aligned} \quad (5.8)$$

This immediately causes concern, as anyone familiar with the representation theory of the Virasoro algebra will notice, that since $h_{33} \to 0$ as $c \to 1$, the state

$$L_{-1} | h_{33} \rangle \,, \quad (5.9)$$

will become a null state at $c = 1$. Since we require the space of states to have a smooth limit, we must rescale this state to obtain a *new* primary field of weight 1,

$$d_3 = \lim_{p \to \infty} \frac{1}{\sqrt{2h_{33}}} L_{-1} | h_{33} \rangle \,. \quad (5.10)$$

It turns out that this field is entirely well behaved – it might have been the case that some of its correlation functions diverged, but in fact they remain finite in the limit $c \to 1$.

So, in the limit $c \to 1$ of $B_{(2,2)}$ we have (amongst other primary fields)

$$\begin{aligned} &2 \text{ fields of weight } 0 \\ &3 \text{ fields of weight } 1 \end{aligned} \quad (5.11)$$

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These are exactly the numbers of such boundary fields that exist on a superposition of the boundary conditions[2]

$$B_{(1,1)} + B_{(3,1)} . \qquad (5.12)$$

On this superposition one expects to have (amongst other primary fields of higher weight) two scalar fields

$$\mathbb{1}_{(1,1)}, \ \mathbb{1}_{(3,1)}, \qquad (5.13)$$

being the projectors onto the individual boundary conditions, and three fields of weight one

$$\psi^{(1,1)\to(3,1)}, \ \psi^{(3,1)\to(1,1)}, \ \phi^{(3,1)}, \qquad (5.14)$$

where the fields $\psi^{(1,1)\to(3,1)}$ and $\psi^{(3,1)\to(1,1)}$ interpolate the two boundary conditions, while the field $\phi^{(3,1)}$ lives on the $B_{(3,1)}$ boundary condition.

One can check [12] that the combinations

$$\mathbb{1}_{(2,2)} = \mathbb{1}_{(1,1)} + \mathbb{1}_{(3,1)} ,$$
$$\phi_{(3,3)} = \sqrt{3} \cdot \mathbb{1}_{(1,1)} - \frac{1}{\sqrt{3}} \cdot \mathbb{1}_{(3,1)} , \qquad (5.15)$$

of the fields on the $B_{(1,1)} + B_{(3,1)}$ boundary have exactly the same operator product expansions as their namesakes on the $B_{(2,2)}$ boundary. Similarly one can form linear combinations of the weight one fields on the superposition which have the same operator product expansions with themselves and with the weight 0 fields as do those on the single $B_{(2,2)}$ boundary. In this way we find substantial evidence to support our conjecture that these two boundary conditions become equal at $c = 1$.

We can now return to our discussion of RG flows generated by $\phi_{(3,3)}$. If we consider a perturbation by $\phi_{(3,3)}$, then the corresponding Hamiltonian is

$$H = H_0 + \mu \, \phi_{(3,3)} . \qquad (5.16)$$

If we now rewrite this using (5.15), we find

$$H = H_0 + \mu \left[\sqrt{3} \cdot \mathbb{1}_{(1,1)} - \frac{1}{\sqrt{3}} \cdot \mathbb{1}_{(3,1)} \right] . \ (5.17)$$

Hence, on states in the component boundary condition $B_{(1,1)}$

$$H = H_0 + \sqrt{3} \, \mu , \qquad (5.18)$$

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[2]The boundary conditions $(r, 1)$ and $(1, r)$ have no similar problems and in fact have the same (smooth) limit as $c \to 1$.

and on states in the component boundary condition $B_{(3,1)}$

$$H = H_0 - \frac{1}{\sqrt{3}} \, \mu . \qquad (5.19)$$

In other words the perturbation of $B_{(2,2)}$ by $\phi_{(3,3)}$ just leads to the linear splitting of the energies of the states in the $B_{(1,1)}$ and $B_{(3,1)}$ boundary conditions seen in figure 13.

There is of course no such simple interpretation of the $\phi_{(3,3)}$ perturbations for $c < 1$, but it appears that the identity of the IR endpoints of the perturbation of $B_{(2,2)}$ by $\pm\phi_{(3,3)}$ remains unchanged all the way down to $p = 4$ which is the smallest value of p for which this boundary perturbation exists in the unitary minimal models.

We believe that this pattern extends to all the perturbations of 'low' weight, i.e. by fields of type $\phi_{(r,r)}$. That is these fields can all be written at $c = 1$ as linear combinations of projectors, and that the IR endpoints are unchanged as one passes to the models with $c < 1$.

6. Conclusions

We have seen that the perturbations of the unitary minimal model boundary conditions by fields of type $\phi_{(1,3)}$ can be analysed in perturbation theory and the boundary condition of the perturbative IR fixed point identified. In general this is a superposition of boundary conditions. This results can then be checked by the numerical TCSA method.

We have also seen how perturbations by fields of type $\phi_{(3,3)}$, and in general of type $\phi_{(r,r)}$, can be understood as deformations of especially simple RG flows at $c = 1$. Again these can be checked by TCSA methods.

Finally we should mention that TBA equations for the function $g_\alpha(L)$ have been proposed in [17] for the integrable perturbations $B_{(1,r)} \pm \phi_{(1,3)}$. So far these have not been subjected to quantitative tests, but they agree with the IR fixed point calculated using perturbation theory (for one direction of the flow) and with TCSA (for the other direction).

This still leaves many questions open:

For purely boundary flows some questions which occur are: Can one understand any other RG flows as deformations of $c = 1$ flows? (Some work towards this for flows by weight one fields is contained in [13].) Can one find TBA systems to describe the remaining $\phi_{(1,3)}$ RG flows with superpositions of boundary conditions at the IR endpoints? Finally, can one find a simple picture such as the Landau-Ginzburg picture for bulk flows which will give the qualitative pattern of flows and fixed points?

Some recent results for joint bulk and boundary perturbations will be presented in the talk by Roberto Tateo [20], but in general the situation is even less clear than that for purely boundary perturbations.

Acknowledgments

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