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From Reflection Amplitudes to One-point Functions in Non-simply Laced Affine Toda Theories and Applications to Coupled Minimal Models

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ABSTRACT: The reflection amplitudes in non-affine Toda theories which possess extended conformal symmetry are calculated. Considering affine Toda theories as perturbed non-affine Toda theories and using reflection relations which relate different fields with the same conformal dimension, we deduce the vacuum expectation values of local fields for all dual pairs of non-simply laced affine Toda field theories. As an application, we calculate the leading term in the short and long distance predictions of the two-point correlation functions in the massive phase of two coupled minimal models. The central charge of the unperturbed models ranges from $c = 1$ to $c = 2$, where the perturbed models correspond to two magnetically coupled Ising models and Heisenberg spin ladders, respectively.

1. Introduction

Among the family of known integrable quantum field theories (QFT)s the affine Toda field theories (ATFT)s have attracted much attention, both classically and at the quantum level, due to their remarkable properties and interesting algebraic structure. They are generally described by the action in Euclidean space :

$$\mathcal{A} = \int d^2x \Big[\frac{1}{8\pi} (\partial_\mu \boldsymbol{\varphi})^2 + \sum_{i=0}^{r} \mu_{\mathbf{e}_i} e^{b\mathbf{e}_i \cdot \boldsymbol{\varphi}} \Big], \quad (1.1)$$

where $\{\mathbf{e}_i\} \in \Phi_{\mathbf{s}}(\mathcal{G})$ ($i = 1, ..., r$) is the set of simple roots of the finite Lie algebra $\mathcal{G}$ of rank r and $-\mathbf{e}_0$ is a maximal root satisfying $\mathbf{e}_0 + \sum_{i=1}^{r} n_i \mathbf{e}_i = 0$. We also introduce the scale parameters $\mu_{\mathbf{e}_i}$. This class of models which can be considered as perturbed conformal field theories (CFT)s appears in various physical contexts. The ultraviolet (UV) behaviour of these integrable theories is encoded in the CFT data

while the large distance properties are defined by the S-matrix data. In such models a representation of the basic CFT primary fields is generally provided in terms of exponential fields. The CFT data also includes the "reflection amplitudes" (RA) [1] which define the linear transformations between different exponential fields possessing the same quantum numbers. In particular, these RA play a crucial role for the description of the zero-mode dynamics which determines the UV asymptotics of the ground state energy $E(R)$ (or effective central charge $c_{\mathrm{eff}}(R)$) for the system on the circle of size R. In [2], we compared this result at small R to one obtained from the S-matrix data using the TBA method. Both results agree which can be considered as a non-trivial test for the S-matrix amplitudes proposed in [3]. On the other hand, the RA are the main objects for the calculation of the one-point functions of local fields, i.e. vacuum expectation values (VEV)s in the bulk. These latter quantities play an important role in QFT and statistical mechanics [4, 5]. In statistical mechanics the "generalized susceptibilities" i.e. the linear re-

sponse of the system to external fields, are determined by such quantities. In QFT defined as perturbed CFT, they also constitute the basic ingredients for multipoint correlation functions, using short-distance expansions [5, 6]. Over the past four years, important progress has been made in the calculations of the VEVs in two dimensional integrable QFT [7, 8]. In particular, the VEVs for ATFTs associated with simply laced cases have been calculated in [8].

In this talk, I will present the method we used in [2, 9] to get the VEVs for all non-simply laced ATFTs :

$$G(\boldsymbol{a}) = < \exp(\boldsymbol{a} \cdot \boldsymbol{\varphi})(x) > \qquad (1.2)$$

from the RA of the Toda field theories (TFT)s calculated in [2]. The VEVs proposed are checked both non-perturbatively and perturbatively. Finally, by considering the specific case of $C_2^{(1)}$ and its quantum group restriction, I calculate VEVs of primary operators in two planar systems corresponding to two coupled minimal models $\mathcal{M}_{p/p'}$. This provides a numerical estimation of the two-point correlation function between operators belonging to different models [10].

2. Conformal field theory data : reflection amplitudes

The stress-energy tensor $T(z)$, where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$ are complex coordinates,

$$T(z) = -\frac{1}{2}(\partial_z \boldsymbol{\varphi})^2 + \mathbf{Q} \cdot \partial_z^2 \boldsymbol{\varphi} \qquad (2.1)$$

generates the conformal invariance of the action (1.1) when the term with the zeroth root is omitted. Here, we introduce a background charge :

$$\mathbf{Q} = b\boldsymbol{\rho} + \frac{1}{b}\boldsymbol{\rho}^\vee \qquad (2.2)$$

where $\boldsymbol{\rho} = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}$ and $\boldsymbol{\rho}^\vee = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}^\vee$ are respectively the Weyl and dual Weyl vectors of $\mathcal{G}$. The sums in their definitions run over all positive roots $\{\boldsymbol{\alpha}\} \in \Phi_+$, dual positive roots $\{\boldsymbol{\alpha}^\vee\} \in \Phi_+^\vee$. Defining $\boldsymbol{a} = (a_1, ..., a_r)$, the exponential fields

$$V_{\boldsymbol{a}}(x) = \exp(\boldsymbol{a} \cdot \boldsymbol{\varphi})(x) \qquad (2.3)$$

are spinless conformal primary fields with dimensions $\Delta(\boldsymbol{a}) = \frac{\mathbf{Q}^2}{2} - \frac{(\boldsymbol{a}-\mathbf{Q})^2}{2}$. By analogy with the Liouville field theory, the physical space of states $\mathcal{H}$ in TFT consists of the continuum variety of primary states associated with the exponential fields (2.3) and their conformal descendents with $\boldsymbol{a} = i\mathbf{P} + \mathbf{Q}$ and $\mathbf{P} \in R^r$.

Besides the conformal invariance the TFTs also possess an extended symmetry generated by the $W(\mathcal{G})$-algebra [11]. The full chiral $W(\mathcal{G})$-algebra contains r holomorphic fields $W_j(z)$ ($W_2(z) = T(z)$) with spins j which follow the exponents of the Lie algebra $\mathcal{G}$. The primary fields Φ_w of the $W(\mathcal{G})$ algebra are classified by r eigenvalues w_j, $j = 1, \ldots, r$, of the operator $W_{j,0}$ (the zeroth Fourier component of the current $W_j(z)$):

$$W_{j,0}\Phi_w = w_j\Phi_w, \qquad W_{j,n}\Phi_w = 0, \quad n > 0.$$

The fields $V_{\boldsymbol{a}}$ are also primary with respect to the full chiral algebra $W(\mathcal{G})$ with the eigenvalues w_j depending on $\boldsymbol{a}$. These functions $w_j(\boldsymbol{a})$, which define the representation of the $W(\mathcal{G})$-algebra, are invariant with respect to the Weyl group $\mathcal{W}$ of the Lie algebra $\mathcal{G}$ [11], i.e. $w_j(\mathbf{Q}+\hat{s}(\boldsymbol{a} - \mathbf{Q})) = w_j(\boldsymbol{a})$ where $\hat{s} \in \mathcal{W}$ is arbitrary.

Then one defines the primary operators $\Phi_{\boldsymbol{a}}(x)$ in the TFT in terms of (2.3) by introducing the numerical factors $N(a)$:

$$\Phi_{\boldsymbol{a}}(x) = N^{-1}(a) \exp(\boldsymbol{a} \cdot \boldsymbol{\varphi})(x) \qquad (2.4)$$

such that the conformal normalization condition:

$$< \Phi_{\boldsymbol{a}}(x)\Phi_{\boldsymbol{a}}(y) >_{TFT} = \frac{1}{|x - y|^{4\Delta(\boldsymbol{a})}}$$

is satisfied. Indeed, the fields $V_{\mathbf{Q}+\hat{s}(\boldsymbol{a}-\mathbf{Q})}(x)$ are reflection images of each other and are related by the linear transformation :

$$V_{\boldsymbol{a}}(x) = R_{\hat{s}}(\boldsymbol{a})V_{\mathbf{Q}+\hat{s}(\boldsymbol{a}-\mathbf{Q})}(x) \qquad (2.5)$$

where $R_{\hat{s}}(\boldsymbol{a}) \equiv N(\boldsymbol{a})/N(\mathbf{Q} + \hat{s}(\boldsymbol{a} - \mathbf{Q}))$ is called the "reflection amplitude", an important object in CFT which defines the two-point functions of the operator $V_{\boldsymbol{a}}$. In [2] we obtained the following expression for the reflection amplitude $R_{\hat{s}}(\boldsymbol{a})$ in non-simply laced TFT:

$$R_{\hat{s}}(\boldsymbol{a}) = \frac{A(\hat{s}\mathbf{P})}{A(\mathbf{P})} \qquad (2.6)$$

where

$$A(\mathbf{P}) = \prod_{i=1}^{r} [\pi\mu_{\mathbf{e}_i}\gamma(\mathbf{e}_i^2 b^2/2)]^{i\boldsymbol{\omega}_i^\vee \cdot \mathbf{P}/b}$$
$$\times \prod_{\boldsymbol{\alpha}>0} \Gamma(1-i\mathbf{P}\cdot\boldsymbol{\alpha} b)\Gamma(1-i\mathbf{P}\cdot\boldsymbol{\alpha}^\vee/b)$$

contains the fundamental dual weights $\boldsymbol{\omega}_i^\vee$ and we denote $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ as usual. We accept eq. (2.6) as the proper analytical continuation of the function $R_{\hat{s}}(\boldsymbol{a})$ for all $\boldsymbol{a}$.

3. Vacuum expectation values in non-simply laced ATFTs

In the conformal perturbation theory (CPT) approach to ATFT, one can formally rewrite any N-point function of local operators $\mathcal{O}_a(x)$ as :

$$< \mathcal{O}_{a_1}(x_1)...\mathcal{O}_{a_N}(x_N) >=$$
$$Z_\lambda^{-1} < \mathcal{O}_{a_1}(x_1)...\mathcal{O}_{a_N}(x_N)e^{-\lambda\int d^2x\Phi_{pert}(x)} >_0$$

where $Z_\lambda =< e^{-\lambda\int d^2x\Phi_{pert}(x)} >_0$, Φ_{pert} is the perturbing local field, λ is the CPT expansion parameter which characterize the strength of the perturbation, and $< ... >_0$ denotes the expectation value in the TFT. Whereas vertex operators (2.3) satisfy reflection relations (2.5) in the CFT, the CPT framework provides similar relations among their one-point functions in the perturbed case. In other words, if dots stand for any local insertion one has :

$$< V_{\boldsymbol{a}}(x)(...) >_0= R_{\hat{s}}(\boldsymbol{a}) < V_{\mathbf{Q}+\hat{s}(\boldsymbol{a}-\mathbf{Q})}(x)(...) >_0 .$$

Indeed, using CPT one expects that similar relations hold for $G(\boldsymbol{a})$. It is then crucial to notice that each ATFT Lagrangian representation in (1.1), denoted $\mathcal{L}_b[\Phi_{\mathbf{s}}(\mathcal{G})]$ with coupling constant b, can be rewritten as two different perturbed TFTs. Let us denote by $\boldsymbol{\eta}$ the extra-root associated with the perturbation and let $\{\epsilon_i\}$ be

an orthogonal basis $(\epsilon_i.\epsilon_j = \delta_{ij})$ in R^r :

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(B_r^{(1)})] \equiv \mathcal{L}_b[\Phi_{\mathbf{s}}(B_r) \oplus \boldsymbol{\eta} \equiv -\epsilon_1 - \epsilon_2],$$
$$\equiv \mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(D_r) \oplus \boldsymbol{\eta} \equiv -\epsilon_r];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(C_r^{(1)})] \equiv \mathcal{L}_b[\Phi_{\mathbf{s}}(C_r) \oplus \boldsymbol{\eta} \equiv -2\epsilon_1],$$
$$\equiv \mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(C_r) \oplus \boldsymbol{\eta} \equiv -2\epsilon_r];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(F_4^{(1)})] \equiv \mathcal{L}_b[\Phi_{\mathbf{s}}(F_4) \oplus \boldsymbol{\eta} \equiv -\epsilon_1 - \epsilon_2],$$
$$\equiv \mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(B_4)$$
$$\oplus \boldsymbol{\eta} \equiv -\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)];$$

$$\mathcal{L}_b[\Phi_{\mathbf{s}}(G_2^{(1)})] \equiv \mathcal{L}_b[\Phi_{\mathbf{s}}(G_2) \oplus \boldsymbol{\eta} \equiv -\sqrt{2}\epsilon_1],$$
$$\equiv \mathcal{L}_{-b}[\overline{\Phi}_{\mathbf{s}}(A_2) \oplus \boldsymbol{\eta} \equiv -\sqrt{2/3}\epsilon_2]$$

where the different sets of simple roots can be found in [12]. Here, we introduced also the notation
$$\Phi_{\mathbf{s}}(A_2) = \{\sqrt{2}\epsilon_2, \ \sqrt{3/2}\epsilon_1 - 1/\sqrt{2}\epsilon_2\};$$
$$\Phi_{\mathbf{s}}(G_2) = \{\sqrt{2/3}\epsilon_2, \ 1/\sqrt{2}\epsilon_1 - \sqrt{3/2}\epsilon_2\};$$
$$\overline{\Phi}_{\mathbf{s}}(C_r) = \Phi_{\mathbf{s}}(C_r)|_{\epsilon_i\leftrightarrow\epsilon_{r+1-i}};$$
$$\overline{\Phi}_{\mathbf{s}}(D_r) = \Phi_{\mathbf{s}}(D_r)|_{\epsilon_i\leftrightarrow\epsilon_{r+1-i}};$$
$$\overline{\Phi}_{\mathbf{s}}(A_2) = \Phi_{\mathbf{s}}(A_2)|_{\epsilon_1\leftrightarrow\epsilon_2};$$
$$\overline{\Phi}_{\mathbf{s}}(B_4) = \Phi_{\mathbf{s}}(B_4)|_{\epsilon_i\leftrightarrow-\epsilon_i, \ i\in\{2,3,4\}}.$$

From the previous remarks, we conclude that the VEV (1.2) must satisfy *simultaneously* two irreducible systems of functional equations corresponding to two different sets $\mathcal{W}_{\mathbf{s}}$, i.e.

$$G(\tau\boldsymbol{a}) = R_{\hat{s}_j}(\boldsymbol{a})G(\tau(\mathbf{Q} + \hat{s}_j(\boldsymbol{a} - \mathbf{Q}))) \quad (3.1)$$

for all $\hat{s}_j \in \mathcal{W}_{\mathbf{s}}$ where
- $B_r^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $\mathcal{G} \equiv B_r$
and $(\tau)_{ij} = -\delta_{i \ r+1-j}$ for $\mathcal{G} \equiv D_r$;
- $C_r^{(1)}$: $(\tau)_{ij} = \delta_{ij}$
and $(\tau)_{ij} = -\delta_{i \ r+1-j}$ for $\mathcal{G} \equiv C_r$;
- $F_4^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $\mathcal{G} \equiv F_4$
and $(\tau)_{ij} = \delta_{ij}(\delta_{2j} + \delta_{3j} + \delta_{4j} - \delta_{1j})$ for $\mathcal{G} \equiv B_4$;
- $G_2^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $\mathcal{G} \equiv G_2$
and $(\tau)_{ij} = -\delta_{i \ 3-j}$ for $\mathcal{G} \equiv A_2$.

The spectrum of *real* non-simply laced ATFTs can be expressed in terms of one mass parameter

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$\overline{m}$ as :

$$
\begin{aligned}
B_r^{(1)} : \ & M_a = 2\overline{m}\sin(\pi a/H), \quad a = 1, 2, \ldots, r-1, \\
& M_r = \overline{m} \ ; \\
C_r^{(1)} : \ & M_a = 2\overline{m}\sin(\pi a/H), \quad a = 1, 2, \ldots, r \ ; \\
G_2^{(1)} : \ & M_1 = \overline{m}, \quad M_2 = 2\overline{m}\cos(\pi(1/3 - 1/H)) \ ; \\
F_4^{(1)} : \ & M_1 = \overline{m}, \quad M_2 = 2\overline{m}\cos(\pi(1/3 - 1/H)), \\
& M_3 = 2\overline{m}\cos(\pi(1/6 - 1/H)), \\
& M_4 = 2M_2\cos(\pi/H)
\end{aligned}
$$

with the "deformed" Coxeter number [3] $H = h(1-B)+h^\vee B$ for $B = \frac{b^2}{1+b^2}$. For the non-simply laced cases (except $BC_r \equiv A_{2r}^{(2)}$ - $r \geq 1$ - for which three different parameters are necessary), we have only two different parameters : μ which is associated with the set of standard roots of length $|\mathbf{e}_i|^2 = 2$ whereas μ' is associated with the set of non-standard roots of length $|\mathbf{e}_i|^2 = l^2 \neq 2$. Exact relations between these parameters and the mass parameter $\overline{m}$ in the above spectra were obtained in [2] using the Bethe ansatz (BA) method :

$$
\begin{aligned}
\left(-\pi\mu\gamma(1+b^2) \right)^{h-z} & \left(-\pi\mu'\gamma(1+b^2 l^2/2) \right)^z \\
& = \left(\overline{m}k(\mathcal{G})\kappa(\mathcal{G}) \right)^{2H(1+b^2)}
\end{aligned}
\tag{3.2}
$$

where we define $z = \frac{2(h-h^\vee)}{(2-l^2)}$. Also, we have :

$$
k(B_r^{(1)}) = \frac{2^{-2/H}}{\Gamma(1/H)}, \quad k(C_r^{(1)}) = \frac{2^{2B/H}}{\Gamma(1/H)},
$$

$$
k(F_4^{(1)}) = k(G_2^{(1)}) = \frac{\Gamma(2/3)}{2\Gamma(1/2)\Gamma(1/6 + 1/H)}
$$

and $\kappa(\mathcal{G}) = \frac{\Gamma((1-B)/H)\Gamma(1+B/H)}{2}$.

Finally, using (3.2) we proposed the "minimal" meromorphic solution of (3.1) for all untwisted ATFTs :

$$
\begin{aligned}
G(\boldsymbol{a}) = & \left[\overline{m}k(\mathcal{G})\kappa(\mathcal{G}) \right]^{-\boldsymbol{a}^2} \\
& \times \left[\frac{\mu\gamma(1+b^2)}{\mu'\gamma(1+b^2 l^2/2)} \right]^{\frac{\mathbf{d}\cdot\boldsymbol{a}(1-B)}{Hb}} \\
& \times \left[\frac{\left(-\pi\mu\gamma(1+b^2) \right)^{l^2/2}}{-\pi\mu'\gamma(1+b^2 l^2/2)} \right]^{\frac{\mathbf{d}\cdot\boldsymbol{a}B}{Hb}} \\
& \times \exp\int_0^\infty \frac{dt}{t} \Big(\boldsymbol{a}^2 e^{-2t} - \sum_{\boldsymbol{\alpha}>0} \frac{\sinh(a_{\boldsymbol{\alpha}}bt)\psi_{\boldsymbol{\alpha}}(\boldsymbol{a},t)}{\sinh(t)\sinh(\frac{b^2|\boldsymbol{\alpha}|^2}{2}t)} \Big)
\end{aligned}
\tag{3.3}
$$

with

$$
\begin{aligned}
\psi_{\boldsymbol{\alpha}}(\boldsymbol{a},t) = & \sinh\Big(\big(\frac{b^2|\boldsymbol{\alpha}|^2}{2} + 1 \big)t \Big) \\
& \times \frac{\sinh\big((a_{\boldsymbol{\alpha}}b - 2Q_{\boldsymbol{\alpha}}b + H(1+b^2))t \big)}{\sinh(H(1+b^2)t)}
\end{aligned}
$$

where we denote $a_{\boldsymbol{\alpha}} = \boldsymbol{a}\cdot\boldsymbol{\alpha}$ and define $\mathbf{d} = \frac{\boldsymbol{\rho}^\vee h^\vee - \boldsymbol{\rho}h}{1-l^2/2}$. The integral in (3.3) is convergent iff :

$$
\boldsymbol{\alpha}\cdot\mathbf{Q} - H(b+1/b) \ < \ Re(\boldsymbol{\alpha}\cdot\boldsymbol{a}) \ < \ \boldsymbol{\alpha}\cdot\mathbf{Q}
$$

for all $\boldsymbol{\alpha} \in \Phi_+$ and is defined via analytic continuation outside this domain.

It is straightforward to show that the VEV associated with the twisted ATFTs is obtained from (3.3) using the duality relation for the parameters $\mu_{\mathbf{e}_i}$ and $\mu_{\mathbf{e}_i}^\vee$ associated with the dual pairs of ATFTs :

$$
\pi\mu_{\mathbf{e}_i}\gamma\big(\frac{b^2\mathbf{e}_i^2}{2} \big) \ = \ \left[\pi\mu_{\mathbf{e}_i}^\vee\gamma\big(\frac{\mathbf{e}_i^{\vee 2}}{2b^2} \big) \right]^{b^2\mathbf{e}_i^2/2}
$$

and the change $b \to 1/b$.

In [2], the bulk free energy for all non-simply laced cases have been calculated using the BA approach. On the other hand, VEVs (3.3) can also be used to derive the bulk free energy in ATFT $f_{\widehat{\mathcal{G}}} = -\lim_{V\to\infty}\frac{1}{V}\ln Z$, where V is the volume of the 2D space and Z is the singular part of the partition function associated with the action (1.1). For specific values $\boldsymbol{a} \in b\{\mathbf{e}_i\}$, with $\{\mathbf{e}_i\} \in \Phi_{\mathbf{s}}$ ($i = 1,...,r$) or $\mathbf{e}_0$, the integral in (3.3) can be evaluated explicitly. Using (3.2) and the obvious relations $\partial_\mu f(\mu) = \sum_{\{i\}} < e^{b\mathbf{e}_i\cdot\boldsymbol{\varphi}} >$ or $\partial_{\mu'}f(\mu') = \sum_{\{i'\}} < e^{b\mathbf{e}_{i'}\cdot\boldsymbol{\varphi}} >$ where $\{i\}$ and $\{i'\}$ denote respectively the whole set of long and short roots, we obtain the following bulk free energy :

$$
\begin{aligned}
f_{\widehat{\mathcal{G}}} = & \frac{\overline{m}^2\sin(\pi/H)}{8\sin(\pi B/H)\sin(\pi(1-B)/H)} ; \\
f_{\widehat{\mathcal{G}}} = & \frac{\overline{m}^2\cos(\pi(1/3 - 1/H))}{16\cos(\pi/6)\sin(\pi B/H)\sin(\pi(1-B)/H)}
\end{aligned}
$$

for $\widehat{\mathcal{G}} = B_r^{(1)}$, $C_r^{(1)}$ and $\widehat{\mathcal{G}} = G_2^{(1)}$, $F_4^{(1)}$, respectively. With the change $B \to (1-B)$ one obtains the dual cases. Both approach are in agreement.

One important check consists in expanding the vacuum expectation value (3.3) in a power series in b and comparing each coefficient with the

one obtained from standard Feynman perturbation theory associated with the action (1.1). In [9] we checked that $< \boldsymbol{\varphi} > = \frac{\delta}{\delta \boldsymbol{a}} G(\boldsymbol{a})|_{\boldsymbol{a}=0}$ and the composite operator $<< \boldsymbol{\varphi}^a \boldsymbol{\varphi}^b >> \equiv < \boldsymbol{\varphi}^a \boldsymbol{\varphi}^b > - < \boldsymbol{\varphi}^a >< \boldsymbol{\varphi}^b > = \frac{1}{2} \frac{\delta^2 \ln G(\boldsymbol{a})}{\delta \boldsymbol{a}^a \delta \boldsymbol{a}^b} |_{\boldsymbol{a}=0}$ agreed in both approaches.

4. Related perturbed CFTs : coupled minimal models

The action (1.1) corresponding to the affine Lie algebra $C_2^{(1)}$ with real coupling b is :

$$\mathcal{A} = \int d^2 x \Big[\frac{1}{8\pi} (\partial_\mu \boldsymbol{\varphi})^2 + \mu' e^{-2b\varphi_1} + \mu' e^{2b\varphi_2} + \mu e^{b(\varphi_1 - \varphi_2)} \Big] \quad (4.1)$$

where we have chosen the convention that the length squared of the long roots is four. In the ATFT approach to perturbed CFT, one usually identifies the perturbation with the affine extension of the Lie algebra $\mathcal{G}$. Instead, the perturbation will be associated here with the standard (length 2) root of $C_2^{(1)}$. Removing the last term in the action (4.1) leaves a model associated with $D_2 = SO(4) = SU(2) \otimes SU(2)$, i.e. two decoupled Liouville models. To associate the two first terms of the $C_2^{(1)}$ Toda potential to two decoupled conformal field theories, we introduce for each one a specific background charge at infinity. Then, the exponential fields $e^{-2b\varphi_1}$ and $e^{2b\varphi_2}$ have conformal dimensions 1. As is well known, the "minimal model" $\mathcal{M}_{p/p'}$ with central charge $c = 1 - 6\frac{(p-p')^2}{pp'}$ can be obtained from the Liouville case. Consequently, the D_2 CFT can be identified with two decoupled minimal models by the substitutions $b \to i\beta$, $\mu \to -\mu$, $\mu' \to -\mu'$ and the choice (a) : $\beta^2 = \beta_+^2 = p/2p'$ or (b) : $\beta^2 = \beta_-^2 = p'/2p$ with $p < p'$. We define $\{\Phi_{rs}^{(1)}\}$ and $\{\Phi_{r's'}^{(2)}\}$ as the two sets of primary fields with conformal dimensions :

$$\Delta_{rs} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'} \quad (4.2)$$

for $1 \leq r < p$, $1 \leq s < p'$ and $p < p'$. They are simply related to the vertex operators of each minimal model through the relation :

$$\Phi_{rs}^{(i)}(x) = N_{rs}^{(i)-1} \exp(i\eta_i^{rs} \varphi_i(x)) \quad (4.3)$$

with $\eta_1^{rs} = -\eta_2^{rs} = \frac{(1-r)}{2\beta} - (1-s)\beta$, and where we have introduced the normalization factors $N_{rs}^{(i)}$ for each model. These numerical factors depend on the normalization of the primary fields. Here, they are chosen in such a way that they satisfy the conformal normalization condition :

$$< \Phi_{rs}^{(i)}(x) \Phi_{rs}^{(i)}(y) >_{CFT} = \frac{1}{|x-y|^{4\Delta_{rs}}}$$

for $i \in \{1, 2\}$. For further convenience, we write these coefficients $N_{rs}^{(i)} = N^{(i)}(\eta_i^{rs})$ where :

$$N^{(1)}(\eta) = \big[-\pi \mu' \gamma(-2\beta^2) \big]^{\frac{\eta}{2\beta}}$$
$$\times \Big[\frac{\nu(2\beta^2 + 2\eta\beta)\nu(1/2\beta^2 - \eta/\beta)}{\nu(2\beta^2)\nu(1/2\beta^2)} \Big]^{\frac{1}{2}}$$

and we define $\nu(x) = \Gamma(x)/\Gamma(2-x)$. Notice that $N^{(2)}(\eta) = N^{(1)}(-\eta)$.

For imaginary values $b = i\beta$ the resulting model is very different from the *real* $C_2^{(1)}$ ATFT (4.1) in its physical content : it contains solitons, breathers and excited solitons. However, there are good reasons [1] to believe that the expectation values obtained in the real coupling case (3.3) provide also the expectation values for imaginary coupling. In this latter case, after performing the quantum group restriction of (4.1), the action becomes :

$$\tilde{\mathcal{A}} = \mathcal{M}_{p/p'} + \mathcal{M}_{p/p'} + \lambda \int d^2 x \ \Phi_{pert} \quad (4.4)$$

where we have respectively (a) : $\Phi_{pert} \equiv \Phi_{12}^{(1)} \Phi_{12}^{(2)}$ or (b) : $\Phi_{pert} = \Phi_{21}^{(1)} \Phi_{21}^{(2)}$ and the parameter λ characterizes the strength of the interaction. To express the final result for the VEV in terms of the parameter λ in the action (1.1), we need the exact relation between λ and the parameters μ, μ' in the $C_2^{(1)}$ ATFT with imaginary coupling. We obtain $\lambda = \frac{\pi \mu \mu'}{(4\beta^2 - 1)^2} \gamma(4\beta^2) \gamma^2(1 - 2\beta^2)$.

In case (a), using (3.3) and (4.3) the outcome for the VEV of primary operators is :

$$< 0_{j\tilde{j}} | \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(2)}(x) | 0_{j\tilde{j}} > =$$
$$d_{rs,r's'}^{j\tilde{j}} \Big[\frac{-\pi \lambda \gamma(\frac{1}{1+\xi})(1+\xi)^{\frac{4-2\xi}{1+\xi}}}{\gamma(\frac{3\xi-1}{1+\xi}) \gamma(\frac{1-\xi}{1+\xi})} \Big]^{\frac{(1+\xi)}{2-\xi}(\Delta_{rs} + \Delta_{r's'})}$$
$$\times \exp \mathcal{Q}_{12}((1+\xi)r - 2\xi s, (1+\xi)r' - 2\xi s') (4.5)$$

---

[1] The calculation of the VEVs in both cases (b real or imaginary) within the standard perturbation theory agree through the identification $b = i\beta$.

where $d_{rs,r's'}^{j\tilde{j}} = \frac{\sin(\frac{\pi(2j+1)}{p}|p'r-ps|)}{\sin(\frac{\pi(2j+1)}{p}(p'-p))} \frac{\sin(\frac{\pi(2\tilde{j}+1)}{p}|p'r'-ps'|)}{\sin(\frac{\pi(2\tilde{j}+1)}{p}(p'-p))}$.
Here, degenerate vacua, $|0_{j\tilde{j}}> (j + \tilde{j} \in Z)$, are associated with the $\mathcal{U}_q(D_2) \subset \mathcal{U}_q(D_3^{(2)})$ representation where the spin-$j(\tilde{j})$ representation of $SU(2)$ has dimension $2j + 1$ $(2\tilde{j} + 1)$. The integral $\mathcal{Q}_{12}(\theta, \theta')$ for $|\theta \pm \theta'| < 4\xi$ and $\xi > \frac{1}{3}$ writes :

$$\int_0^\infty \frac{dt}{t} \Big(\frac{\Psi_{12}(\theta, \theta', t)}{\sinh((1+\xi)t)\sinh(2t\xi)\sinh((4-2\xi)t)}$$
$$- \frac{\theta^2 + \theta'^2 - 2(1-\xi)^2}{4\xi(\xi+1)} e^{-2t} \Big)$$

with

$$\Psi_{12}(\theta, \theta', t) = \Big[\cosh((\theta+\theta')t)\cosh((\theta-\theta')t)$$
$$- \cosh((2-2\xi)t) \Big] \sinh((1-\xi)t)\cosh((4-2\xi)t)$$
$$+ \Big[\cosh((\theta+\theta')t) + \cosh((\theta-\theta')t)$$
$$- \cosh((2-2\xi)t) - 1 \Big] \sinh(t)\cosh(t\xi)$$

and defined by analytic continuation outside this domain. The relation between M and λ is :

$$M = \frac{2^{\frac{\xi}{2-\xi}} \Gamma(\frac{\xi}{4-2\xi}) \Gamma(\frac{1}{4-2\xi})}{\pi \Gamma(\frac{1+\xi}{4-2\xi})} \Big[\frac{-\pi\lambda\gamma(\frac{1}{1+\xi})}{\gamma(\frac{3\xi-1}{1+\xi})\gamma(\frac{1-\xi}{1+\xi})} \Big]^{\frac{1+\xi}{4-2\xi}}.$$

Consequently, provided $\beta^2 < 2/3$, the perturbed CFTs develop a massive spectrum for :

$\qquad (i) \quad \lambda > 0$ i.e. $0 < \frac{p}{p'} < \frac{1}{2}$,
$\qquad (ii) \quad \lambda < 0$ i.e. $\frac{1}{2} < \frac{p}{p'} < 1$
where $\xi = \frac{p}{2p'-p}$.

In case (b) the outcome for the VEV is readily obtained from (4.5) through the change $p \leftrightarrow p'$, $(r, r') \leftrightarrow (s, s')$, $\xi \to \frac{1+\xi}{3\xi-1}$.

The VEV $< 0_{j\tilde{j}} | \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(2)}(x) | 0_{j\tilde{j}} > \equiv \mathcal{G}_{rs,r's'}^{j\tilde{j}}$ controls both short and long distance of any two-point correlation functions of primary operators:

$$< 0_{j\tilde{j}} | \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(2)}(0) | 0_{j\tilde{j}} > \underset{|x| \to 0}{\longrightarrow} \mathcal{G}_{rs,r's'}^{j\tilde{j}},$$

$$< 0_{j\tilde{j}} | \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(i)}(0) | 0_{j\tilde{j}} > \underset{|x| \to \infty}{\longrightarrow} \mathcal{G}_{rs,11}^{j\tilde{j}} \mathcal{G}_{11,r's'}^{j\tilde{j}}$$

The case (a) with $p = 4$, $p' = 5$ in (4.4) describes two tricritical Ising models which interact through their leading energy density operators $\Phi_{12}^{(i)} = \epsilon^{(i)}$ of conformal dimension $\Delta_\epsilon = $

1/10. Among other primary operators, each minimal model also contains $\Phi_{22}^{(i)} = \sigma^{(i)}$ with $\Delta_\sigma = 3/80$. For instance, for any vacuum $|j\tilde{j} >$ (up to $d_{rs,r's'}^{j\tilde{j}}$):

$$< \sigma^{(1)}(0)\sigma^{(2)}(0) >_{j\tilde{j}} \sim 1.315726811...(-\lambda)^{3/32};$$
$$< \sigma^{(1)}(0)\sigma^{(2)}(\infty) >_{j\tilde{j}} \sim 1.310238901...(-\lambda)^{3/32};$$
$$< \epsilon^{(1)}(0)\epsilon^{(2)}(0) >_{j\tilde{j}} \sim 2.419476973...(-\lambda)^{1/4};$$
$$< \epsilon^{(1)}(0)\epsilon^{(2)}(\infty) >_{j\tilde{j}} \sim 2.340491994...(-\lambda)^{1/4}$$

where the parameter λ is related to the mass of the lowest kink by $\lambda = -0.2566343706...M^{8/5}$. The case (b) with $p = 5$, $p' = 6$ in (4.4) describes two 3-state Potts models coupled [13] by their energy density operator $\Phi_{21}^{(i)} = \epsilon^{(i)}$ with conformal dimension $\Delta_{21} = 2/5$. Each minimal model also contains the primary operator $\Phi_{23}^{(i)} = \sigma^{(i)}$ - the spin operator - with $\Delta_{23} = 1/15$. We obtain for instance (up to $d_{23,23}^{j\tilde{j}}$) :

$$< \sigma^{(1)}(0)\sigma^{(2)}(0) >_{j\tilde{j}} \sim 4.50...(-\lambda)^{2/3};$$
$$< \sigma^{(1)}(0)\sigma^{(2)}(\infty) >_{j\tilde{j}} \sim 3.64...(-\lambda)^{2/3}$$

where $\lambda = -0.2612863655...M^{2/5}$. Other integrable coupled models [13] can be worked out along the same lines. For examples, four coupled minimal models (restricted $D_4^{(1)}$ ATFT), two coupled WZNW $SO(n)$ models (restricted $D_{2n}^{(1)}$ ATFT), etc...

5. Concluding remarks

Although the non-simply laced BC_r ATFT is different from all the other cases as it possesses three parameters μ, μ' and μ'' it is clear that the VEV can be obtained using the previous approach and it satisfies *simultaneously* B_r and C_r reflection relations. The mass-μ combination :

$$\big(-2\pi\mu\gamma(1+b^2/2) \big)^2 \big(-\pi\mu'\gamma(1+b^2) \big)^{2(r-1)}$$
$$\times \big(-\pi\mu''\gamma(1+2b^2) \big) = \Big(\frac{\overline{m}\kappa(BC_r)}{\Gamma(1/H)} \Big)^{2H(1+b^2)}$$

is proven to be very useful where the mass of the particles in BC_r are $M_a = 2\overline{m}\sin(\pi a/H)$ for $a = 1,...,r$ and $H = 2r + 1$. Using the exact VEV, the self-dual bulk free energy follows,

$$f_{BC_r} = \frac{\overline{m}^2 \sin(\pi/H)}{8\sin(\pi B/H)\sin(\pi(1-B)/H)}.$$

To conclude, the calculation of VEVs using RA appears to be a very powerful tool which can also be applied to descendent fields [14, 15].

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