Hidden Virasoro Symmetry of the Sine Gordon Theory

Davide Fioravanti *

S.I.S.S.A.-I.S.A.S. and I.N.F.N. Sez. di Trieste
Via Beirut 2-4 34013 Trieste, Italy
E-mail: dfiora@sissa.it

ABSTRACT: In the framework of the Sine-Gordon (SG) theory we will present the construction of a dynamical Virasoro symmetry which has nothing to do with the space-time Virasoro symmetry of 2D CFT. Although, it is non-local in the SG field theory, nevertheless it gives rise to a local action on specific N-soliton solution variables. These *analytic* variables possess a beautiful geometrical meaning and enter the Form Factor expressions. At the end, we will also give some preliminary hints about the quantisation.

KEYWORDS: Integrable System, Symmetries, Conserved Charges..

1. Introduction.

The 2D Sine-Gordon (SG) model, defined by the action:

$$S = \frac{\pi}{\gamma}\int Ld^2x \quad , \quad L = (\partial_\nu\phi)^2 - m^2(cos(2\phi)-1), \tag{1.1}$$

where γ is the coupling constant and m is a mass scale, is one of the simplest massive Integrable Quantum Field Theories (IQFT's). Nevertheless, it possesses all the fetures peculiar to the most general IQFT: *e.g.* an infinite number of (local) conserved charges I_{2n+1}, $n \in \mathbb{Z}$ in involution, an infrared factorized scattering theory with solitons, antisolitons and a number of bound states called "breathers" etc.[1]. Despite this on-shell information, the off-shell Quantum Field Theory is much less understood. In particular, the computation of the correlation functions is still a very important open problem. Actually, some progress toward the direction of an approximative calculation in the Infra-Red (IR) and Ultra-Violet (UV) regions has been made recently. For instance, the exact Form-Factors (FF's) of the exponential fields $< 0|\exp[\alpha\phi(0)]|\beta_1, ..., \beta_n >$ were computed [2]. This allows one to make predictions about the long-distance behaviour of the corresponding correlation functions. On the other hand, some efforts have been made to estimate the short distance behaviour of the theory in the context of the so-called Conformal Perturbation Theory (CPT) [3]. Good estimates of interesting physical quantities are possible by combining the previous IR and UV techniques ([4] and reference therein). In addition, the exact expression for the Vacuum Expectation Values (VEV's) of the exponential fields (and some descendents) were proposed in [5]. What still remains unclear however is the explicit form of the correlation functions, in particular their intermediate distance behaviour and analytic properties. Some exact results exist only at the so-called free-fermion point $\gamma = \frac{\pi}{2}$ [6].

A radically different approach to the Sine-Gordon theory consists in searchig for additional infinite-dimentional symmetries and is inspired by the success the Virasoro symmetry had in the 2D CFT. In fact, it has been shown in [7] that the Sine-Gordon theory possesses an infinite dimensional symmetry provided by the (level 0) affine quantum algebra $\widehat{sl}(2)_q$. Actually, this symmetry is not useful for determining equations constraining correlation functions. However, there are some evidences that another kind of infinite dimensional symmetry should be present in the

*Although this contribution is deeply based on a series of papers in collaboration with M. Stanishkov, it presents new results and alternative derivations of results contained in this series of papers.

Sine-Gordon theory [8], a sort of limit of the so-called Deformed Virasoro Algebra (DVA) [9].

In this talk we present a construction of a new Virasoro symmetry in the Sine-Gordon theory. Although it will be clear how to implement it in the general field theory, we are mainly concerned here with the construction of this symmetry in the case of classical the N-soliton solutions, since – among the other resons – the symmetry in this case is much simpler realised (in particular it becomes *local* contrary to the field theory case) and the soliton phace space can be quantised in a simpler manner, as shown in two beautiful papers [10], [11].

2. The Virasoro Symmetry in Field Theory.

Let us recall the construction of the Virasoro symmetry in the context of (m)KdV theory [12, 13]. It was shown in [13], following the so-called algebraic approach, that it appears as a generalization of the ordinary dressing transformations of integrable models. Here we briefly recall the main results of this article. Being integrable, the mKdV system admits a zero-curvature representation:

$$[\partial_t - A_t, \partial_x - A_x] = 0, \tag{2.1}$$

where the Lax connections A_x , A_t belong to $A_1^{(1)}$ loop algebra and contain the field $\phi(x)$ and its derivatives: for instance

$$A_x = \begin{pmatrix} \phi' & \lambda \\ \lambda & -\phi' \end{pmatrix}. \tag{2.2}$$

The usual KdV variable $u(x)$ is connected to the mKdV field ϕ by the Miura transformation:

$$u = \frac{1}{2}(\phi')^2 + \frac{1}{2}\phi'' \tag{2.3}$$

(we denote by prime the derivative with respect to the *space variable* x of KdV). Of great importance in our construction is the solution $T(x, \lambda)$ of the so called associated linear problem:

$$(\partial_x - A_x(x, \lambda))T(x, \lambda) = 0 \tag{2.4}$$

which is usually referred to as transfer matrix. A formal (suitably normalized) solution of (2.4) can be easily found:

$$T_{reg} \quad (x, \lambda) = \tag{2.5}$$
$$e^{H\phi(x)}\ \mathcal{P}\exp\left(\lambda \int_0^x dy(e^{-2\phi(y)}E + e^{2\phi(y)}F)\right).$$

It is obvious that this solution defines $T(x, \lambda)$ as an infinite series in positive powers of λ with an infinite radius of convergence (we shall often refer to (2.6) as *regular expansion*).It is also clear from (2.6) that $T(x, \lambda)$ possesses an essential singularity at infinity where it is governed by the corresponding asymptotic expansion. It has been derived in detail in [14]

$$T(x, \lambda)_{asy} = KG(x, \lambda)e^{-\int_0^x dyD(y)}, \tag{2.6}$$

where K and G and D are written explicitely in [14, 15]. In particular the matrix

$$D(x, \lambda) = \sum_{i=-1}^{\infty} \lambda^{-i} d_i(x) H^i \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7}$$

contains the local conserved densities d_{2n+2} such that $I_{2n+1} = \int_0^L d_{2n+2}dx$.

Obviously, the zero-curvature form (2.1) is preserved by a gauge transformation for A_x

$$\delta A_x(x, \lambda) = [\theta(x, \lambda), \mathcal{L}] \tag{2.8}$$

and a similar one for A_t and we have solely to pay attention to the fact that the r.h.s. of the previous equation is independent of λ since the l.h.s is (2.2). Hence, a suitable choice for the gauge parameter θ_n goes through the construction of the following object

$$Z^X(x, \lambda) = T(x, \lambda)XT(x, \lambda)^{-1}, \tag{2.9}$$

where X is such that

$$[\partial_x, X] = 0. \tag{2.10}$$

Indeed, it is obvious from the previous definition that it satisfies the *resolvent condition*

$$[\mathcal{L}, Z^X(x, \lambda)] = 0, \tag{2.11}$$

for the first Lax operator $\mathcal{L} = \partial_x - A_x(x, \lambda)$. Now, this property means that

$$[\mathcal{L}, (Z^X(x, \lambda))_-] = -[\mathcal{L}, (Z^X(x, \lambda))_+], \tag{2.12}$$

where the subscript – (+) means that we restrict the series only to negative (non-negative) powers of λ, and is important for the construction of a consistent gauge parameter defined as

$$\theta^X(x,\lambda) = (Z^X(x,\lambda))_- \quad or$$
$$\theta^X(x,\lambda) = (Z^X(x,\lambda))_+. \qquad (2.13)$$

Moreover, we have to impose one more consistency condition impied by the explicit form of A_x (2.2), namely δA_x must be diagonal:

$$\delta^X A_x = H\delta^X\phi'. \qquad (2.14)$$

This implies restrictions on the indices of the transformations [15]. By using $T = T_{reg}$ we obtain the so–called dressing symmetries [14] and the indices are even for $X = H$ and odd for $X = E, F$. Instead, by using $T = T_{asy}$ we get for $X = H$ the commuting (m)KdV flows (or mKdV hierarchy).

At this point we want to make an important observation. Let us consider the KdV variable x as a *space direction* x_- of some more general system (and $\partial_- = \partial_x$ as a space derivative). let us introduce the *time* variable x_+ through the corresponding evolution flow

$$\partial_+ = (\delta^E_{-1} + \delta^F_{-1}). \qquad (2.15)$$

Then, it can be proved [14] that the equation of motion for ϕ becomes:

$$\partial_+\partial_-\phi = 2\sinh(2\phi) \qquad (2.16)$$

or if $\phi \to i\phi$

$$\partial_+\partial_-\phi = 2sin(2\phi) \qquad (2.17)$$

i.e. the Sine-Gordon equation! We consider this observation very important since it provides a *global* introduction of Sine-Gordon dynamics in the KdV system – a fact which was not known before. Finally, let us note that it can be shown that these two kind of symmetries (regular and asymptotic) commute with each other. In this sence the non-local regular transformations provide a true symmetry of the KdV hierarchy. In particullar the ∂_+ (2.15) evolution of SG equation (as well as the *space* derivative ∂_-) commutes with the entire hierarchy.

Now, let us explain how the Virasoro symmetry appears in the KdV system [13]. The main idea is that one may dress not only the generators of the underlying $A_1^{(1)}$ algebra but also an arbitrary differential operator in the spectral parameter. We take for example $\lambda^{m+1}\partial_\lambda$ which, as it is well known, are the vector fields of the diffeomorphisms of the unit circumference and close a Virasoro algebra. Then we proceed in the same way as above:

$$Z^V(x,\lambda) = T(x,\lambda)\partial_\lambda T(x,\lambda)^{-1}. \qquad (2.18)$$

When we consider the asymptotic case, *i.e.* we take $T = T_{asy}$ in (2.18), we obtain the non-negative Virasoro flows: *e.g.* the first ones can be written as

$$\begin{aligned}\delta^V_0\phi'(x) &= (x\partial+1)\phi'(x),\\ \delta^V_2\phi' &= 2xa'_3 - (\phi')^3 + \frac{3}{4}\phi''' + 2a'_1\int_0^x d_1,\\ \delta^V_4\phi' &= 2xa'_5 + (\phi')^5 - \frac{5}{2}\phi'''(\phi')^2 - \frac{27}{8}(\phi'')^2\phi' +\\ &+ \frac{5}{16}\phi^V + 2a'_3\int_0^x d_1 + 6a'_1\int_0^x d_3, \qquad (2.19)\end{aligned}$$

where the a_j are defined in [14]. We also constructed the negative Virasoro flows by taking $T = T_{reg}$ in (2.18) [13], but here we are not interested in these flows because only the non-negative part of the Virasoro flows commute with the light-cone SG dynamics ∂_+ and ∂_-.

Now we would like to extend the construction presented above in (m)KdV theory to the case of Sine-Gordon. We have already seen in (2.15) how to extend the (m)KdV dynamics to the SG ∂_+ flow and for obvious resons we will rename in the following the KdV variable u with

$$u^- = \tfrac{1}{2}(\partial_-\phi)^2 + \tfrac{1}{2}\partial_-^2\phi. \qquad (2.20)$$

Hence, after looking at the symmetric rôle that the derivatives ∂_- and ∂_+ play in the Sine-Gordon equation, we can obtain the negative (m)KdV flows of the variables $\partial_+\phi$ and

$$u^+ = \tfrac{1}{2}(\partial_+\phi)^2 + \tfrac{1}{2}\partial_+^2\phi \qquad (2.21)$$

in the same way as before but with the changes $x_- \to x_+$ and $\partial_- \to \partial_+$ [16]. Similarly, we get anhoter half Virasoro algebra by using the same construction as above but with x_- interchanged with x_+. Of course, it is not obvious at all that the two different halfs will recombine into a unique Virasoro algebra, but they do! [17] In conclusion, we have found an entire Virasoro algebra commuting with both light-cone SG dynamics flows $\partial_\pm$. Moreover, we prefer to examine in detail the action of this symmetry in the simple and useful case of soliton solutions.

3. The Virasoro Symmetry on soliton solution phase space.

We start with a brief description of the well known soliton solutions of SG equation (mKdV equation). They are best expressed in terms of the so-called *tau-function.* In the case of N-soliton solution of SG (mKdV) it has the form:

$$\tau(X_1, ..., X_N|B_1, ..., B_N) = \det(1+V) \quad (3.1)$$

where V is a matrix:

$$V_{ij} = 2\frac{B_i X_i(x)}{B_i + B_j} \quad , \quad i,j = 1, ..., N. \quad (3.2)$$

The SG (mKdV) field is then expressed as

$$e^{\phi} = \frac{\tau_-}{\tau_+}, \quad (3.3)$$

where

$$\tau_\pm(x) = \tau(\pm X(\ldots, t_{-3}, t_{-1}, t_1, t_3, \ldots)|B) \quad (3.4)$$

and $X_i(t_{2k+1})$ contains all the *times* (*e.g.* $t_{-1} = x_+, t_1 = x_-, t_3 = t$):

$$X_i(x) = x_i \exp(2 \sum_{k=-\infty}^{+\infty} B_i^{2k+1} t_{2k+1}). \quad (3.5)$$

The variables B_i and x_i are the parameters describing the solitons: $\beta_i = \log B_i$ are the so-called rapidities and x_i are related to the positions. By putting all the negative (positive) *times* to zero, we rediscover the usual mKdV hierarchy (the other with $x_- \to x_+$).

Our final goal will be the quantization of solitons and of the Virasoro symmetry. It was argued in [10] that this is best performed in another set of variables $\{A_i, B_i\}$. The implicit map from $\{X_i, B_i\}$ to these new variables is

$$X_j \prod_{k\neq j} \frac{B_j - B_k}{B_j + B_k} = \prod_{k=1}^{N} \frac{B_j - A_k}{B_j + A_k} \quad , \quad j = 1, ..., N. \quad (3.6)$$

The $\{A_i, B_i\}$ are the soliton limit of certain variables describing the more general quasi-periodic finite-zone solutions of SG (mKdV) [18]: in that context B_i are the branch points (i.e. define the complex structure) of the hyperelliptic Riemann surface describing the solution and A_i are the zeroes of the so-called Baker-Akhiezer function defined on it. In view of the nice geometrical meaning of these variables they were called analytical variables in [10]. In terms of these variables the tau functions have still a complicated form

$$\tau_+ = 2^N \prod_{j=1}^{N} B_j \{\frac{\prod_{i<j}(A_i + A_j)\prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)}\}$$

$$\tau_- = 2^N \prod_{j=1}^{N} A_j \{\frac{\prod_{i<j}(A_i + A_j)\prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)}\},$$

but, from (3.3), the SG (m-KdV) field enjoys a simple expression

$$e^{\phi} = \prod_{j=1}^{N} \frac{A_j}{B_j}. \quad (3.7)$$

One can verify that, as a consequence, the two components of the stress-energy tensor (the usual KdV variable) are expressed as

$$u^- = \sum_{j=1}^{N} A_j^2 - \sum_{j=1}^{N} B_j^2$$

$$u^+ = \sum_{j=1}^{N} A_j^{-2} - \sum_{j=1}^{N} B_j^{-2}. \quad (3.8)$$

In conclusion, we want to restrict the Virasoro symmetry of SG equation constructed above to the case of soliton solutions. This has been done in [16], but here we will follow a different path, which underlines the geometrical meaning of this symmetry. Indeed, our starting point is the transformation of the rapidities under the Virasoro symmetry. It can be easily deduced as a soliton limit of the Virasoro action on the Riemann surface describing the finite-zone solutions [19]:

$$\delta_{2n} B_i = B_i^{2n+1} \quad , \quad n \geq 0. \quad (3.9)$$

Similarly, for negative transformations we obtain [16]

$$\delta_{-2n} B_i = -B_i^{-2n+1} \quad , \quad n > 0 \tag{3.10}$$

where we have put an additional – sign in the r.h.s. for preserving the self-consistency of the construction. In other words, the Virasoro action changes the complex structure. What remains is to obtain the transformations of the A_i variables. In [16] we have deduced them from the transformations of the fields $\delta_{2n}\phi$, $\delta_{2n}\phi'$, $\delta_{2n}u$ *etc.* (2.19) restricted to the soliton solutions using (3.7,3.8). Instead, in this talk we will show that those are consequences of (3.9) and (3.10) if applied to the implicit map (3.6) by using the explicit expression of X_j in terms of B_i (3.5). The problem is simplified by the fact that the Virasoro algebra is freely generated, *i.e.* we need to compute only the δ_0, $\delta_{\pm 2}$ and $\delta_{\pm 4}$ transformations, since the higher vector fields are then obtained by commutation. As shown in [16], although the transformations of $\partial_\pm \phi$ and $u^\pm$ in the SG theory are quasi-local, since they contain certain indefinite integrals, the corresponding integrands become total derivatives when restricted to the soliton solutions. Therefore the Virasoro transformations become *local* when restricted to the soliton solutions! We feel more natural and more compact to express the Virasoro action on A_i by using the *equation of motions* of A_i derived from (3.5) and (3.6), for instance:

$$\begin{aligned}
\delta_{-1} A_i &= \partial_+ A_i = \prod_{j=1}^{N} \frac{A_i^2 - B_j^2)}{B_j^2} \prod_{j\neq i} \frac{A_j^2}{(A_i^2 - A_j^2)}, \\
\delta_1 A_i &= \partial_- A_i = \prod_{j=1}^{N} (A_i^2 - B_j^2) \prod_{j\neq i} \frac{1}{(A_i^2 - A_j^2)}, \\
\frac{1}{3}\delta_3 A_i &= (\sum_{j=1}^{N} B_j^2 - \sum_{k\neq i} A_k^2)\partial_- A_i, \\
\frac{1}{5}\delta_5 A_i &= (\sum_{j=1}^{N} B_j^4 - \sum_{k\neq i} A_k^4)\partial_- A_i - \\
&\quad - \sum_{j\neq i}(A_i^2 - A_j^2)\partial_- A_i \partial_- A_j .
\end{aligned} \tag{3.11}$$

In conclusion, the calculation is straightforward but quite tedious and we present here only the final result:

$$\begin{aligned}
\delta_{-2}\ A_i &= \frac{1}{3}x_+ \delta_{-3} A_i - A_i^{-1} - (\sum_{j=1}^{N} A_j^{-1})\partial_+ A_i - \\
&- \ x_- \partial_+ A_i \\
\delta_{-4}\ A_i &= \frac{1}{5}x_+ \delta_{-5} A_i - A_i^{-3} - \qquad (3.12) \\
&- \ \{\sum_{j\neq i}^{N} \frac{1}{A_i}(\frac{1}{A_i^2} - \frac{1}{A_j^2}) + \sum_{j=1}^{N} \frac{1}{A_j} \sum_{k=1}^{N} \frac{1}{B_k^2}\}\partial_+ A_i - \\
&- \ x_- \delta_{-3} A_i
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\delta_0 A_i &= (x_- \partial_- - x_+ \partial_+ + 1)A_i, \\
\delta_2 A_i &= \frac{1}{3}x_- \delta_3 A_i + A_i^3 - (\sum_{j=1}^{N} A_j)\partial_- A_i - x_+ \partial_- A_i, \\
\delta_4 A_i &= \frac{1}{5}x_- \delta_5 A_i + A_i^5 - \{\sum_{j\neq i} A_i(A_i^2 - A_j^2) + \\
&\quad + \sum_{j=1}^{N} A_j \sum_{k=1}^{N} B_k^2\}\partial_- A_i - x_+ \delta_3 A_i .
\end{aligned} \tag{3.14}$$

At this point, two important checks are are relevant. The firts concerns the commutation of the δ_{2m} (with $m \in \mathbb{Z}$) acting on A_i with the light-come SG flow $\partial_\pm$. This check admits a positive answer. The second is verifying the algebra of the δ_{2m} (with $m \in \mathbb{Z}$) acting on A_i and is non trivial because we have derived all the transformations (3.13) and (3.14) from the Virasoro algebra on B_i written in (3.9) and (3.10). The action on A_i is again a representation of the centerless Virasoro algebra:

$$[\delta_{2n}, \delta_{2m}]A_i = (2n-2m)\delta_{2n+2m} A_i \quad , \quad n, m \in \mathbb{Z}. \tag{3.15}$$

4. Comments about quantisation.

Of course, we are interested in the quantum Sine-Gordon theory. In the case of solitons there is a standard procedure, the canonical quantisation of the N-soliton solutions. In fact, let us introduce, following [10], the canonically conjugated variables to the *analytical variables* A_i:

$$P_j = \prod_{k=1}^{N} \frac{B_k - A_j}{B_k + A_j} \quad , \quad j = 1..., N. \tag{4.1}$$

In these variables one can perform the canonical quantisation of the N-soliton system introducing the deformed commutation relations between the operators A_i and P_i :

$$P_j A_j = q A_j P_j ,$$
$$P_k A_j = A_j P_k \quad for \quad k \neq j , \qquad (4.2)$$

where $q = \exp(i\xi)$, $\xi = \frac{\pi\gamma}{\pi-\gamma}$. It is very intriguing to understand how the Virasoro symmetry is deformed after the quantisation!

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