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Null Vectors in Logarithmic Conformal Field Theory

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ABSTRACT: The representation theory of the Virasoro algebra in the case of a logarithmic conformal field theory is considered. Here, indecomposable representations have to be taken into account, which has many interesting consequences. We study the generalization of null vectors towards the case of indecomposable representation modules and, in particular, how such logarithmic null vectors can be used to derive differential equations for correlation functions. We show that differential equations for correlation functions with logarithmic fields become inhomogeneous.

DURING THE LAST few years, so-called logarithmic conformal field theory (LCFT) established itself as a well-defined new animal in the zoo of conformal field theories in two dimensions [1]. By now, quite a number of applications have been pursued, and sometimes longstanding puzzles in the description of certain theoretical models could be resolved, e.g. the Haldane-Rezzayi state in the fractional quantum Hall effect [2], multifractality, etc. (see [3] for examples).

However, the computation of correlation functions within an LCFT still remains difficult, and only in a few cases, four-point functions (or even higher-point functions) could be obtained explicitly. The main reason for this obstruction is that the representation theory of the Virasoro algebra is much more complicated in the LCFT case due to the fact that there exist indecomposable but non-irreducible representations (Jordan cells). This fact has many wide ranging implications. First of all, it is responsible for the appearance of logarithmic singularities in correlation functions. Furthermore, it makes it necessary to generalize almost every notion of (rational) conformal field theory, e.g. characters, highest-weight modules, null vectors etc.

Null vectors are the perhaps most important tool in conformal field theory (CFT) to explicitly calculate correlation functions. In certain CFTs, namely the so-called minimal models, a subset of highest-weight modules possess infinitely many null vectors which, in principle, allow to compute arbitrary correlation functions involving fields only out of this subset. It is well known that global conformal covariance can only fix the two- and three-point functions up to constants. The existence of null vectors makes it possible to find differential equations for higher-point correlators, incorporating local conformal covariance as well. This paper will pursue the question, how this can be translated to the logarithmic case.

For the sake of simplicity, we will concentrate on the case where the indecomposable representations are spanned by rank two Jordan cells with respect to the Virasoro algebra. The abbreviation LCFT will refer to this case. To each such highest-weight Jordan cell $\{|h;1\rangle, |h;0\rangle\}$ belong two fields, and ordinary primary field $\Phi_h(z)$, and its logarithmic partner $\Psi_h(z)$. In particular, one then has $L_0|h;1\rangle = h|h;1\rangle + |h;0\rangle$, $L_0|h;0\rangle = h|h;0\rangle$. Furthermore, the main scope will lie on the evaluation of four-point functions.

1. $SL(2,\mathbb{C})$ Covariance

In ordinary CFT, the four-point function is fixed

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by global conformal covariance up to an arbitrary function $F(x,\bar{x})$ of the harmonic ratio of the four points, $x = \frac{z_{12}z_{34}}{z_{14}z_{32}}$ with $z_{ij} = z_i - z_j$. As usual, we consider only the chiral half of the theory, although LCFTs are known not to factorize entirely in chiral and anti-chiral halfs.

In LCFT, already the two-point functions behave differently, and the most surprising fact is that the propagator of two primary fields vanishes, $\langle\Phi_h(z)\Phi_{h'}(w)\rangle = 0$. In particular, the norm of the vacuum, i.e. the expectation value of the identity, is zero. On the other hand, it can be shown [5] that all LCFTs possess a logarithmic field $\Psi_0(z)$ of conformal weight $h = 0$, such that with $|\tilde{0}\rangle = \Psi_0(0)|0\rangle$ the scalar product $\langle 0|\tilde{0}\rangle = 1$. More generally, we have

$$\begin{aligned}\langle\Phi_h(z)\Psi_{h'}(w)\rangle &= \delta_{hh'}\frac{A}{(z-w)^{h+h'}}\,, \qquad (1.1)\\ \langle\Psi_h(z)\Psi_{h'}(w)\rangle &= \delta_{hh'}\frac{B-2A\log(z-w)}{(z-w)^{h+h'}}\,,\end{aligned}$$

with A, B free constants. In an analogous way, the three-point functions can be obtained up to constants from the Ward-identities generated by the action of $L_{\pm1}$ and L_0. Note that the action of the Virasoro modes is non-diagonal in the case of an LCFT,

$$\begin{aligned}&L_n\langle\phi_1(z_1)\ldots\phi_n(z_n)\rangle = \qquad (1.2)\\ &\sum_i z^n\left[z\partial_i + (n+1)(h_i+\delta_{h_i})\right]\langle\phi_1(z_1)\ldots\phi_n(z_n)\rangle\end{aligned}$$

where $\phi_i(z_i)$ is either $\Phi_{h_i}(z_i)$ or $\Psi_{h_i}(z_i)$ and the off-diagonal action is $\delta_{h_i}\Psi_{h_j}(z) = \delta_{ij}\Phi_{h_j}(z)$ and $\delta_{h_i}\Phi_{h_j}(z) = 0$. Therefore, the action of the Virasoro modes yields additional terms with the number of logarithmic fields reduced by one. This action reflects the transformation behavior of a logarithmic field under conformal transformations,

$$\phi_h(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h\left(1+\log(\partial_z f(z))\delta_h\right)\phi_h(f(z))\,. \qquad (1.3)$$

An immediate consequence of the form of the two-point functions and the cluster property of a well-defined quantum field theory is that $\langle\Phi_{h_1}(z_1)\ldots\Phi_{h_n}(z_n)\rangle = 0$, if all fields are primaries. Actually, this is only true if a correlator is considered, where all fields belong to Jordan cells. LCFTs do contain other primary fields, which themselves are not part of Jordan cells, and whose correlators are non-trivial. These are the twist-fields, which sometimes are also called pre-logarithmic fields (see ref. 4 in [1]). Twist fields introduce non-trivial boundary conditions, since they behave exactly like branch cuts. Fusion of a twist with the corresponding anti-twist annihilates the branch cut but may leave a puncture, where for example screening integral contours may get pinched (for details see [6]). As a consequence, operator product expansions of two conjugate twist fields will produce contributions from Jordan cells of primary fields and their logarithmic partners. However, since the twist fields behave as ordinary primaries with respect to the Virasoro algebra, the computation of correlation functions of twist fields only can be performed as in the common CFT case. The solutions, however, may exhibit logarithmic divergences as well. In this paper, we will compute correlators with logarithmic fields, instead.

Another consequence is that

$$\begin{aligned}&\langle\Psi_{h_1}(z_1)\Phi_{h_2}(z_2)\ldots\Phi_{h_n}(z_n)\rangle \qquad (1.4)\\ &= \langle\Phi_{h_1}(z_1)\Psi_{h_2}(z_2)\Phi_{h_3}(z_3)\ldots\Phi_{h_n}(z_n)\rangle\\ &= \ldots = \langle\Phi_{h_1}(z_1)\ldots\Phi_{h_{n-1}}(z_{n-1})\Psi_{h_n}(z_n)\rangle\,.\end{aligned}$$

Thus, if only one logarithmic field is present, it does not matter, where it is inserted. Note that the action of the Virasoro algebra does not produce additional terms, since correlators without logarithmic fields vanish. Therefore, a correlator with precisely one logarithmic field can be evaluated as if the theory would be an ordinary CFT.

It is an easy task to find the general form for four-point functions. The final expressions are the more complicated the more logarithmic fields are present. One obtains

$$\begin{aligned}\langle\Psi_1\Phi_2\Phi_3\Phi_4\rangle &= \prod_{i<j} z_{ij}^{\mu_{ij}}F^{(0)}(x)\,, \qquad (1.5)\\ \langle\Psi_1\Psi_2\Phi_3\Phi_4\rangle &= \prod_{i<j} z_{ij}^{\mu_{ij}}\left[F^{(1)}_{12}(x) - 2F^{(0)}(x)\log(z_{12})\right]\,,\\ \langle\Psi_1\Psi_2\Psi_3\Phi_4\rangle &= \prod_{i<j} z_{ij}^{\mu_{ij}}\Big[F^{(2)}_{123}(x)\\ &- \sum_{1\le k<l\le 3}\tilde{F}^{(1)}_{kl}(x)\log(z_{kl}) + 2F^{(0)}(\log(z_{12})\log(z_{13})\\ &\quad + \log(z_{12})\log(z_{23}) + \log(z_{13})\log(z_{23}))\\ &- F^{(0)}(\log^2(z_{12}) + \log^2(z_{13}) + \log^2(z_{23}))\Big]\,,\end{aligned}$$

where we omit the very lengthy expression for $\langle\Psi_1\Psi_2\Psi_3\Psi_4\rangle$. Other choices for the logarithmic fields are simply obtained by renaming the indices. The correct combinations are $\tilde{F}^{(1)}_{ij}(x) = F^{(1)}_{ik}(x) + F^{(1)}_{jk}(x) - F^{(1)}_{ij}(x)$ with k the remaining index of the third logarithmic field. Therefore, the full solution for the four-point function of an LCFT involves twelve different functions $F^{(r)}_{i_1\ldots i_{r+1}}(x)$, $0 \leq r \leq 3$. In a similar way, one can make an $SL(2,\mathbb{C})$ covariant ansatz for a generic n-point function of Jordan cell fields. These results generalize the expressions obtained in [4] for the $h = 0$ Jordan cell of the identity field.

2. Null vectors in LCFT

In an earlier work [5], all null vectors up to level five were explicitly computed, which are built on rank two Jordan cell representations. A main feature of these null vectors is that they consist of two different descendants, i.e.

$$|\chi^{(n)}_{h,c}\rangle = \sum_{|\{m\}|=n} L_{-\{m\}}\left(\beta^{\{m\}}|h;1\rangle + \beta'^{\{m\}}|h;0\rangle\right) \tag{2.1}$$

in an obvious multi-index notation. Within a correlator, such a null vector will automatically translate into an inhomogeneous differential equation. The homogeneous part is the same as for an ordinary level n null field descendant of Ψ_h, while the inhomogeneity is given as solution of another differential equation, corresponding to a non-trivial descendant of Φ_h. On the other hand, if we consider the differential equation for a null field on the primary Φ_h, we still end up with an inhomogeneous differential equation due to the other logarithmic fields (there must be at least one!) in the correlator.

Thus, the coefficients $\beta^{\{m\}}$ are determined as functions in h, c by the linear system of equations

$$L_{\{p\}}\sum_{|\{m\}|=n}\beta^{\{m\}}L_{-\{m\}}|h;1\rangle = 0 \quad \forall \quad |\{p\}| = n \tag{2.2}$$

in the usual way. Using the commutation relations of the Virasoro algebra, these equations are reduced to equations involving solely L_0 and the central charge, i.e.

$$\sum_{|\{m\}|=n}\beta^{\{m\}}f_{\{p\},\{m\}}(L_0, C)|h;1\rangle = 0\,. \tag{2.3}$$

Now, due to the off-diagonal part of the action (1.2) of the Virasoro algebra, one gets additional contributions proportional to $|h;0\rangle$ which have to be canceled by the new coefficients $\beta'^{\{m\}}$. With $L_0|h;1\rangle = (h+\delta_h)|h;1\rangle$, one can show that these equations take the form

$$\sum_{|\{m\}|=n}\beta^{\{m\}}f_{\{p\},\{m\}}(h,c)|h;1\rangle = 0\,, \tag{2.4}$$

$$\sum_{|\{m\}|=n}(\beta'^{\{m\}} + \beta^{\{m\}}\partial_h)f_{\{p\},\{m\}}(h,c)|h;0\rangle = 0\,.$$

A solution to these equations is given by putting $\beta'^{\{m\}}(h, c = c'(h)) = \partial_h\beta^{\{m\}}(h, c = c(h))$ where the condition $c(h) = c'(h)$ fixes the possible values of the central charge to a discrete set. Often, the simpler null state conditions $L_p|\chi^{(n)}_{h,c}\rangle = 0$ for $p = 1,\ldots n$ are used. Although they are equivalent to the above conditions in ordinary CFT, they only provide sufficient but not necessary conditions in the LCFT case, as can already seen at level three.

For example, the conditions for logarithmic null states at level two are firstly the well-known ones for an ordinary level two null state, $\beta^{\{2\}} = -\frac{2}{3}(2h+1)\beta^{\{1,1\}}$, $\beta^{\{1,1\}} = const$, and $c = 2h(5-8h)/(2h+1)$. In addition, the off-diagonal contributions yield $\beta'^{\{2\}} = -\frac{4}{3}h\beta^{\{1,1\}}$, $\beta'^{\{1,1\}} = 0$, and $c = 5-16h$. The two different conditions for the central charge have two common solutions, namely $(h = -\frac{5}{4}, c = 25)$ and $(h = -\frac{1}{4}, c = 1)$.

3. Correlation Functions

With the generalization of null vectors to the logarithmic case at hand, the next question is how to effectively compute correlation functions involving fields from non-trivial Jordan cells. As an example, we consider a four-point function with such a primary field which is degenerate at level two. To simplify the formulæ, we fix the remaining three points in the standard way, i.e. we consider $G_4 = \langle\phi_1(\infty)\phi_2(1)\Phi_{h_3}(z)\phi_4(0)\rangle$. According to (1.2), the level two descendant yields

$$\left[\frac{3\,\partial_z^2}{2(2h_3+1)} + \sum_{w\neq z}\left(\frac{\partial_w}{w-z} - \frac{h_w + \delta_{h_w}}{(w-z)^2}\right)\right]G_4 = 0\,. \tag{3.1}$$

If there is only one logarithmic field, δ_h will produce a four-point function without logarithmic fields, i.e. won't yield an additional term. Hence, after rewriting this equation as an ordinary differential equation solely in z, we can express the conformal blocks in terms of hypergeometric functions Putting without loss of generality the logarithmic field at infinity, we can rewrite

$$\begin{aligned} G_4 &= z^{p+\mu_{34}}(1-z)^{q+\mu_{23}}F^{(0)}(z)\,, \\ p &= \tfrac{1}{6}-\tfrac{2}{3}h_3-\mu_{34}-\tfrac{1}{6}\sqrt{r_4}\,, \\ q &= \tfrac{1}{6}-\tfrac{2}{3}h_3-\mu_{23}-\tfrac{1}{6}\sqrt{r_2}\,, \\ r_i &= 1-8h_3+16h_3^2+48h_ih_3+24h_i\,, \end{aligned} \tag{3.2}$$

with $F^{(0)}$ being a solution of the hypergeometric system ${}_2F_1(a,b;c;z)$ given by

$$\begin{aligned} a &= \tfrac{1}{2}-\tfrac{1}{6}\sqrt{r_2}-\tfrac{1}{6}\sqrt{r_4}-\tfrac{1}{6}\sqrt{r_1}\,, \\ b &= \tfrac{1}{2}-\tfrac{1}{6}\sqrt{r_2}-\tfrac{1}{6}\sqrt{r_4}+\tfrac{1}{6}\sqrt{r_1}\,, \\ c &= 1-\tfrac{1}{3}\sqrt{r_4}\,. \end{aligned} \tag{3.3}$$

The next complicated case is the presence of two logarithmic fields. The ansatz now reads

$$G_4 = z^{p+\mu_{34}}(1-z)^{q+\mu_{23}}\left(F^{(1)}_{ij}(z)-2\log(w_{ij})F^{(0)}(z)\right)\,. \tag{3.4}$$

Surprisingly, if the two logarithmic fields are put at $w_2=1$ and $w_4=0$, the additional term in the new ansatz vanishes. However, the δ_h operators in (3.1) create two terms such that the standard hypergeometric equation becomes inhomogeneous,

$$\begin{aligned} &\left[z(1-z)\partial_z^2+(c-(1+a+b)z)\partial_z-ab\right]F^{(1)}_{24}(z) \\ &\qquad = \frac{\frac{2}{3}(2h_3+1)}{z(1-z)}F^{(0)}(z)\,. \end{aligned} \tag{3.5}$$

The solution of this inhomogeneous equation cannot be given in closed form, it involves integrals of products of hypergeometric functions. But for special choices of the conformal weights, simple solutions can be obtained. The best known LCFT certainly is the CFT with central charge $c=c_{2,1}=-2$. The field of conformal weight $h=h_{2,1}=1$ in the Kac table possesses a logarithmic partner. Choosing all weights in the four-point function to be equal to h, we find with ${}_2F_1(-4,-1;-2;z)=A(2z-1)+Bz^3(z-2)\equiv Af_1+Bf_2$ the solutions

$$\begin{aligned} F^{(0)}(z) &= [z(1-z)]^{-4/3}(Af_1+Bf_2)\,, \\ F^{(1)}_{24}(z) &= [z(1-z)]^{-4/3}\,[Cf_1+Df_2 \\ &\quad +\left(\tfrac{2}{3}(B-2A)f_2-\tfrac{2}{3}Af_1\right)\log(z) \\ &\quad -\left(\tfrac{2}{3}(B-2A)f_2-\tfrac{2}{3}Af_1\right)\log(1-z) \\ &\quad +\tfrac{1}{9}(6z^2-6z-7)Af_1+(-\tfrac{2}{3}z^3+\tfrac{5}{9}f_1)B]\,. \end{aligned} \tag{3.6}$$

Note that $F^{(0)}$ does not depend on which field is the logarithmic one (hence the omitted lower index), since only the contraction of *two* logarithmic fields causes logarithmic divergences. A nice example for this is the twist field $\mu(z)$ in the $c=-2$ LCFT, which has $h=-1/8$. Although its OPE with itself yields a logarithmic term, $\mu(z)\mu(w)\sim\tilde{\mathbb{I}}(w)+\log(z-w)\mathbb{I}$, no logarithm shows up in its two-point function. At least four twist fields are necessary to get a logarithm in a correlation function, which is equivalent to two logarithmic fields, since $\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)\sim -2\log(z-w)\tilde{\mathbb{I}}(w)-\log^2(z-w)\mathbb{I}(z)$.

So far, we have considered correlation functions with logarithmic fields, but where the null field condition was exploited for a primary field. As mentioned at the beginning of this section, a null vector descendant on the full Jordan cell (not on its irreducible subrepresentation) is more complicated. For example, the logarithmic partner of the $h=1$ field in the $c=-2$ LCFT turns out to be the $h=h_{1,5}$ field in the Kac table. In deed, as shown in [5], there exists a null vector of the form

$$\begin{aligned} &|\chi^{(5)}_{h=1,c=-2}\rangle \\ &= [\tfrac{16}{3}L_{-1}L_{-2}^2+\tfrac{52}{3}L_{-2}L_{-3}-12L_{-1}L_{-4} \\ &\quad +\tfrac{148}{3}L_{-5}]|h;0\rangle \\ &+[L_{-1}^5-10L_{-1}^3L_{-2}+36L_{-1}^2L_{-3}-L_{-1}L_{-4} \\ &\quad +16L_{-1}L_{-2}^2-40L_{-2}L_{-3}+160L_{-5}]|h;1\rangle \end{aligned} \tag{3.7}$$

The first descendant is precisely the same as for a primary field degenerate of level five. However, a remarkable fact in LCFT is that the null descendant factorizes,

$$\begin{aligned} &|\chi^{(5)}_{h=1,c=-2}\rangle = (\ldots)|h;0\rangle+ \\ &(L_{-1}^3-8L_{-1}L_{-2}+20L_{-3})(L_{-1}^2-2L_{-2})|h;1\rangle \end{aligned} \tag{3.8}$$

namely into the level two null descendant times a level three descendant which turns out to be the null descendant of a primary field of conformal weight $h_{3,1}=3$. Hence, the level two descendant of the logarithmic field is a null descendant only up to a primary field of weight $h_{3,1}=h_{1,5}+2$.

Presumably, this is a general LCFT feature [8]: Namely, the typical LCFT case is that the logarithmic partner constituting a Jordan cell representation is degenerate of level $n+k$ with n the level where the primary has its null state, and $k>0$. On the other hand, the conformal properties of the logarithmic field are identical to the ones of its primary partner up to the non-diagonal contributions. Hence the two fields cannot be distinguished if these additional contributions were ignored. It follows that in a correlator without any further logarithmic fields (where the off-diagonal part of the null field does not contribute), the logarithmic field must behave exactly as its primary partner, i.e. must possess the same null field. The only way this can happen consistently is that the diagonal part of the null vector factorizes.

Another important point is that the additional descendant on the primary partner is not unique. If again the logarithmic partner constituting a Jordan cell representation is degenerate of level $n+k$, then the descendant of the primary field is determined only up to an arbitrary contribution $\sum_{|\{m\}|=k}\alpha^{\{m\}}L_{-\{m\}}|\chi^{(n)}_{h,c}\rangle$.

That the $(1,5)$ entry of the Kac table does indeed refer to the logarithmic partner of the $h=1$ primary $(2,1)$ field can be seen from the solutions of the homogeneous differential equation resulting from (3.7) when there are no off-diagonal contributions. Of course, the resulting ordinary differential equation of degree five has, among others, the same solutions as the hypergeometric equation above for the $(2,1)$ field. These are the correct solutions, if there is no other logarithmic field. The other three solutions turn out to have logarithmic divergences. Therefore, they cannot be valid solutions for this case, but must constitute solutions for a correlator with two logarithmic fields. However, in this case one has to take into account that the full null state has an additional contribution from the primary partner of the $(1,5)$ field. The full inhomogeneous equation reads (with the "simplest" choice for the primary part of the descendant)

$$\begin{aligned}0 =\ & \big[z^3(1-z)^3\partial^5 + 8z(z-1)(z^2-z+1)\partial^3 \\ & - 4(2z-1)(5z^2-5z+2)\partial^2 + 24(2z-1)^2\partial \\ & - 48(2z-1)\big]\, F^{(1)}_{34}(z) \\ & + \big[-\tfrac{16}{3}z(z-1)(2z-1)^2\partial^3 \\ & + \tfrac{44}{3}(2z-1)(5z^2-5z+2)\partial^2 \\ & - \tfrac{8}{z(z-1)}(57z^4-114z^3+90z^2-333z+5)\partial \\ & + \tfrac{16}{z(z-1)}\,(2z-1)(18z^2-18z+5)\big]\, F^{(0)}(z)\end{aligned} \tag{3.9}$$

in the case of one further logarithmic field put at zero. Similar equations can be written down for all three choices $F^{(1)}_{3j}(z)$ as well as for higher numbers of logarithmic fields. In general, there is one part of the differential equation for $F^{(r)}_I$ with $I=\{3,i_1,\ldots,i_r\}$, and the inhomogeneity is given by $F^{(r-1)}_{I-\{3\}}$. It is clear from this that the full set of solutions can be obtained in a hierarchical scheme, where one fist solves the homogeneous equations and increases the number of logarithmic fields one by one.

In the example above, $F^{(0)}$ is given as in (3.6). Then the inhomogeneity reads $80(3z^2-3z+1)A+16z(z^2-9z+3)B$. With this, the solution is finally obtained to be

$$\begin{aligned}F^{(1)}_{34} =\ & C_1f_1 + C_2f_2 + C_3[3f_1\log(\tfrac{z}{z-1})-6] \\ & + C_4[3f_2\log(z-1)-12z^3] \\ & + C_5[3(f_1+f_2)\log(z)+12z(z^2-3z+1)] \\ & + \big[\tfrac{2}{9}(3f_1-2f_2)\log(z)+\tfrac{2}{9}(7f_1+2f_2)\log(z-1) \\ & \quad + \tfrac{1}{27}\,(12z^3-18z^2+32z-1)\big]\,A \\ & + \big[\tfrac{2}{9}(f_2-f_1)\log(z)-\tfrac{2}{9}(4f_1+f_2)\log(z-1) \\ & \quad + \tfrac{1}{27}\,(36z^2-6z^3-17f_1)\big]\,B\,.\end{aligned} \tag{3.10}$$

As is obvious from the above expression, correlation functions involving more than one logarithmic field become quite complicated. Although the two logarithmic fields were chosen to be located at $z,0$, the above solution also contains terms in $\log(z-1)$. This is a consequence of the associativity of the OPE and duality of the four-point function.

In principle, the full set of four-point functions can be evaluated in this way. Care must be taken with the solutions of the homogeneous equation. As indicated above, not all of them might be valid solutions. If the correlator does contain only one logarithmic field, then there cannot be any logarithmic divergences in the solution. However, it is instructive to find the reason, why already the homogeneous equation admits logarithmic solutions. Firstly, one should remember that a similar situation arises in minimal models. All primary fields come in pairs in the Kac table, which are usually identified with each other, $(r,s) \equiv (q-r,p-s)$ if the central charge is $c = c_{p,q}$. So, in principle, one and the same correlator can be evaluated by exploiting two different null state conditions, which in general will be of different degrees, $rs \neq rs + qp - (qs + pr)$. Therefore, the physical solutions are given by the intersection of the two sets of solutions.

In the logarithmic case, the typical BPZ argument that only the common set of fusion rules can be non-vanishing [7], has to be modified. The $(2,1)$ field has the formal BPZ fusion rules $[(2,1)] \times [(2,1)] = [(1,1)] + [(3,1)]$, but the last term must vanish due to dimensional reasons, since $h_{3,1} = 3 > 2h_{2,1} = 2\cdot 1$. On the other hand, one has in a formal way $[(2,1)]\times[(1,5)] = [(1,1)]$, meaning that the OPE of the logarithmic field with its own primary partner won't yield a logarithmic dependency. Note that a logarithmic field can be considered as the normal ordered product of its primary partner with the logarithmic partner of the identity, i.e. $\Psi_h(z) = {:}\Phi_h\tilde{\mathbb{I}}{:}(z)$. As long as an OPE of such a field with a primary field is considered, one can evaluate it in the usual way, and then take the normal ordered product of the right hand side with $\tilde{\mathbb{I}}$, since the latter field behaves almost as the identity field with respect to fusion with primary fields. But as soon as the OPE of two logarithmic fields is taken, one gets a new term: $[(1,5)] \times [(1,5)] = [(1,1)]+[(1,3)]+[(1,5)]$, where all terms are omitted which must vanish due to dimensional reasons. Now, the (1,3) field $\tilde{\mathbb{I}}$ itself appears in the OPE, which is correct because the OPE of two such normal ordered products will involve the well-known OPE $\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w) \sim -2\log(z-w)\tilde{\mathbb{I}}(w)$. This proves that the logarithmic solutions of the conformal blocks of the four-point function can only be valid when sufficiently many logarithmic fields are involved.

This leads back to the above mentioned observation that the null state of a logarithmic field of level $n+k$ factorizes into the level n null descendant of its primary partner times the level k null state of a primary field of conformal weight $h+n$. Indeed, it is a nice exercise to show that in our $c=-2$ example the Virasoro modes of the level two null descendant, acting on the logarithmic $\Psi_{h=1}$ field, produce a field which transforms as a primary field of conformal weight $h = 3$. The reason is that the derivative, acting on a logarithmic field, eats up the fermionic zero modes. Indeed, in

$$\begin{aligned}[L_{-n},\Psi_h(z)] &= z^n((n+1)h + z\partial)\Psi_h(z) \qquad (3.11)\\ &= (n+1)h\Psi_h(z) + {:}(\partial\Phi_h)\tilde{\mathbb{I}}{:}(z) + {:}\Phi_h(\partial\tilde{\mathbb{I}}){:}(z)\,.\end{aligned}$$

where the δ_h part is omitted, the derivative first acts as derivative on the primary part of the logarithmic field, and then acts on the field $\tilde{\mathbb{I}}$. In the $c = -2$ LCFT this basic logarithmic field can be constructed out of two anticommuting scalar fields,

$$\theta^\alpha(z) = \sum_{n\neq 0}\theta^\alpha_n z^{-n} + \theta^\alpha_0\log(z) + \xi^\alpha\,, \qquad (3.12)$$

$\alpha = \pm$, whose zero modes are responsible for all the logarithms. Then $\tilde{\mathbb{I}}(z) = -\frac{1}{2}\epsilon_{\alpha\beta}{:}\theta^\alpha\theta^\beta{:}(z)$. Therefore, the derivative will eat up zero modes, e.g. $\tilde{\mathbb{I}}(0)|0\rangle = \xi^+\xi^-|0\rangle$ and $\partial\tilde{\mathbb{I}}(0)|0\rangle = (\theta^+_{-1}\xi^- + \theta^-_{-1}\xi^+)|0\rangle$. By considering states, one can show that the level two null descendant applied to the state $\Psi_{h=1}(0)|0\rangle$ yields a state proportional to a highest-weight state of weight $h = 3$.

4. Conclusion

Taking into account the proper action of the Virasoro algebra on logarithmic fields, i.e. working with Jordan cell representations as generalizations of irreducible highest-weight representations, allows to evaluate correlations functions in LCFT in a similar fashion as in ordinary CFT. The main difference is that each n-point function represents a full hierarchy of conformal blocks involving $r+1 = 1,\ldots,n$ logarithmic fields. The solution of this hierarchy can be obtained step by step, where the case with one logarithmic field only is worked out in the same way as in ordinary CFT. In each further step, the differential equations are inhomogeneous, with the inhomogenity determined by the conformal blocks of correlators with fewer logarithmic fields. A more detailed exposition including twist fields will appear elsewhere [8]. This fills one of the few remaining gaps to put LCFT on equal footing with better known ordinary CFTs such as minimal models.

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