Sigma models on Supermanifolds and Theory of Disorder.

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ABSTRACT: It is discussed how averaging over quenched disorder leads to field theories on supersymmetric manifolds.

Supersymmetric (SUSY) field theories in general and SUSY sigma models in particular appear in condensed matter physics in the context of theory of quenched disorder.

In general any many-body system is characterized by its observables which correlation functions. Whenever randomness is present in the Hamiltonian (for example, in a form of a random potential $V(x)$), the correlation functions $G(x_1, ...x_N; [V])$ depend on the randomness. In many cases when these functions are not self-averaging, it is not sufficient to only consider the Green functions averaged over disorder, but one needs to study the complete distribution functions of $G(x_1, ...x_N; [V])$. This occurs when the Green's function averaged over disorder is not equal to the typical Green's function:

$$\overline{G(x_1, ...x_N; [V])} \neq \exp[\overline{\ln G(x_1, ...x_N; [V])}] \quad (1)$$

Distribution functions are usually characterized by their moments, that is in theory of disorder one is concerned with studying averages of various powers of G.

In order to understand the origin of SUSY let us consider a simplest system when particles are non-interacting. The technical difficulty one encounters in dealing with quenched disorder, is that one has to average Green functions which, being derivatives of logarithm of the generating functional, contain randomness both in the numerator and the denominator. Indeed, let us take a look at the path integral representation of the N-point Green's function:

$$G(1, ..., 2N) = \frac{\int D\bar{\psi}D\psi\psi(1)...\bar{\psi}(2N)e^{-i\bar{\psi}\hat{H}\psi}}{\int D\bar{\psi}D\psi \exp[-i\bar{\psi}\hat{H}\psi]}$$

(here $\bar{\psi}, \psi$ are Grassmann fields). It is awkward to average this expression over V. To deal with this problem one can employ a trick of rewriting the denominator as a path integral over c-number fields $\bar{\beta}, \beta$. Then we obtain

$$G(1, ..., 2N) = \int D\bar{\psi}\ D\psi\psi(1)...\bar{\psi}(2N)\, e^{-i\bar{\psi}\hat{H}\psi} \times \int D\bar{\beta}D\beta \exp[-i\bar{\beta}\hat{H}\beta]$$

Now the disorder is only in the numerator (the denominator is equal to one!) and one can avarage over it for a price of having a path integral over c-number and Grassmann fields (this is what is called SUSY in condensed matter!). Details of this approach one can find in the book by Efetov [1]. An excellent review on the SUSY approach belongs to Mirlin [2].

Thus a simultaneous presence of c-number and Grassmann fields is a characteristic feature of field theories describing disordered systems. Needless to say these theories are non-unitary (the partition function is equal to one!).

Another characteristic feature of field theory of disordered systems is that the same system is described not by one, but by an entire set of field theories. Everything depends of what quantity one wants to calculate. Indeed, as I have said, one is interested in distribution functions of G's.

To calculate moments of such distribution functions one has to calculate averages of powers $\overline{G^p}$. To do this it is necessary to introduce several copies of the fields. Hence different moments of the same distribution function are described by SUSY theories with different number of fields!

An interesting family of theories of disorder consists of critical two-dimensional models. Among those there are models which are described by field theories with local Lagrangians. Here one can apply powerful machinery of conformal field theory (CFT) to get a complete and essentially non-perturbative description.

An alternative of the SUSY approach is the replica approach. Here before performing disorder average one introduces r copies of the Hamiltonian and uses the identity

$$\ln Z = \lim_{r\to 0}\frac{Z^r - 1}{r} \tag{2}$$

Such limiting procedure requires a careful definition and its status remains controvercial. The replica approach faithfully reproduces perturbation theory results, but outside of the perturbation theory limits it has often been questioned (see, for example, [1], [3]). There have been attempts to repair the deficiencies of replica trick by adopting various symmetry breaking schemes. Recently such a scheme has been successfully used to reproduce the level-level correlation functions in the random matrix theory [4]. The question is whether one can formulate general principles of replica symmetry breaking. To get an insight to develop such principles one needs to have as many non-perturbative results as possible.

Whether in SUSY or in replica representation, critical models of disorder share certain general features.

• All these models are non-unitary CFTs with central charge $C = 0$. This is related to the fact that the partition functions of all these models is equal to one by construction.

• All these theories are logarithmic, that is the corresponding Hamiltonian cannot be completely diagonalized; the best one can do is to reduce it to the Jourdan cell form. The general properties of logarithmic CFTs were described by Gurarie [5], Caux *et al.*[7] and Gurarie and Ludwig [8]. As has been recently demonstrated by Gurarie [6] and Cardy [9], the presence of logarithms looks completely natural and unavoidable both in SUSY and in the replica representation.

The most famous example of the critical model of disorder is the model describing the plateau transition in Integer Quantum Hall (IQH) effect. The corresponding field theoretical formulation is given either by the Pruisken's model [10] or by the superspin model derived from the Chalker-Coddington network model [11],[12],[13]. The SUSY formulation of Pruisken's model is a sigma model on the manifold $\mathrm{SU}(1,1|2)/\mathrm{U}(1|1)\times \mathrm{U}(1|1)$. The action may be written in the following form:

$$S = \int d^2x\,\mathrm{Str}\left[-\frac{1}{8\alpha}\left(\partial_\mu Q\right)^2 + \frac{1}{8}\sigma^0_{xy}\left(Q\left[\partial_x Q, \partial_y Q\right]\right) + \frac{1}{2}\pi\rho_0\eta\Sigma^3 Q\right],$$

where Q is a 4×4 supermatrix satisfying the conditions,

$$\mathrm{Str}Q = 0, \quad Q^2 = 1, \tag{3}$$

$\Sigma^3 = \mathrm{diag}(1,-1,1,-1)$ (in the boson-fermion supermatrix representation), $\sigma^0_{xy}(E)$ is the bare Hall conductance at energy E, ρ_0 is the average density of states at energy E, and η is the imaginary frequency which serves as a symmetry breaking field. The coupling constant α is related to the disorder strength.

The second term appearing in (3) is topological, and despite the fact that its presence is crucial for the critical behaviour, its effect cannot be spotted in a perturbative expansion in powers of α; it does not contribute to the equations of motion and hence does not contribute to the loop expansion of the beta function. The effects of the topological term become visible only for samples of size greater than $\xi \sim l\exp[\pi{\sigma^0_{xx}}^2]$, where l is the electron mean free path. In the model (3), with $\sigma^0_{xy} = 0$, the length scale ξ corresponds to the localization length; with $\sigma^0_{xy} = 1/2$, this scale is the *transmutation length* (in field-theoretic jargon) and signifies a crossover to the regime of *universal* critical fluctuations.

Thus the effective field theory for the plateau transition is not known at present, but there is a plausible hypothesis made by Zirnbauer that such theory is PSL(2/2) principal chiral model [3]. In our paper [14] we have shown that this idea is indeed very promising and can explain some known facts about the transition.

Other examples of disordered critical points are the Ising model with a weak bond disorder (this problem can also be formulated as a problem of 2D Majorana fermions with a random mass) [15] and the model of electrons with spin-orbit scattering (the simplectic random ensamble). In both these cases the disorder renormalizes to zero and can be treated perturbatively. For the Ising model the disorder average of the n-th power of the spin-spin correlation function is given by [16],[17]

$$\overline{G(R)^n} \equiv \overline{< \sigma(R)\sigma(0) >^n} = (a/R)^{n/4}[\ln(R/a)]^{n(n-1)/8}$$

which is compatible with the following distribution function:

$$P[G(R)] \sim [G(R)]^{-1} \exp\left[-\frac{4\ln^2[G(R)/G_0(R)]}{\ln\ln(R/a)}\right]$$
$$G_0(R) = (a/R)^{1/4}[\ln(R/a)]^{-1/8}$$

Here the Green's function is self-averaging:

$$\overline{G(R)} = (a/R)^{1/4}G_{typical}(R) = (a/R)^{1/4}\frac{\exp[2\sqrt{\ln\ln(R/a)/\pi}]}{\ln^{1/8}(R/a)}$$

For a long time these three problems had remained the only known examples of the disordered critical systems; in recent years the list of such models has extended. Among the new additions to the list of critical models are (i) the model of Dirac electrons in a random gauge potential (DRGP problem), (ii) the random XY model, (iii) the model of plateau transiton in Spin Hall effect and (iv) the field theory describing the vicinity of the so-called Nishimori line in the random-bond Ising model. In all these models the critical points are non-perturbative and are described by non-trivial field theories. The amount of information about critical behaviour available in each case is rather different for each of these theories. Ideally CFT is capable of giving a complete information about all correlation functions together with a complete basis of all primary fields. In principle one can use this information to describe the vicinity of the critical point. In problems of disorder related to localization of quantum particles one needs to depart from the critical point to calculate the diffusion propagator or (for problems with a singular density of states) the energy dependence of the density of states. This program has been fulfilled only for the random XY model. The SUSY version of the random XY model has been solved by Guruswamy *et. al* [24] who constructed the free-field representation of the model and even calculated the frequency dependence of the average density of states. Quite an advanced level of understanding has also been achieved for DRGP model, especially in the Abelian case. The model was formulated in [18] for the Abelian case and for the non-Abelian case in [19]. The various aspects of the solution have been subsequently refined in [7],[20],[21],[22]. The problem of the Spin Hall plateau transition has been proven to be equivalent to the critical percolation by Gruzberg *et. al.* These authors have also found the exact exponents [25]. The universal conductance has been obtained by Cardy [26]. The work on the correlation functions and operator expansion for this model is still in progress [27]. The field theory for the Nishimori line in the random bond Ising model has recently been suggested in [28]. No results are available for the critical region.

Only in the DRGP model the replica solution has been compared with solutions obtained by other means. The action of the non-Abelian DRGP problem contains N species of massless Dirac fermions subject to a random static SU(N)-symmetric gauge potential. In this most interesting case of non-Abelian disorder we have a luxury to compare several non-perturbative solutions obtained by three (!) different methods. Thus the replica version of the model was solved exactly by Caux *et al.* [7].

Another approach to the problem was formulated in [22],[20]. It is based on the fact that for zero energy the Dirac equation can be solved explicitely for an arbitrary gauge potential. The corresponding Green's functions (or normalized wave functions) can (at least in principle) be averaged over the disorder. A comparison with the replicas is overshadowed by the fact that the average has been performed only in the limit of infinite disorder strength. In this case the distribution function of disorder potentials is equal to one. Though the solution obtained in this way differs from the replica result [22], one may wonder whether it is legitimate to compare solutions obtained for qualitatively different disorder distributions.

Recently Bernard and LeClair have suggested a solution of the DRGP problem based on the SUSY approach [29]. This solution is derived in the limit of weak bare disorder strength and therefore can be directly compared with the replicas. All three solutions turn out to be different [31], though the difference between the SUSY solution and the infinite disorder limit is less profound than the difference between these two and the replica solution.

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