The Drinfel'd Twisted XYZ Model

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ABSTRACT: We construct a factorizing Drinfel'd twist for a face type model equivalent to the XYZ model. Completely symmetric expressions for the operators of the monodromy matrix are obtained

1. Introduction

Several integrable quantum spin chain models within the range of the algebraic Bethe ansatz method have a distinguished basis of states, which minimizes quantum effects. That is, the quasi-particle creation and annihilation operators in this basis have an appearance devoid of polarization clouds. For the XXX and XXZ models with underlying group $sl(2)$ the bases in question were found by Maillet and Sanchez de Santos [1] through the construction of a generalized Drinfeld twist [2]. The ensuing representation of the quantum monodromy matrices coincides, as noted by Terras [3], for the case of the rational XXX model with the representation provided by Sklyanin's functional Bethe ansatz method [4]. An obvious generalization of Sklyanin's method (substituting polynomials in the spectral parameter by polynomials in the exponential of the spectral parameter) leads us to the conclusion that an analogous coincidence holds true for the trigonometric XXZ model. The purpose of this talk (based on the [7]) is to report on the generalization of the above results to the elliptic $sl(2)$-XYZ model. For this sake we make use of Baxter's map of the XYZ model onto an ice type model [5]. This brings us formally near to the XXX and XXZ models and allows us to construct the corresponding F-transformation in the quantum space and write down explicit expressions for the monodromy matrix elements in this new basis

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2. XYZ model and its relation to ice-type models

In the framework of the Algebraic Bethe Ansatz [8] the XYZ-model is determined by the elliptic solution of the Yang-Baxter equation

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)R_{13}(\lambda_1-\lambda_3)R_{12}(\lambda_1-\lambda_2) \tag{2.1}$$

with

$$R^{xyz}(\lambda-\mu) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \tag{2.2}$$

where

$$a = \frac{\Theta(2\eta)\Theta(\lambda-\mu)}{\Theta(0)\Theta(\lambda-\mu+2\eta)},\quad b = \frac{\Theta(2\eta)H(\lambda-\mu)}{\Theta(0)H(\lambda-\mu+2\eta)},\quad c = \frac{H(2\eta)\Theta(\lambda-\mu)}{\Theta(0)H(\lambda-\mu+2\eta)},\quad d = \frac{H(2\eta)H(\lambda-\mu)}{\Theta(0)\Theta(\lambda-\mu+2\eta)} \tag{2.3}$$

with the notation $H(u) = \vartheta_1\left(\frac{u}{2K},q\right)$, $\Theta(u) = \vartheta_4\left(\frac{u}{2K},q\right)$ and $\vartheta_4(z,q) = \sum_{m\in Z}(-1)^n q^{n^2} e^{2\pi inz}$,

$\vartheta_1(z,q) = -iq^{\frac{1}{4}}e^{i\pi z}\theta_4(z+\tau/2,q)$ are the standard theta-functions of a single complex variable. The somewhat different parametrization as compared to [8] is due to the normalization in order to achieve unitarity of the R-matrix.

The monodromy matrix $T(\lambda,\{\lambda_i\})$ (generalized to the inhomogeneous chain [9], [10]) is given as the ordered product of Lax operators $L_i(\lambda-\lambda_i)=R_{0i}(\lambda-\lambda_i)$

$$\begin{aligned} T(\lambda,\{\lambda_i\}) &= \\ L_N(\lambda-z_N)\dots L_2(\lambda-z_2)L_1(\lambda-z_1) &= \\ \begin{pmatrix} A(\lambda,\{\lambda_i\}) & B(\lambda,\{\lambda_i\}) \\ C(\lambda,\{\lambda_i\}) & D(\lambda,\{\lambda_i\}) \end{pmatrix} & . \end{aligned} \quad (2.4)$$

The presence of the Boltzmann weight d in Eq. (2.2) reflects the non-conservation of spin, which is responsible for the absence of a local vacuum for the Lax operator associated with the above R-matrix.
To circumvent the problems arising from the eight vertex nature, we use the vertex–face map established by Baxter [5] to obtain a XXZ type (six vertex) R-matrix by exploiting the relation

$$\begin{aligned} R^{xyz}(\lambda-\mu)\phi_{l,l'}\otimes z_{m',l'} &= \\ \sum_m w(m,m'|l,l')\phi_{m,m'}\otimes z_{m,l} & \end{aligned} \quad (2.5)$$

valid for all integers l,l',m,m' such that $|l-l'|=|l'-m'|=1$ and the summation on the r.h.s. is over integers m s.t. $|m-m'|=|m-l|=1$.
The two dimensional vectors $\phi,\ z$ are given by

$$\begin{aligned} \phi_{l,l+1} &= X(s_l+\mu);\ z_{l+1,l}=X(s_l+\lambda) \\ \phi_{l+1,l} &= X(t_{l+1}-\mu);\ z_{l-1,l}=X(t_l-\lambda) \end{aligned} \quad (2.6)$$

with $X(u)=\binom{H(u)}{\Theta(u)}$ and the abbrevation $s_l=s+2\eta l$, where s,t are arbitrary complex parameters.

The Boltzmann weights are given by ($h(u)=H(u)\Theta(u)$; $\omega_l=(\frac{s+t}{2}+2\eta l-K)$):

$$\begin{aligned} a_l &= a'_l = 1; \\ b_l &= \frac{h(\lambda)h(\omega_{l-1})}{h(\lambda+2\eta)h(\omega_l)}; \\ b'_l &= \frac{h(\lambda)h(\omega_{l+1})}{h(\lambda+2\eta)h(\omega_l)}; \\ c_l &= \frac{h(2\eta)h(\omega_l-\lambda)}{h(\lambda+2\eta)h(\omega_l)}; \\ c'_l &= \frac{h(2\eta)h(\omega_l+\lambda)}{h(\lambda+2\eta)h(\omega_l)}, \end{aligned} \quad (2.7)$$

where we have introduced the notations: $a_l = w(l-1,l|l-2,l-1)$, $b_l = w(l+1,l|l,l-1)$, $c_l=w(l+1,l|l,l+1)$, $a'_l=w(l+1,l|l+2,l+1)$, $b'_l = w(l-1,l|l,l+1)$, $c'_l = w(l-1,l|l,l-1)$. These weights can be arranged into a matrix

$$R_{12}(l) = \begin{pmatrix} a_l & 0 & 0 & 0 \\ 0 & b_l & c_l & 0 \\ 0 & c'_l & b'_l & 0 \\ 0 & 0 & 0 & a'_l \end{pmatrix} \quad (2.8)$$

which fulfills the modified Yang-Baxter equation [11], [12]

$$\begin{aligned} R_{12}(l-\sigma_3)R_{13}(l)R_{23}(l-\sigma_1) &= \\ R_{23}(l)R_{13}(l-\sigma_2)R_{12}(l) & . \end{aligned} \quad (2.9)$$

The monodromy matrix related to this modified Yang-baxter equation is

$$\begin{aligned} T_{0,1\dots N}(l) &= \\ R_{0N}(l-\sigma_1-\dots-\sigma_{N-1})\dots R_{02}(l-\sigma_1)R_{01}(l), & \end{aligned} \quad (2.10)$$

where 0 denotes the horizontal auxiliary space (with the asssociated spectral parameter λ_0), the positive integers $1,\dots,N$ label the vertical local quantum spaces which span the physical Hilbert space $\mathcal{H}_N$ (with associated local inhomogeneities $\{\lambda_i\}$), and σ_i equals ±1 depending on whether the arrow in the i-th space is up or down (right/ left for the horizontal space). It also sets our convention to associate the integer in the right lower corner of the graphical representation of the monodromy matrix. From (2.9) follows the equation for the monodromy matrices

$$\begin{aligned} R_{00'}(l-\sigma_1-\dots-\sigma_N)T_0(l)\otimes T_{0'}(l-\sigma_0) &= \\ T_{0'}(l)\otimes T_0(l-\sigma_{0'})R_{00'}(l). & \end{aligned} \quad (2.11)$$

It is easy to check that the unitarity relation $R_{21}R_{12}=\mathbb{I}$ is satisfied too.

The construction of the eigenvalues of the transfer matrix obtained from the initial monodromy matrix (2.4) using the modified monodromy matrix (2.10) is explained in reference

[8] (One has however to keep in mind that our monodromy matrix (2.10) differs from that of [8] by an additional change of basis in the quantum space).
We will concentrate in what follows on the computation of a factorizing F-matrix for the monodromy matrix (2.10).

3. The F basis

The factorizing F-matrix for two sites defined by the relation $F_{21}R_{12} = F_{12}$ is

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c' & b' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[12]} . \tag{3.1}$$

The proof of the factorization property amounts to checking the same relations as those in the proof of the unitarity of the R-matrix above.

The factorizing F-matrix for N sites (N quantum spaces) turns out to be given by formally the same expression as found in [6] for the XXX model

$$\begin{aligned} F_{1\ldots N}(l) &= \sum_{\alpha\in Z_2^N} P_\alpha R^{\sigma_\alpha}_{1\ldots N}(l)(z_1,\ldots,z_N) \\ P_\alpha &= \prod_{i=1}^{N} P_i^{\alpha_i}, \end{aligned} \tag{3.2}$$

where $P_i^{\alpha_i}$ projects on the α_i-th component in the i-th space and the permutation σ_α is uniquely determined through the conditions

$$\begin{aligned} \alpha_{\sigma_\alpha(i+1)} &\geq \alpha_{\sigma_\alpha(i)} \quad \text{if} \quad \sigma_\alpha(i+1) > \sigma_\alpha(i) \\ \alpha_{\sigma_\alpha(i+1)} &> \alpha_{\sigma_\alpha(i)} \quad \text{if} \quad \sigma_\alpha(i+1) < \sigma_\alpha(i) . \end{aligned} \tag{3.3}$$

The modification of the Yang-Baxter equation (2.9) enforces a particular rule for the handling of the integer valued parameter l in the formation of the intertwining matrix $R^\sigma(l)$ (related to the permutation σ), which can be read off from the modified composition law

$$\begin{aligned} R^{\sigma\sigma_i}(l) &= R_{\sigma(i),\sigma(i+1)}(\tilde{l}_i)R^\sigma(l); \\ \tilde{l}_i &= l - \sigma_{\sigma(1)} - \ldots - \sigma_{\sigma(i-1)}, \end{aligned} \tag{3.4}$$

where σ_i is the transposition of $i, i+1$, and σ an arbitrary permutation.
$R^\sigma(l)$ has the intertwining property

$$R^\sigma(l)T_{0,1\ldots N}(l) = T_{0,\sigma(1)\ldots\sigma(N)}(l)R^\sigma(l-\sigma_0) .$$

The matrix $F_{1\ldots N}(l)$ satisfies the factorizing equation

$$R^\sigma_{1\ldots N}(l) = F^{-1}_{\sigma(1\ldots N)}(l)F_{1\ldots N}(l) . \tag{3.5}$$

A proof of the latter equation can be found in [7].

The operators of the monodromy matrix (2.10) in the F basis are obtained by using a recursion relation, which enables one to express the monodromy matrix for N sites in terms of that for $N-1$ sites [7]:

$$\begin{aligned} \tilde{T}_{0,1\ldots N}(l) = &\begin{pmatrix} \mathbb{1} & 0 \\ \tilde{C}^1_{2\ldots N}(l) & \tilde{D}^1_{2\ldots N}(l) \end{pmatrix}_{[1]} \times \\ &\tilde{T}_{0,2\ldots N}(l-\sigma_1)R_{01}(l) \times \\ &\begin{pmatrix} \mathbb{1} & 0 \\ \tilde{C}^1_{2\ldots N}(l-\sigma_0) & \tilde{D}^1_{2\ldots N}(l-\sigma_0) \end{pmatrix}^{-1}_{[1]} . \end{aligned} \tag{3.6}$$

This relation can be solved recursively starting with the one site monodromy matrix which coincides with the Lax operator $L_i = R_{0i}$ (2.8).

The solution of the recursion relation finally yields the following result for the operators of the monodromy matrix (we use a slight change in notation: $b(\lambda)=\frac{h(\lambda)}{h(\lambda+2\eta)}$ and denote $k=l-\sum_{i=1}^{N}\sigma_i$):

$$\tilde{D}_l(\lambda_0) = \frac{h(\omega_{l+1})}{h(\omega_{1+\frac{k+l-N}{2}})} \otimes_{i=1}^{N} \begin{pmatrix} b(\lambda_{0i}) & 0 \\ 0 & 1 \end{pmatrix}_{[i]} ;$$

$$\tilde{B}_l(\lambda_0) = \frac{h(\omega_{l+1})}{h(\omega_k)} \sum_{i=1}^{N} c_{k-1}(\lambda_{0i})\sigma_i^- \times \otimes_{j\neq i}^{N} \begin{pmatrix} b(\lambda_{0j}) & 0 \\ 0 & b^{-1}(\lambda_{ji}) \end{pmatrix}_{[j]} ;$$

$$\tilde{C}_l(\lambda_0) = \sum_{i=1}^{N} c'_l(\lambda_{0i})\sigma_i^+ \times \otimes_{j\neq i}^{N} \begin{pmatrix} b(\lambda_{0j})b^{-1}(\lambda_{ij}) & 0 \\ 0 & 1 \end{pmatrix}_{[j]} ;$$

$$\tilde{A}_l(\lambda_0) = \frac{h(\omega_{\frac{k+l-N}{2}})}{h(\omega_k)} \Bigg\{ \otimes_{i=1}^{N} \begin{pmatrix} 1 & 0 \\ 0 & b(\lambda_{i0})^{-1} \end{pmatrix}_{[i]} + \sum_{i=1}^{N} \frac{c_{k-1}(\lambda_{0i}) c_l'(\lambda_{0i})}{b(\lambda_{0i})} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{[i]} \times \otimes_{j\neq i} \begin{pmatrix} \frac{b(\lambda_{0j})}{b(\lambda_{ij})} & 0 \\ 0 & b(\lambda_{ji})^{-1} \end{pmatrix}_{[j]} + \sum_{i\neq j}^{N} \frac{c_{k-1}(\lambda_{0i}) c_l'(\lambda_{0j})}{b(\lambda_j - \lambda_k)} \sigma_-^i \otimes \sigma_+^j \times \otimes_{k\neq i,j} \begin{pmatrix} \frac{b(\lambda_{0k})}{b(\lambda_{jk})} & 0 \\ 0 & b(\lambda_{ki})^{-1} \end{pmatrix}_{[k]} \Bigg\}, \quad (3.7)$$

where $\lambda_{ij}{=}\lambda_i - \lambda_j$.

The above mentioned basis transformation (2.5) amounts to splitting the model into sectors with a fixed number of turned spins. To obtain the spectrum of the XYZ model one uses the operators $\tilde{B}_l(\lambda), \tilde{C}_l(\lambda)$ to construct eigenvectors in the form proposed by [8] (there denoted by $B_{k,l}(\lambda)$ etc.).

4. Conclusion

The form of the F-matrix, Eq. (3.2) and the appearance of the monodromy matrix in the basis supplied by the F-matrix, Eq's (3.7), , are completely analogous to what has been found in [1] and [6] for the rational and trigonometric models. The concrete expressions for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are in particular manifestly symmetric with respect to exchanges of the local inhomogeneity parameters λ_i. The quasiparticle operators $\tilde{B}$ and $\tilde{C}$ are free from polarization effects due to non-local exchange terms.

The argument used in [6] - borrowed from [1] - concerning the identification of operators corresponding to different entries of the monodromy matrix relied on the $sl(n)$ symmetry of the rational model. It is not available for the trigonometric and elliptic model. The recursive procedure followed instead in the preceding section is equally applicable to the XXX, XXZ and XYZ model.

It seems rather plausible in view of the formal similarities of the rational, trigonometric and elliptic models that some version of Sklyanin's functional Bethe ansatz should also be feasible in the latter case as has already been achieved for the XYZ Gaudin magnet in [10].

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References

[1] Maillet, J. M., Sanchez de Santos, J.: Drinfel'd twists and algebraic Bethe ansatz, q-alg/9612012.

[2] Drinfel'd, V. G.: Quasi–Hopf algebras, *Leningrad Math. J.* **1**, (1990), 1419.

[3] Terras, V.: Drinfel'd twists and functional Bethe ansatz, *Lett. Math. Phys.* **48**, (1999), 263-276, math-ph/9902009.

[4] Sklyanin, E. K.: Functional Bethe Ansatz, in: *Integrable and Superintegrable Systems*, B. Kupershmidt (ed.), World Scientific, Singapore, 1990, pp. 8-33.

[5] Baxter, R. J.: Eight vertex model in lattice statistics and one-dimmensional anisotropic Heisenberg chain I-III, *Ann. Phys. (N.Y.)* **76**, (1973), 1-71.

[6] Albert, T.-D., Boos, H., Flume, R., Ruhlig, K.: Resolution of the nested hierarchy for rational sl(n) models, nlin-SI/0002027.

[7] Albert, T.-D., Boos, H., Flume, R., Poghossian, R. , Ruhlig, K.: The Drinfel'd twisted XYZ model, nlin-SI/0005023.

[8] Faddeev, L. D., Takhtadzhan, L. A.: The quantum version of the inverse problem and the Heisenberg XYZ model, *Russ. Math. Surveys* **34**, (1979), 11-68.

[9] Babujian, H., Lima-Santos, A., Poghossian, R. H.: Knizhnik-Zamolodchikov-Bernard equations connected with the eight-vertex model, *Int. Journ. Mod. Phys.* **A 14**, (1999), 615-630, nlin-SI/9804015.

[10] Sklyanin, E. K., Takebe, T.: Algebraic Bethe ansatz for the XYZ Gaudin magnet, *Phys. lett.*, **A 219**, (1996), 217-225, q-alg/9601028.

[11] Gervais, J. L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville field theory, *Nucl. Phys.* **B 238**, (1984), 125.

[12] Felder, G.: Elliptic quantum groups, hep-th/9412207.