Steps towards the QCD calculation for $e^+e^- \to$ 3 jets at NNLO

Sven Moch, Peter Uwer

Institut für Theoretische Teilchenphysik, Universität Karlsruhe,
76128 Karlsruhe, Germany
E-mail: moch@particle.uni-karlsruhe.de, uwer@particle.uni-karlsruhe.de

Stefan Weinzierl*

Dipartimento di Fisica, Università di Parma, INFN Gruppo Collegato di Parma,
43100 Parma, Italy
E-mail: stefanw@fis.unipr.it

Abstract: High precision analyses of experimental data for e^+e^- annihillation, such as determination of jet rates or event shape observables, call for complete next-to-next-to-leading order (NNLO) perturbative QCD predictions. In this talk, we discuss the various ingredients entering the calculation of the NNLO corrections in e^+e^- annihillation.

1. Introduction

Perturbation theory is a powerful tool for precise theoretical predictions on the outcome of high-energy experiments. The state-of-the-art is the transition to fully differential next-to-next-to-leading order (NNLO) calculations for jet physics. The most prominent processes where a complete NNLO calculation is desirable are Bhabha scattering, $pp \to 2$ jets and $e^+e^- \to 3$ jets. A NNLO calculation of $e^+e^- \to 3$ partons is expected to reduce the theoretical uncertainty in the extraction of α_s down to 1% [1]. Furthermore, it allows to model the jet structure more accurately and should improve the knowledge on the interplay between perturbative and power corrections.

In the past years there has been tremondous progress towards this goal: The relevant master integrals for $pp \to 2$ jets and $e^+e^- \to 3$ jets have been calculated, either with the help of a Mellin-Barnes representation [2] or by solving differential equations [3]. Numerical results for these master integrals serve as a useful cross-check [4, 5]. With these results the two-loop amplitudes for Bhabha scattering [6], for $pp \to 2$ jets [7, 8], for $pp \to \gamma\gamma$ [9] and for light-by-light scattering [10] have been calculated.

*Speaker.

However, the two-loop amplitudes are only one part of the story. Loop amplitudes in a massless theory like QCD lead to infrared singularities. These divergences cancel against corresponding singularities resulting from amplitudes with additional unresolved partons. Presently, the singular behaviour of double-unresolved tree amplitudes [11, 12] and of one-loop amplitudes with one unresolved parton is understood [13, 14], as well as the structure of the poles of two-loop amplitudes up to $1/\varepsilon$ [15]. But despite this progress there is still a fair amount of work to be done:

- The two-loop amplitude for $e^+e^- \to 3$ jets remains to be calculated, although recently there has been reported progress [16]. Due to multiple scales, the amplitude will depend on the ratio of two kinematical invariants, for example s_{12}/s_{123} and s_{23}/s_{123}. In contrast, the two-loop amplitudes calculated so far depend only on one ratio of invariants, say s/t.
- A method to cancel IR divergences has to be set up. This requires the extension of the subtraction or slicing method to NNLO. In particular analytic integrations over unresolved regions in phase space have to be carried out.
- The final numerical computer program requires stable and efficient numerical methods, in particular for the Monte Carlo integration of the double unresolved contributions.

In this talk we focus on the first two problems.

2. Two-loop amplitudes

Let us briefly review how the calculation of the two-loop amplitudes for the single scale problems has been performed. Starting from the Feynman diagrams, which yield two-loop tensor integrals, i.e. integrals with powers of the loop momentum in the numerator, one performs the tensor reduction. Using a Schwinger parametrization these tensor integrals can be related to scalar integrals with higher powers of the propagators and different values of the dimension D [17]. In a second step, integration-by-parts [18], Lorentz-invariance identities [3] or form-factor relations [19] can be used to eliminate propagators and to reduce the integrals to a few basic topologies. At this stage, the basic topologies occur in various dimensions and with various powers ν_i of the propagators. The last step relates these basic topologies to the (much smaller) set of master integrals, which are in most cases just the basic topologies for $D = 4 - 2\varepsilon$ and all $\nu_i = 1$. Again, integral relations derived with integration-by-parts or form-factor relations are used for this step. Looking at the computational cost, one finds that the first two steps are rather easy to perform, whereas the third step requires to solve large systems of equations. In particular in the presence of multiple scales this becomes rather involved.

We therefore would like to advocate a different approach, where basic topologies with arbitrary powers of the propagators and arbitrary dimensions are evaluated directly [20]. As an advantage, this approach keeps the size of the intermediate expressions much smaller and allows for a very efficient implementation of the reduction scheme. The starting point is the observation that these integrals can be written as nested sums involving Gamma functions. For example, a basic integral occuring in $e^+e^- \to 3$ jets is the so-called C-topology, which can be written as [20]

$$
\begin{aligned}
I = & \frac{\Gamma(2m-2\varepsilon-\nu_{1235})\Gamma(1+\nu_{1235}-2m+2\varepsilon)\Gamma(2m-2\varepsilon-\nu_{2345})\Gamma(1+\nu_{2345}-2m+2\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(3m-3\varepsilon-\nu_{12345})} \\
& \times \frac{\Gamma(m-\varepsilon-\nu_5)\Gamma(m-\varepsilon-\nu_{23})}{\Gamma(2m-2\varepsilon-\nu_{235})} (-s_{123})^{2m-2\varepsilon-\nu_{12345}} \sum\limits_{i_1=0}^{\infty} \sum\limits_{i_2=0}^{\infty} \frac{x_1^{i_1}}{i_1!} \frac{x_2^{i_2}}{i_2!} \\
& \times \left[\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+\nu_2)\Gamma(i_1+i_2-2m+2\varepsilon+\nu_{12345})\Gamma(i_1+i_2-m+\varepsilon+\nu_{235})}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+\nu_{23})} \right. \\
& -x_1^{2m-2\varepsilon-\nu_{1235}} \\
& \times \frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+\nu_2)\Gamma(i_1+i_2+\nu_4)\Gamma(i_1+i_2+m-\varepsilon-\nu_1)}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{15})} \\
& -x_2^{2m-2\varepsilon-\nu_{2345}} \\
& \times \frac{\Gamma(i_1+\nu_3)\Gamma(i_2+2m-2\varepsilon-\nu_{345})\Gamma(i_1+i_2+\nu_1)\Gamma(i_1+i_2+m-\varepsilon-\nu_4)}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{45})} \\
& +x_1^{2m-2\varepsilon-\nu_{1235}} x_2^{2m-2\varepsilon-\nu_{2345}} \frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+2m-2\varepsilon-\nu_{345})}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})} \\
& \left. \times \frac{\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{235})\Gamma(i_1+i_2+3m-3\varepsilon-\nu_{12345})}{\Gamma(i_1+i_2+4m-4\varepsilon-\nu_{12345}-\nu_5)} \right], \qquad (2.1)
\end{aligned}
$$

where we set $x_1 = (-s_{12})/(-s_{123})$ and $x_2 = (-s_{23})/(-s_{123})$. Here, the result for the integral holds for arbitrary dimensions $D = 2m - 2\varepsilon$ and any (not necessarily integer) power ν_i of the propagators.

The Gamma functions can be expanded systematically in ε, thus allowing to solve these nested sums algorithmically to any given order in ε and to express the result in the basis of Z-sums, defined by

$$
Z(n; m_1, ..., m_k; x_1, ..., x_k) = \sum\limits_{n \ge i_1 > i_2 > ... > i_k > 0} \frac{x_1^{i_1}}{i_1^{\ m_1}} \cdots \frac{x_k^{i_k}}{i_k^{\ m_k}}. \qquad (2.2)
$$

Important special cases of this definition for Z-sums are multiple polylogarithms (in which we express our final results) [21] - [23]

$$
\mathrm{Li}_{m_k,...,m_1}(x_k, ..., x_1) = Z(\infty; m_1, ..., m_k; x_1, ..., x_k) \qquad (2.3)
$$

and Euler-Zagier sums (which occur in the expansion of Gamma functions)

$$
Z_{m_1,...,m_k}(n) = Z(n; m_1, ..., m_k; 1, ..., 1). \qquad (2.4)
$$

Z-sums form an algebra. In fact, Z-sums can be considered as generalizations of Euler-Zagier sums or harmonic sums. Many algorithms known for the latter [24] carry over to Z-sums. The usefulness of the Z-sums lies in the fact, that they interpolate between multiple polylogarithms and Euler-Zagier sums, the interpolation being compatible with the algebra structure.

3. Subtraction method

The NNLO cross section receives contributions from double unresolved tree amplitudes, one-loop amplitudes with one unresolved parton and two-loop amplitudes:

$$
\begin{aligned}
d\sigma_{n+2}^{(0)} &= \left(\mathcal{A}_{n+2}^{tree\,*} \mathcal{A}_{n+2}^{tree} \right) \theta_{n+2} d\phi_{n+2}, \\
d\sigma_{n+1}^{(1)} &= \left(\mathcal{A}_{n+1}^{tree\,*} \mathcal{A}_{n+1}^{1-loop} + \mathcal{A}_{n+1}^{1-loop\,*} \mathcal{A}_{n+1}^{tree} \right) \theta_{n+1} d\phi_{n+1}, \\
d\sigma_{n}^{(2)} &= \left(\mathcal{A}_{n}^{tree\,*} \mathcal{A}_{n}^{2-loop} + \mathcal{A}_{n}^{2-loop\,*} \mathcal{A}_{n}^{tree} + \mathcal{A}_{n}^{1-loop\,*} \mathcal{A}_{n}^{1-loop} \right) \theta_{n} d\phi_{n},
\end{aligned} \tag{3.1}
$$

where θ_n is the jet-defining function for n partons. Taken separately, each part gives a divergent contribution. Only the sum of all contributions is infrared finite. This problem already occurs in NLO calculations and can be solved by adding and subtracting a suitable chosen term. For example the NLO cross section for $(n+1)$ partons is written as [25]

$$
\int d\sigma_{n+2}^{(0)} + \int d\sigma_{n+1}^{(1)} = \int \left(d\sigma_{n+2}^{(0)} - d\sigma_{n+1}^{A} \right) + \int \left(d\sigma_{n+1}^{(1)} + d\sigma_{n+1}^{A} \right). \tag{3.2}
$$

Here $d\sigma_{n+1}^{A}$ acts as a local counterterm for single unresolved configurations to $d\sigma_{n+2}^{(0)}$ in D dimensions and is integrable over a one-parton subspace.

A similar subtraction scheme is needed for the NNLO cross section. The more complicated part is the subtraction term to $d\sigma_{n+2}^{(0)}$ for double unresolved configurations. In addition, the one-loop corrections $d\sigma_{n+1}^{(1)}$ require a subtraction term $d\sigma_{n}^{A,loop}$ which approximates in D dimensions the one-loop corrections when one parton becomes unresolved. We shortly outline how to obtain the subtraction terms for $d\sigma_{n}^{A,loop}$. We first decompose the amplitudes into primitive amplitudes (e.g. with a fixed cyclic ordering and a definitive routing of the fermion lines). Primitive one-loop amplitudes factorize in the collinear limit where one parton becomes unresolved as [13, 14]

$$
\begin{aligned}
& \mathcal{A}_{n+1}^{tree\,*} \mathcal{A}_{n+1}^{1-loop} + \mathcal{A}_{n+1}^{1-loop\,*} \mathcal{A}_{n+1}^{tree} \rightarrow \\
& \quad \mathrm{Sing}^{tree} \left(\mathcal{A}_{n}^{tree\,*} \mathcal{A}_{n}^{1-loop} + \mathcal{A}_{n}^{1-loop\,*} \mathcal{A}_{n}^{tree} \right) + \mathrm{Sing}^{1-loop} \mathcal{A}_{n}^{tree\,*} \mathcal{A}_{n}^{tree}
\end{aligned} \tag{3.3}
$$

The singular function Sing^{tree} is already known from NLO calculations and poses no problem. The function Sing^{1-loop} gives a new contribution. However, it is relatively simple to write down an appropriate subtraction term for each primitive structure. As an example we consider the splitting $g \to \bar{q}q$. For the primitive part where the fermion entering the loop turns right we have as subtraction term

$$
\begin{aligned}
& \langle \mu | V_{\bar{q}_i q_j, k}^{1-loop} | \nu \rangle = \\
& \quad \left[-g^{\mu\nu} - \frac{4}{2 p_i p_j} S^{\mu\nu} \right] c_\Gamma \left(\frac{\mu^2}{-2 p_i p_j} \right)^{\varepsilon} \left\{ \frac{1}{\varepsilon^2} + \frac{3}{2} \frac{1}{\varepsilon(1-2\varepsilon)} + \frac{1}{1-2\varepsilon} \right\}
\end{aligned} \tag{3.4}
$$

Here $S^{\mu\nu}$ denotes the spin correlation tensor. This subtraction term has to integrated over a one-parton phase space. For the more complicated integrals we can rely on the technique of nested sums [26]. The integrated counter-part reads:

$$\int d\phi_{dipole} \frac{1}{2p_i p_j} V^{1-loop}_{\bar{q}_i q_j, k} = \left(-g^{\mu\nu}\right) c_\Gamma^2 \left(\frac{\mu^2}{P^2}\right)^{\varepsilon} \left(\frac{\mu^2}{-P^2}\right)^{\varepsilon}$$
$$\cdot \left\{ -\frac{1}{3}\frac{1}{\varepsilon^3} - \left(\frac{23}{18} + \frac{1}{3}\gamma\right)\frac{1}{\varepsilon^2} - \left(\frac{1}{6}\gamma^2 + \frac{23}{18}\gamma + \frac{118}{27}\right)\frac{1}{\varepsilon} \right.$$
$$\left. - \left(\frac{7}{9}\zeta_3 + \frac{1}{18}\gamma^3 + \frac{23}{36}\gamma^2 + \frac{118}{27}\gamma + \frac{1}{6}\pi^2 + \frac{4075}{324}\right)\right\} + O(\varepsilon) \qquad (3.5)$$

We note that in general for one-loop amplitudes with one unresolved parton the soft radiation pattern is more complicated as compared to a NLO calculation. Generally, a simple dipole picture with emitter and spectator will not be sufficient.

4. Outlook and conclusions

The NNLO calculation for $e^+e^- \to 3$ jets is a challenging project. We have reported on recent progress in the evaluation of two-loop amplitudes with multiple scales using the novel approach of nested sums. All algorithms for the solution of the nested sums are suitable for implementation in computer algebra systems like GiNaC [27] or FORM [28]. We have outlined the road to the cancellation of infrared divergences. This involves the analytic integration over the single and double unresolved partonic phase space and we believe the technique of nested sums to be of great value here as well. We have not addressed any issues [29] concerning the final numerical integration of the fully differential NNLO cross section by means of Monte Carlo techniques.

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