Results of the $\mathcal{O}(\alpha_s)$ two-loop virtual corrections to $B \to X_s \ell^+\ell^-$ in the standard model

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Abstract: We present the results of the $\mathcal{O}(\alpha_s)$ two-loop virtual corrections to the differential decay width $d\Gamma(B \to X_s\ell^+\ell^-)/d\hat{s}$, where $\hat{s}$ is the invariant mass squared of the lepton pair, normalized to m_b^2. Those contributions from gluon bremsstrahlung which are needed to cancel infrared and collinear singularities are also included. Our calculation is restricted to the range $0.05 \leq \hat{s} \leq 0.25$ where the effects from resonances are small. The new contributions drastically reduce the renormalization scale dependence of existing results for $d\Gamma(B \to X_s\ell^+\ell^-)/d\hat{s}$. The renormalization scale uncertainty of the corresponding branching ratio (restricted to $0.05 \leq \hat{s} \leq 0.25$) gets reduced from $\sim \pm 13\%$ to $\sim \pm 6.5\%$.

1. Introduction

After the observation of the penguin-induced decay $B \to X_s\gamma$ [1] and corresponding exclusive channels such as $B \to K^*\gamma$ [2], rare B-decays have begun to play an important role in the phenomenology of particle physics. They put strong constraints on various extensions of the standard model. The inclusive decay $B \to X_s\ell^+\ell^-$ has not been observed so far, but is expected to be detected at the currently running B-factories.

The next-to-leading logarithmic (NLL) result for $B \to X_s\ell^+\ell^-$ suffers from a relatively large ($\pm 16\%$) dependence on the matching scale μ_W [3, 4]. The NNLL corrections to the Wilson coefficients remove the matching scale dependence to a large extent [5], but leave a $\pm 13\%$-dependence on the renormalization scale μ_b, which is of $\mathcal{O}(m_b)$. In order to further improve the result, we have recently calculated the $\mathcal{O}(\alpha_s)$ two-loop corrections to the matrix elements of the operators O_1 and O_2 as well as the $\mathcal{O}(\alpha_s)$ one-loop corrections to O_7,..., O_{10} [6]. Because of large resonant contributions from $\bar{c}c$ intermediate states, we restrict the invariant lepton mass squared s to the region $0.05 \leq \hat{s} \leq 0.25$, where $\hat{s} = s/m_b^2$. In the following we present a summary of the results of these calculations.

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2. Theoretical Framework

The appropriate tool for studies on weak B-mesons decays is the effective Hamiltonian technique. The effective Hamiltonian is derived from the standard model by integrating out the t-quark, the Z_0- and the W-boson. For the decay channels $b \to s\ell^+\ell^-$ $(\ell = \mu, e)$ it reads

$$\mathcal{H}_{\text{eff}} = -\frac{4\,G_F}{\sqrt{2}}\,V_{ts}^*\,V_{tb}\sum_{i=1}^{10} C_i\,O_i\,,$$

where O_i are dimension six operators and C_i denote the corresponding Wilson coefficients. The operators can be chosen as [5]

$$\begin{array}{ll}
O_1 = (\bar{s}_L\gamma_\mu T^a c_L)(\bar{c}_L\gamma^\mu T^a b_L) & O_2 = (\bar{s}_L\gamma_\mu c_L)(\bar{c}_L\gamma^\mu b_L) \\
O_3 = (\bar{s}_L\gamma_\mu b_L)\sum_q(\bar{q}\gamma^\mu q) & O_4 = (\bar{s}_L\gamma_\mu T^a b_L)\sum_q(\bar{q}\gamma^\mu T^a q) \\
O_5 = (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\sigma b_L)\sum_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\sigma q) & O_6 = (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\sigma T^a b_L)\sum_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\sigma T^a q) \\
O_7 = \frac{e}{g_s^2} m_b(\bar{s}_L\sigma^{\mu\nu} b_R)F_{\mu\nu} & O_8 = \frac{1}{g_s} m_b(\bar{s}_L\sigma^{\mu\nu} T^a b_R)G^a_{\mu\nu} \\
O_9 = \frac{e^2}{g_s^2}(\bar{s}_L\gamma_\mu b_L)\sum_\ell(\bar{\ell}\gamma^\mu\ell) & O_{10} = \frac{e^2}{g_s^2}(\bar{s}_L\gamma_\mu b_L)\sum_\ell(\bar{\ell}\gamma^\mu\gamma_5\ell)\,.
\end{array}$$

The subscripts L and R refer to left- and right- handed fermion fields. We work in the approximation where the combination $(V_{us}^* V_{ub})$ of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements is neglected. The CKM structure factorizes therefore.

3. Virtual Corrections to the Operators O_1, O_2, O_7, O_8, O_9 and O_{10}

Using the naive dimensional regularization scheme in $d = 4-2\,\epsilon$ dimensions, ultraviolet and infrared singularities both show up as $1/\epsilon^n$-poles $(n = 1, 2)$. The ultraviolet singularities cancel after including the counterterms. Collinear singularities are regularized by retaining a finite strange quark mass m_s. They are cancelled together with the infrared singularities at the level of the decay width, when taking the bremsstrahlung process $b \to s\ell^+\ell^- g$ into account. Gauge invariance implies that the QCD-corrected matrix elements of the operators O_i can be written as

$$\langle s\ell^+\ell^-|O_i|b\rangle = \hat{F}_i^{(9)}\langle O_9\rangle_{\text{tree}} + \hat{F}_i^{(7)}\langle O_7\rangle_{\text{tree}}\,,$$

where $\langle O_9\rangle_{\text{tree}}$ and $\langle O_7\rangle_{\text{tree}}$ are the tree-level matrix elements of O_9 and O_7, respectively.

3.1 Virtual corrections to O_1 and O_2

For the calculation of the two-loop diagrams associated with O_1 and O_2 we mainly used a combination of Mellin-Barnes technique [6, 7] and of Taylor series expansion in s. For $s < m_b^2$ and $s < 4\,m_c^2$, most diagrams allow the latter. The unrenormalized form factors $\hat{F}^{(7,9)}$ of O_1 and O_2 are then obtained in the form

$$\hat{F}^{(7,9)} = \sum_{i,j,l,m} c_{ijlm}^{(7,9)}\,\hat{s}^i\,\ln^j(\hat{s})\,\left(\hat{m}_c^2\right)^l\ln^m(\hat{m}_c)\,,$$

where $\hat{m}_c = \frac{m_c}{m_b}$. The indices i, j, m are non-negative integers and $l = -i, -i+\frac{1}{2}, -i+1, \ldots$.

Besides the counterterms from quark field, quark mass and coupling constant (g_s) renormalization, there are counterterms induced by operator mixing. They are of the form

$$C_i \cdot \sum_j \delta Z_{ij} \langle O_j \rangle \quad \text{with} \quad \delta Z_{ij} = \frac{\alpha_s}{4\,\pi} \left[a_{ij}^{01} + \frac{a_{ij}^{11}}{\epsilon} \right] + \frac{\alpha_s^2}{(4\,\pi)^2} \left[a_{ij}^{02} + \frac{a_{ij}^{12}}{\epsilon} + \frac{a_{ij}^{22}}{\epsilon^2} \right] + \mathcal{O}(\alpha_s^3)\,.$$

A complete list of the coefficients a_{ij}^{lm} used for our calculation can be found in [6]. The operator mixing involves also the evanescent operators

$$O_{11} = (\bar{s}_L \gamma_\mu \gamma_\nu \gamma_\sigma T^a c_L)\,(\bar{c}_L \gamma^\mu \gamma^\nu \gamma^\sigma T^a b_L) - 16\, O_1 \quad \text{and}$$
$$O_{12} = (\bar{s}_L \gamma_\mu \gamma_\nu \gamma_\sigma c_L)\,(\bar{c}_L \gamma^\mu \gamma^\nu \gamma^\sigma b_L) - 16\, O_2\,.$$

3.2 Virtual corrections to O_7, O_8, O_9 and O_{10}

The renormalized contributions from the operators O_7, O_8 and O_9 can all be written in the form

$$\langle s\ell^+\ell^- | C_i O_i | b \rangle = \widetilde{C}_i^{(0)} \left(-\frac{\alpha_s}{4\pi} \right) \left[F_i^{(9)} \langle \widetilde{O}_9 \rangle_{\text{tree}} + F_i^{(7)} \langle \widetilde{O}_7 \rangle_{\text{tree}} \right],$$

with $\widetilde{O}_i = \frac{\alpha_s}{4\,\pi}\, O_i$, $\quad \widetilde{C}_{7,8}^{(0)} = C_{7,8}^{(1)}$ and $\quad \widetilde{C}_9^{(0)} = \frac{4\,\pi}{\alpha_s} \left(C_9^{(0)} + \frac{\alpha_s}{4\,\pi} C_9^{(1)} \right)$.

The formally leading term $\sim g_s^{-2} C_9^{(0)}(\mu_b)$ to the amplitude for $b \to s\ell^+\ell^-$ is smaller than the NLL term $\sim g_s^{-2} [g_s^2/(16\,\pi^2)]\, C_9^{(1)}(\mu_b)$ [8]. We therefore adapt our systematics to the numerical situation and treat the sum of these two terms as a NLL contribution, as indicated by the expression for $\widetilde{C}_9^{(0)}$. The decay amplitude then starts out with a NLL term.

The contribution from O_8 is finite, whereas those from O_7 and O_9 are not, ie $F_7^{(7)}$ and $F_9^{(9)}$ suffer from the same infrared divergent part f_{inf}.

As the hadronic parts of the operators O_9 and O_{10} are identical, the QCD corrected matrix element of O_{10} can easily be obtained from that of O_9.

4. Bremsstrahlung Corrections

It is known [3, 4] that the contribution to the inclusive decay width from the interference between the tree-level and the one-loop matrix elements of O_9 and from the corresponding bremsstrahlung corrections can be written as

$$\frac{d\Gamma_{99}}{d\hat{s}} = \left(\frac{\alpha_{em}}{4\,\pi} \right)^2 \frac{G_F^2\, m_{b,\text{pole}}^5\, |V_{ts}^* V_{tb}|^2}{48\,\pi^3}\, (1-\hat{s})^2\, (1+2\,\hat{s}) \left[2 \left| \widetilde{C}_9^{(0)} \right|^2 \frac{\alpha_s}{\pi}\, \omega_9(\hat{s}) \right].$$

Analogous formulas hold true for the contributions from O_7 and the interference terms between the matrix elements of O_7 and O_9:

$$\frac{d\Gamma_{77}}{d\hat{s}} = \left(\frac{\alpha_{em}}{4\,\pi} \right)^2 \frac{G_F^2\, m_{b,\text{pole}}^5\, |V_{ts}^* V_{tb}|^2}{48\,\pi^3}\, (1-\hat{s})^2\, 4\,(1+2/\hat{s}) \left[2 \left| \widetilde{C}_7^{(0)} \right|^2 \frac{\alpha_s}{\pi}\, \omega_7(\hat{s}) \right],$$
$$\frac{d\Gamma_{79}}{d\hat{s}} = \left(\frac{\alpha_{em}}{4\,\pi} \right)^2 \frac{G_F^2\, m_{b,\text{pole}}^5\, |V_{ts}^* V_{tb}|^2}{48\,\pi^3}\, (1-\hat{s})^2\, 12 \cdot 2\, \frac{\alpha_s}{\pi}\, \omega_{79}(\hat{s})\, \text{Re}\left[\widetilde{C}_7^{(0)} \widetilde{C}_9^{(0)} \right].$$

The function $\omega_9(\hat{s}) \equiv \omega(\hat{s})$ can be found eg in in [3, 4]. For $\omega_7(\hat{s})$ and $\omega_{79}(\hat{s})$ see [6]. All other bremsstrahlung corrections are finite and will be given in [9].

5. Corrections to the Decay Width for $B \to X_s \ell^+ \ell^-$

Combining the virtual corrections discussed in section 3 with the bremsstrahlung contributions considered in section 4, we find for the decay width

$$\frac{d\Gamma(b \to X_s \ell^+ \ell^-)}{d\hat{s}} = \left(\frac{\alpha_{em}}{4\,\pi}\right)^2 \frac{G_F^2\, m_{b,\text{pole}}^5\, |V_{ts}^* V_{tb}|^2}{48\,\pi^3} (1-\hat{s})^2 \times \\ \left((1+2\,\hat{s}) \left[\left|\widetilde{C}_9^{\text{eff}}\right|^2 + \left|\widetilde{C}_{10}^{\text{eff}}\right|^2 \right] + 4\,(1+2/\hat{s}) \left|\widetilde{C}_7^{\text{eff}}\right|^2 + 12\ \text{Re}\left[\widetilde{C}_7^{\text{eff}} \widetilde{C}_9^{\text{eff}*}\right] \right) , \quad (5.1)$$

where the effective Wilson coefficients $\widetilde{C}_7^{\text{eff}}$, $\widetilde{C}_9^{\text{eff}}$ and $\widetilde{C}_{10}^{\text{eff}}$ can be written as

$$\widetilde{C}_9^{\text{eff}} = \left[1 + \frac{\alpha_s(\mu)}{\pi}\,\omega_9(\hat{s})\right] \left(A_9 + T_9\, h(\hat{m}_c^2,\hat{s}) + U_9\, h(1,\hat{s}) + W_9\, h(0,\hat{s})\right) \\ - \frac{\alpha_s(\mu)}{4\,\pi} \left(C_1^{(0)} F_1^{(9)} + C_2^{(0)} F_2^{(9)} + A_8^{(0)} F_8^{(9)}\right) ,$$

$$\widetilde{C}_7^{\text{eff}} = \left[1 + \frac{\alpha_s(\mu)}{\pi}\,\omega_7(\hat{s})\right] A_7 - \frac{\alpha_s(\mu)}{4\,\pi} \left(C_1^{(0)} F_1^{(7)} + C_2^{(0)} F_2^{(7)} + A_8^{(0)} F_8^{(7)}\right) ,$$

$$\widetilde{C}_{10}^{\text{eff}} = \left[1 + \frac{\alpha_s(\mu)}{\pi}\,\omega_9(\hat{s})\right] A_{10} \,.$$

The function $h(\hat{m}_c^2, \hat{s})$ is defined in [5], where also the values of A_7, A_9, A_{10}, T_9, U_9 and W_9 can be found. $C_1^{(0)}$, $C_2^{(0)}$ and $A_8^{(0)} = \widetilde{C}_8^{(0,\text{eff})}$ are taken from [7].

6. Numerical Results

The decay width in eq (5.1) has a large uncertainty due to the factor $m_{b,\text{pole}}^5$. Following common practice, we consider the ratio

$$R_{\text{quark}}(\hat{s}) = \frac{1}{\Gamma(b \to X_c\, e\, \bar{\nu}_e)} \frac{d\Gamma(b \to X_s \ell^+ \ell^-)}{d\hat{s}} \,,$$

in which the factor $m_{b,\text{pole}}^5$ drops out. $\Gamma(b \to X_c\, e\, \bar{\nu}_e)$ can be found eg in [5].

In Fig. 1 we investigate the dependence of $R_{\text{quark}}(\hat{s})$ on the renormalization scale μ_b for $0.05 \le \hat{s} \le 0.25$. The solid lines take the new NNLL contributions into account, whereas the dashed lines include the NLL results combined with the NNLL corrections to the matching conditions [5], only. The lower, middle and upper line each correspond to $\mu_b = 2.5,\ 5$ and 10 GeV, respectively, and $\hat{m}_c = 0.29$. From this figure we conclude that the renormalization scale dependence gets reduced by more than a factor of 2. For the integrated quantity we get

$$R_{\text{quark}} = \int_{0.05}^{0.25} d\hat{s}\, R_{\text{quark}}(\hat{s}) = (1.25 \pm 0.08) \times 10^{-5} \,,$$

where the error is obtained by varying μ_b between 2.5 GeV and 10 GeV. Not including our corrections, one finds $R_{\text{quark}} = (1.36 \pm 0.18) \times 10^{-5}$ [5]. In other words, the renormalization scale dependence got reduced from $\sim \pm 13\%$ to $\sim \pm 6.5\%$. The largest uncertainty due to the input parameters is induced by $\hat{m}_c$. Fig. 2 illustrates the dependence of $R_{\text{quark}}(\hat{s})$ on $\hat{m}_c$. The dashed, solid and dash-dotted lines correspond to $\hat{m}_c = 0.27$, $\hat{m}_c = 0.29$ and $\hat{m}_c = 0.31$, respectively, and $\mu_b = 5$ GeV. We find an uncertainty of $\pm 7.6\%$ due to $\hat{m}_c$.

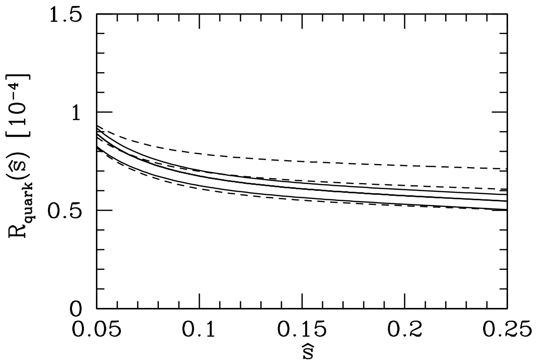

Figure 1: Dependence of $R_{\text{quark}}(\hat{s})$ on μ_b.

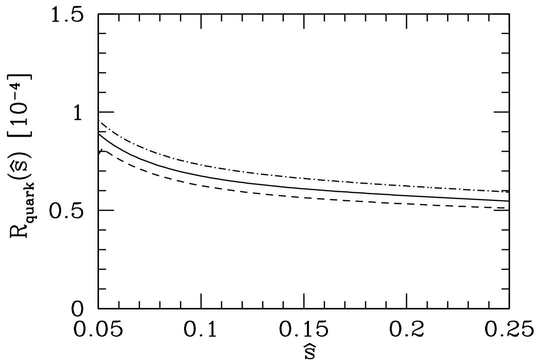

Figure 2: Dependence of $R_{\text{quark}}(\hat{s})$ on $\hat{m}_c$.

We conclude with the remark that the results presented in this exposition have recently been included in a systematic description of the corresponding exclusive decay mode $B \to K^* \ell^+ \ell^-$ [10, 11].

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