# Non-local Duality In B Decays In The t'Hooft Model

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ABSTRACT: We average the decay width of a heavy meson in the t' Hooft model over the heavy quark mass $M$ with a smooth weight function. The averaging has support over a few resonance spacings. We use the previously determined heavy meson decay width which differs from the free quark width by a $1/M$ correction. In contrast, we find that the averaged meson and quark widths differ by a $1/M^2$ correction. We speculate on the relevance of our results to the phenomenologically relevant case of $3+1$ dimensional QCD

Quark-hadron duality has been part of the lore of strong interactions for three decades. Bloom and Gilman[1, 2] (BG) discovered duality in electron-proton inelastic scattering. There, the cross section is given in terms of two Lorentz invariant form factors $W_1$ and $W_2$ which are functions of the invariant mass of the virtual photon, $q^2$, and the energy transfer to the electron, $\nu$. Considering the form factors as functions of the scaling variable $\omega \equiv q^2/2M\nu$, they compared the scaling regime of large $q^2$ (and large $\nu$) with the region of fixed, low $q^2$. They determined that, for each form factor, the low $q^2$ curves oscillate about the scaling curve, that identifiable nucleon resonances are responsible for these oscillations and that the amplitude of a resonant oscillation relative to the scaling curve is independent of $q^2$. Moreover, they introduced sum rules whereby integrals of the form factors at low and large $q^2$ agree and noticed that the agreement was quite good even when the integration involved only a region that spans a few resonances.

Poggio, Quinn and Weinberg[3] (PQW) applied these ideas to electron-positron annihilation. While BG compared experimental curves among themselves, PQW compared the experimental cross section to a scaling curve calculated in QCD. They noticed that the weighted average of the cross section $\sigma(s)$,

$$\bar{\sigma}(s) = \int_0^\infty ds' \frac{\sigma(s')}{s+\Delta} \tag{1}$$

is given in terms of the vacuum polarization of the electromagnetic current with complex argument,

$$\bar{\sigma}(s) = i(\Pi(s+i\Delta) - \Pi(s-i\Delta)), \tag{2}$$

and argued that one can safely use perturbation theory to compute this provided $\Delta$ is large enough. This procedure was better understood with the advent of Wilson's[4] Operator Product Expansion (OPE). It is interesting to point out that the prediction of PQW based on the two generations of quarks and leptons known at the time did not successfully match the experimental results. When PQW allowed for additional matter they found a best match if they supplemented the model with a heavy lepton and a charge 1/3 heavy quark, anticipating the discovery of the tau-lepton and b-quark. It is also interesting that PQW did not include in their average of data the contribution of charm-onium resonances. When this is done the average cross section is raised significantly at low $s$, leaving the higher $s$ region unaffected, as shown in Fig. 1.

In an attempt to understand the origin of quark-hadron duality we have computed both the actual rate and its "scaling limit" from first principles in special situations. In Ref. [5] we computed the semi-leptonic decay rate and spectrum for a heavy hadron in the small velocity (SV) limit. We showed that two channels, $B \to De\nu$ and $B \to D^*e\nu$, give the decay rate to first two orders in an expansion in $1/m_b$ and that to that order the result is identical to the inclusive rate obtained using a heavy quark OPE as introduced in Ref. [6]. The equality holds for the double differential decay rate if it is averaged over a large enough interval of hadronic energies. The computation demonstrates explicitly quark-hadron duality in semi-leptonic $B$-meson decays in



**Figure 1:** Normalized $e^+e^-$ cross section averaged over the squared center of mass energy $s$ as in Eq. (1) with $\Delta = 3\text{GeV}^2$. The red (dashed) curve is from PQW[3]. The green (solid) curve is ours and shows the effect of including the narrow charm-onium resonances.

the SV limit, but really sheds no light into the mechanism for duality. In particular, it is puzzling that duality holds even if the rate is dominated by only two channels.

More recently we attempted to verify duality in hadronic heavy meson decays. In Ref. [7] we considered the width of a heavy meson in a soluble model that in many ways mimics the dynamics of QCD, namely an $SU(N_c)$ gauge theory in $1+1$ dimensions in the large $N_c$ limit. This model, first studied by 't Hooft[8], exhibits a rich spectrum with an infinite tower of narrow resonances for each internal quantum number, making the study of duality viable. We considered a '$B$-meson' with a heavy quark $Q$ and a light (anti-)quark $q$ of masses $M_Q$ and $m$, respectively, which decays via a weak interaction into light $\bar{q}q$ mesons. To leading order in $1/N_c$ the decay rate is dominated by two body final states:
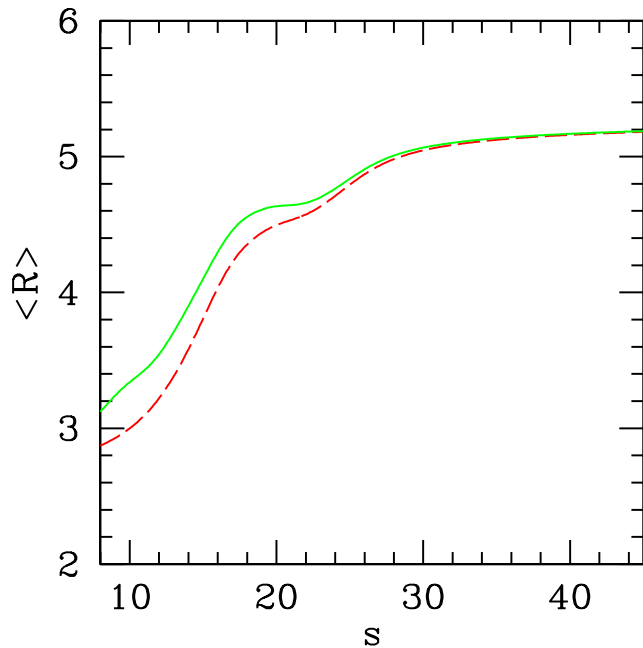
if $\pi_j$ denote the tower of $\bar{q}q$-mesons, the total width is given by $\Gamma(B) = \sum \Gamma(B \to \pi_j \pi_k)$, where the sum extends over all pairing of mesons such that the sum of their masses does not exceed the $B$ mass, $\mu_j + \mu_k < M_B$. The main result of that investigation was that there is rough agreement between $\Gamma(B)$ and the decay rate of a free heavy quark, $\Gamma(Q)$. When considered as functions of $M_Q$ the quark rate is smooth but the meson rate exhibits sharp peaks whenever a threshold for production of a light pair opens up. This is due to the peculiar behavior of phase space in $1 + 1$ dimensions, which is inversely proportional to the momentum of the final state mesons. Nevertheless, in between such peaks it was found that the relation $\Gamma(B) = \Gamma(Q)(1 + 0.14/M_Q)$, in units of $g^2 N_c/\pi = 1$, holds fairly accurately.

In this brief paper we consider the effect of local averaging on the results of Ref. [7]. The main result is that when averaged locally over the heavy mass $M_Q$ the agreement between $\Gamma(B)$ and $\Gamma(Q)$ is parametrically improved. In fact, for the averaged widths we find

$$\langle \Gamma(B) \rangle \approx \langle \Gamma(Q) \rangle \left[ 1 + \frac{0.4}{M_Q^2} + \frac{5.5}{M_Q^3} \right] \tag{3}$$

Remarkably, the correction of order $1/M_Q$ has disappeared.

The computation uses the numerical results of Ref. [7]. The averaging is defined by

$$\langle \Gamma(M) \rangle = \frac{\int_{x_{\min}}^{x_{\max}} dx\, x^n e^{-(x-M)^2/\sigma^2} \Gamma(x)}{\int_{x_{\min}}^{x_{\max}} dx\, x^n e^{-(x-M)^2/\sigma^2}}. \tag{4}$$

The limits of integration are the lowest and highest heavy masses available from Ref. [7]. The width $\sigma$ was taken to be $\sigma = 1$, the scale of the strong interactions in our units. The integer $n$ was varied between $-2$ and $1$ to study the effect of emphasizing low or high masses in the average. The result is shown in Fig. 2 where $\log[\langle \Gamma(B) \rangle - \langle \Gamma(Q) \rangle]$ is plotted versus $\log(M_Q)$. The peculiar behavior at low and large $M_Q$ (outside $1.8 < \log M_Q < 2.3$) is due to the finite endpoints in the integral defining the average as can be readily checked by averaging simple smooth curves. The nearly straight intermediate region is therefore what concerns us. The straight line in Fig. 2 is a linear fit by eye, which is accurate enough since there is some variation in the results depending on the value of the power $n$ used and what range of $\log M_Q$ is fitted. We have checked that small variations in the linear fit do not alter our conclusions. We then find the best fit of the exponential of this line to the function $a + b/M_Q + c/M_Q^2$, yielding $a = -0.01$, $b = 0.47$ and $c = 4.69$. The value of $a$ is consistent with zero, and the quality of the fit is improved by dropping this degree of freedom (and gives Eq. (3)).

Since, when plotted as functions of $M_Q$, $\Gamma(B)$ is (almost) always above $\Gamma(Q)$ and the difference seems constant except at the narrow peaks of $\Gamma(B)$, the question immediately arises as to how the averaging procedure can turn the constant difference into one that decreases as $1/M_Q$. The answer is suggested by the plot in Fig. 1. Recall the average curve of PQW does not include the effect of the narrow charm-onium resonances. Our curve includes these narrow peaks in the averaging and when compared to the curve of PQW it is enhanced at low $s$. The same is apparently occurring in the averaging of $\Gamma(B)$. The area under the narrow peaks at low $M_Q$ is larger than at high $M_Q$, tilting the curve slightly.

The result of Ref. [7] was criticized in Ref. [9] (see also Ref. [10]). The inconsistency between the analytic results of Ref. [9] and the numerical results of Ref. [7] was blamed on numerical inaccuracies of the latter. However, it is clear now that this is unlikely. It seems impossible to understand what error, random or systematic, would conspire to change the behavior of the difference between widths from constant for non-averaged widths to decaying as $1/M_Q$ for averaged widths.

The main result of Ref. [9], that there are no $1/M_Q$ corrections to the partonic width, is easy to derive. Consider the decay $Q \to q + q'\bar{q}'$ where the weak interaction giving rise to the decay is from the operator $\bar{q}\gamma^\mu Q \ \bar{q}'\gamma_\mu q'$, and take the mass of the $q'$ quarks to vanish. In this limit the current $\bar{q}'\gamma_\mu q'$ couples only to the lowest meson in the tower of $\bar{q}'q'$ states, which is massless. This follows from current conservation: from $\langle 0|\bar{q}'\gamma_\mu\gamma_5 q'|p\rangle = fp_\mu$ it follows that $fp^2 = 0$, so $f \neq 0$ only if $p^2 = 0$. It follows that the heavy meson decays into pairs of the form $\phi\pi_n$ where $\phi$ is the massless $\bar{q}'q'$ meson and $\pi_n$ are the mesons in the tower of $\bar{q}q''$ states, where $q''$ is the (light) spectator quark. The amplitude for $B \to \phi\pi_n$ is therefore determined by the $B \to \pi_n$ form-factor at momentum transfer $q^2 = 0$. The calculation of $\Gamma(B)$ is therefore identical
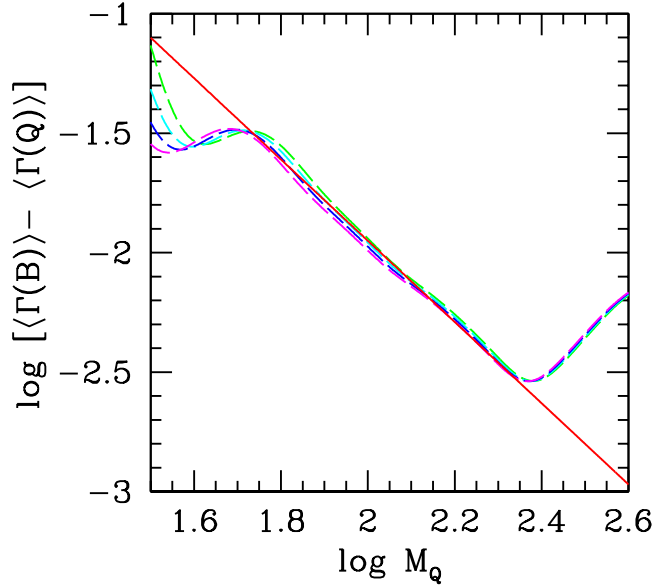


**Figure 2:** Log-difference of averaged meson and quark decay rates as a function of $\log(M_Q)$ (dashed lines), and straight line fit (solid line). The different dashed lines correspond to powers $n = -2, -1, 0$, and $1$ in the average of Eq. (4) ($n = -2$ is highest at small $M_Q$). The straight line is $-1.7\log(M_Q) + 1.45$.

to the semi-leptonic width at zero $e\nu$ invariant mass. This is guaranteed to have no $1/M_Q$ corrections by the argument of Ref. [6].

Ref. [9] attempts to extend this result to the case $m' \neq 0$. This is a very delicate matter and we disagree with the procedure presented there. For example, the perturbative calculations are out of control. The dimensionless expansion parameter in the 't Hooft model is $g^2 N_c/\pi m^2$, where $m$ is the smallest quark mass in the problem. If all light quarks have mass $m$, the leading contribution to the heavy quark width at one loop is (after a lengthy computation)

$$\Gamma(Q)^{(1-\text{loop})} = \frac{3}{2}\left(\frac{g^2 N_c}{\pi m^2}\right)\Gamma(Q)^{(\text{tree})}. \tag{5}$$

It is dangerous to study the case of non-vanishing light quark masses as perturbations about vanishing mass. Still, $\Gamma(Q)$ is a good approximation to $\Gamma(B)$, even though perturbation theory clearly breaks down at small $m$, because the re-summation of the gluon ladder that

gives the quark form factor of a vector current is well approximated by the free quark form factor.

We speculate on the relevance of our results to the phenomenology relevant case of $3 + 1$ dimensional QCD. The heavy meson width is not expected to have singular peaks (as a function of $M_Q$) as in the 't Hooft model. However, it is entirely possible that it oscillates about the quark-dual rate. We cannot rule out, and indeed this work suggests it is entirely possible, that the amplitude of these oscillations decreases with $M_Q$ only as one inverse power. Once averaged over $M_Q$ with a weight function of width of order 1 GeV the rate may display oscillations that decrease faster with $M_Q$, say, as $1/M_Q^2$, as apparently indicated by "practical OPE" arguments[9]. However, it is impractical to average over heavy quark masses, thus leaving us with uncertainties of order $1/M_Q$ in our predictions of hadronic widths. Some evidence for this was presented in Ref. [13] where it was observed that the $b$-quark width agrees better with experimental hadronic widths if the quark mass is replaced by the $B$ or $\Lambda_b$ masses, respectively. In a similar vein, Ref. [14] shows how $1/M$ violations to local, but not global, duality may occur in $B$-meson correlations.

In summary, we have shown that quark-hadron duality in heavy meson decays in the 't Hooft model is accurate only to order $1/M_Q$, but the accuracy is promoted to order $1/M_Q^2$ for local averages over $M_Q$ of the widths. We speculate that the same effect occurs in four dimensional QCD. Since averaging over $b$-quark masses is impossible, it is safe to assume that the heavy meson and baryon widths are at best computed with error of order $1/M_Q$.

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