Bulk Observers In Non-Factorizable Geometries

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Abstract: We study the construction of effective theories at distances much longer than the transverse size of five dimensional non-factorizable geometries. An observer unable to resolve the transverse direction can only measure distances along the parallel dimensions, but the non-factorizable geometry makes the length of a curve along the parallel dimension sensitive to where on the transverse direction the curve lies. We show that all long geodesics that differ in their endpoints only by shifts along the transverse direction have the same length to within the observer's resolution. This allows us to present a consistent interpretation of what is measured by observers that live either on a brane or in the bulk.

Dedicated to the Memory of Detlef Nolte

Introduction and Conclusions. Randall and Sundrum have proposed a solution to the hierarchy problem based on a Z_2 five-dimensional orbifold with 3-branes at the fixed points[1]. There is a negative cosmological constant Λ and tension on the branes, $V_{\rm hid}$ and $V_{\rm vis}$. Then the metric

$$ds^2 = G_{MN}dx^M dx^N = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{1}$$

solves Einstein's equations provided $a(y) = e^{-k|y|}$ and

$$V_{hid} = -V_{vis} = 24M^3k \quad \Lambda = -24M^3k^2 \tag{2}$$

where M is the fundamental 5-dimensional gravitational mass scale. The brane with negative tension, V_{vis}, contains the visible universe and is located at $y = y_c$ while the hidden

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brane, with tension V_{hid}, is at $y = 0$. We will refer to this as the RS model, to the y-coordinate as the transverse direction and to the remaining ones as the parallel directions, since they are parallel to the bounding branes.

In situations of physical interest the transverse direction is microscopic. Macroscopic observers cannot resolve lengths as small as the transverse dimension. Effectively they see a world of four dimensions. Naively one would attempt to describe physics at long distances by ignoring the transverse direction. This is, after all, what is commonly done to infer the spectrum of Kaluza-Klein excitations in factorizable geometries. But for non-factorizable geometries the invariant interval of a line between the points $(0, y_1)$ and $(\Delta x, y_2)$ depends on the transverse direction y. This dependence can give results for the length that vary up to a factor of $\exp(ky_c)$. The macroscopic observer is, however, unable to discern the value of y. We are faced with the question: between two points 0 and Δx what is the length measured by the macroscopic observer? Many physical questions require an answer to this. For example, what does the argument of a field, $\phi(x)$, in an effective theory represent? Since mass is the inverse length for exponential fall-off of a static field, what is the mass of a field? Although much of the physics of these models has been understood[2], most of the research has been focused on the construction of effective field theories for brane observers and not bulk observers. For an observer restricted to a brane the answer is well known. Distances measured by an apparatus made solely of components restricted to a brane correspond to distances inferred by the induced metric on the brane. But an observer living in the bulk will be spread out over the microscopic transverse direction and it is not obvious which $4 - dim$ metric will give the correct physical distances. Our main result is as follows: distances measured by an apparatus made of unrestricted components (bulk fields) correspond to distances measured on the visible brane, provided these distances are much larger than the brane separation. Specifically, the distance between points $(0, y)$ and $(\Delta x, y')$ is, to high accuracy, $\exp(-ky_c)\Delta x$ for any y and y', provided $\exp(-ky_c)\Delta x \gg y_c$.

Brane Observers and Bulk Observers. Effective theories at long distances are meant to accurately describe the world as seen by an observer who has limited resolution. In a non-factorizable geometry we must make a distinction between observers that live on one brane or the other and observers that live in the bulk. We will refer to observers whose measuring devices are constructed solely of fields and particles constrained to a brane as "brane observers," and to those whose measuring devices are built of unconstrained fields and particles as "bulk observers." As we will see there are some qualitative and quantitative differences in the observations they make, so the distinction is important.

We imagine that either observer can make a construction, following Einstein, of locally inertial frames with identical meter sticks and synchronized clocks. To better understand the construction it will be necessary to know how to describe freely falling observers and how observers measure distances and time. The former entails finding geodesics while the latter requires finding the minimum distance between points on a fixed time hypersurface, that is, geodesics of spatial sections.

We write the geodesic equation in the RS model for $0 < y < y_c$, and the case of

$-y_c < y < 0$ can be obtained formally by replacing $k \to -k$:

$$\frac{d^2x^\mu}{d\tau^2} - 2k\frac{dy}{d\tau}\frac{dx^\mu}{d\tau} = 0, \qquad \frac{d^2y}{d\tau^2} - ke^{-2ky}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{3}$$

In terms of coordinate time, $t = x^0$, the solutions are straight line uniform motion in the large three spatial dimensions and accelerated motion in the transverse (fifth) dimension:

$$x^i = x_0^i + \frac{v_0^i}{v_0^0}(t - t_0), \qquad y = \frac{1}{2k}\ln\left(\frac{1}{v_0^2} + \frac{v_0^2}{v_0^{02}}k^2(t-t_0)^2\right). \tag{4}$$

If $\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ then $dy/d\tau$ must be constant. But then $ds^2/d\tau^2 \geq 0$ only if $dy/d\tau = 0$. So this describes only massless particles (light-like geodesics). This is consistent with the observation that the Klein-Gordon equation in the warped background admits y-independent solutions only for massless particles. Light-like geodesics are given, for $\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} > 0$, by

$$x^\mu = x_0^\mu + \frac{v_0^\mu}{k^2v_0^2\tau}, \qquad y = -\frac{1}{k}\ln(\pm k\sqrt{v_0^2}\tau). \tag{5}$$

What is the physical distance between points on this space? Consider two points on a fixed time hypersurface, say $t = 0$, separated by some large coordinate distance Δx. It would seem that the physical distance depends on the transverse coordinate y of these points. If so, how can one build an effective four dimensional theory at physical distances much larger than y_c? The answer, as we now show, is that at large Δx, the physical distance these points is $\exp(-ky_c)\Delta x$, *independent* of the y-coordinates of these points. The correction to this statement is of order y_c, so an observer without the resolution to observe the fifth dimension is also oblivious to this correction.

On a fixed t hypersurface the geodesic equations are

$$\frac{d^2x^i}{d\tau^2} - 2k\frac{dy}{d\tau}\frac{dx^i}{d\tau} = 0, \qquad \frac{d^2y}{d\tau^2} + ke^{-2ky}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0, \tag{6}$$

with solutions

$$x^i(\tau) = x_0^i + \frac{\beta^i}{k^2\beta^2}\tanh[\omega(\tau-\tau_0)], \qquad y(\tau) = -\frac{1}{k}\ln(k\beta\cosh[\omega(\tau-\tau_0)]), \tag{7}$$

where $\beta \equiv \sqrt{\delta_{ij}\beta^i\beta^j}$ and τ is an affine parameter, $\tau \in [0,1]$. The parameter ω is simply the physical length ℓ in units of k along the geodesic, $\omega = k\ell$. To understand the solution consider the case $y(0) \leq y(1)$ which is described by $\tau_0 \geq 1/2$. If $\tau_0 \geq 1$ the coordinate $y(\tau)$ increases monotonically from $y(0)$ to $y(1)$, while for $1/2 \leq \tau_0 < 1$ the solution $y(\tau)$ increases monotonically from $y(0)$ beyond $y(1)$ (at $\tau = 2\tau_0 - 1$) to a maximum at $y(\tau_0)$ and decreases monotonically back to $y(1)$. The transition between these two distinct behaviors occurs at $\tau_0 = 1$, that is, at a critical x separation given by

$$k|\Delta x|_{\rm crit}e^{-ky(1)} = \sqrt{1 - e^{-2k(y(1)-y(0))}}. \tag{8}$$

For $|\Delta x| < |\Delta x|_{\rm crit}$ the solution has $\tau_0 > 1$.

The solutions to the geodesic equation above do not describe the minimum distance path between points in the orbifold of the RS model because it neglects the presence of fixed points. At large separation $|\Delta x|$ the solution above extends to the region $y > y_c$. The shortest path is actually along a $\tau_0 = 1$ geodesic from $y(0)$ to y_c, then along the brane $y = y_c$ and finally back from y_c to $y(1)$ along a second ($\tau_0 = 0$) geodesic. The shortest path between two points has length

$$\ell_{\rm phys} = |\Delta x| e^{-ky_c} + (2y_c - y(0) - y(1) + \ln 4 - 2) + \mathcal{O}(e^{-k(y_c - \min(y(0),y(1)))}). \quad (9)$$

This applies only provided the length along x is large enough that the above geodesic would hit the $y = y_c$ brane. The condition for this is, from Eq. (8), that the length measured along the $y = y_c$ brane be larger than $1/k$.

We can now understand what a brane and a bulk observer are. First, Eq. (4) tells us that if we ignore the y-motion of a particle in the bulk the motion in the transverse space is like that of a free particle in flat space. Bulk observers, being presumably much bigger than y_c, are made of many particles distributed over the y-direction. These particles, however, do not spread in x^i if they have the same initial velocity (a fortunate state of affairs for the observer). Second, Eq. (9) tells us that to an accuracy better than his resolution, this bulk observer's meter stick does not depend on the y-coordinates of its endpoints points.

Third, Eq. (9) shows that the distance measured by the bulk observer between two points is what a negative tension brane observer (that is, an observer on the $y = y_c$ brane) would measure (between the points projected onto the brane). The distance between these points measured by the positive tension brane observer is, however, exponentially larger.

Last, similar conclusions hold for measurement of time, since motion is uniform in $x^i(t)$. Observers can measure two units of time by reflecting a pulse of light off a mirror set at the end of a meter stick.

Interpretation Of The Effective Action. Consider the bulk scalar action

$$S = \frac{1}{2}\int d^4x dy \sqrt{G}\left(G^{AB}\partial_A\Phi\partial_B\Phi - M^2\Phi^2\right). \quad (10)$$

Instead of formally integrating out the y-direction, we study the response of the bulk scalar to arbitrary sources smeared over the resolution of the detecting apparatus and separated by distances larger than the resolution. The Green function of the Klein-Gordon equation, $\Delta(x, y; x', y')$ can be written in terms of the four dimensional Green function for a particle of mass m_n, $\Delta^{(4)}(x - x'; m_n)$, as[3]

$$\Delta(x, y; x', y') = -\sum_n \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot(x-x')}}{q^2 - m_n^2} R_n(y, y') = \sum_n \Delta^{(4)}(x - x'; m_n) R_n(y, y'). \quad (11)$$

Since the residue factorizes, $R_n(y, y') = r_n(y) r_n(y')$, the full Green function $\Delta(x, y; x', y')$ can be obtained in a four dimensional description by coupling a source $j(x)$ to the linear combination $\sum_n r_n(y)\psi_n(x)$. The spectrum, m_n, is determined as solutions to[4]

$$\tilde{N}_\nu(m_n z_1)\tilde{J}_\nu(m_n z_2) - \tilde{J}_\nu(m_n z_1)\tilde{N}_\nu(m_n z_2) = 0, \quad (12)$$

where $z_{1,2} = e^{ky}/k|_{y=0,y_c}$, $\nu = \sqrt{4 + m^2/k^2}$, and $\tilde{Z}_\nu(z) = (1 - \frac{\nu}{2})Z_\nu(z) + \frac{z}{2}Z_{\nu-1}(z)$ for Z_ν a Bessel function, J_ν or N_ν. For small m and n, one has $m_n \sim a(y_c)k$.

By the arguments of the previous section, bulk observers and observers on the visible (negative tension) brane see particles of physical mass $m_n/a(y_c) \sim k$. It is only observers on the hidden (positive tension) brane who see exponentially suppressed masses. Physically, they see masses that have climbed up a potential well and are therefore red-shifted precisely by the warp factor.

Generalizations Similar results are obtained for a wide class of non-factorizable geometries. A simple case is that in which a space built on an S_2/Z_2 orbifold,

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{13}$$

has a warp factor A that is locally a minimum at both fixed points. At large parallel separations the bulk observers see distances as measured along the brane of smaller warp factor.

An interesting case is the metric for a space with a single brane at a fixed point in Anti-de Sitter 5-space (AdS_5) with Anti-de Sitter (AdS_4) sections[5]. The metric is as in Eq. (13), but with η replaced by the metric of AdS_4. The warp factor is

$$A = \log\left(\cosh\frac{c-|y|}{L}\Big/\cosh\frac{c}{L}\right), \tag{14}$$

decreasing from the brane to $y = c$ where it has a minimum and then growing again. Fixed-t geodesics at large parallel separation Δx go mostly very close to the hypersurface $y = c$. There is no brane located there, still distance scales for a bulk observer are as if measured along a $y = c$ brane. The graviton is localized on the brane at $y = 0$[6]. The spectrum of massive excitations of the graviton is observed to be exponentially lighter by a brane observer than by a bulk observer, giving an amusing inverted hierarchy.

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