$K \to \pi\pi$ matrix elements from lattice QCD with Wilson fermions[*]

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ABSTRACT: We present the status of a lattice calculation for the $K \to \pi\pi$ matrix elements of the $\Delta S = 1$ effective weak Hamiltonian which are relevant in the determination of ϵ'/ϵ and for the $\Delta I = 1/2$ rule, directly with two pion in the final state. For $\Delta I = 3/2$ matrix elements, we propose a strategy to study the chiral behaviour at next to leading order in ChPT. Some preliminary results for the matrix elements of Q^+, Q_7 and Q_8 are given. Finally, we briefly discuss the $\Delta I = 1/2$ case which, due to the requirement of non-perturbative subtraction, is much more difficult. We show the signal observed for the matrix elements of Q^- and Q_6.

1. General Strategy

Kaon weak decay amplitudes can be described in terms of matrix elements (ME's) of the $\Delta S=1$ effective weak Hamiltonian. $\mathcal{H}^{\Delta S=1}$ is written as a linear combination of a complete basis of renormalized local operators (OP's) $\hat{Q}_i(\mu)$, where μ is the renormalization scale. The most relevant contributions in the computation of the weak amplitudes $\mathcal{A}_{I=0,2}$ and of ϵ'/ϵ are given by the ME's $\langle\pi\pi|\hat{Q}_i(\mu)|K\rangle_{I=0,2}$ of $\hat{Q}^+$, $\hat{Q}^-$, $\hat{Q}_6$, $\hat{Q}_7$ and $\hat{Q}_8$, which are defined as follows:

$$Q^{\pm} = (\bar{s}u)_L(\bar{u}d)_L \pm (\bar{s}d)_L(\bar{u}u)_L - (u \to c) \qquad Q_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_{q=u,d,s,c} e_q(\bar{q}_\beta q_\beta)_R$$

$$Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_{q=u,d,s,c}(\bar{q}_\beta q_\alpha)_R \qquad Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_{q=u,d,s,c} e_q(\bar{q}_\beta q_\alpha)_R \qquad (1.1)$$

where α, β are colour indices and e_q is the electric charge of q. $(\bar{\psi}_1\psi_2)_{L,R}$ means $\bar{\psi}_1\gamma_\mu(1 \mp \gamma_5)\psi_2$. In order to compute these ME's from lattice QCD, one has to renormalize the bare (divergent) lattice OP's. Two methods are possible on the lattice:

1) Compute $\langle 0|\hat{Q}_i(\mu)|K\rangle$ and $\langle\pi|\hat{Q}_i(\mu)|K\rangle$ and then derive $\langle\pi\pi|\hat{Q}_i(\mu)|K\rangle_{I=0,2}$ using soft pion theorems [1]. In this case, besides the difficulties in controlling chiral behaviour, a major problem is the inclusion of higher order contributions in the chiral expansion which

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should reconstruct the final state interaction (FSI). In fact, in this approach, only $K \to \pi\pi$ ME's at lowest order in ChPT can be obtained (see Refs. [2]).

2) Compute directly $\langle \pi\pi | \hat{Q}_i(\mu) | K \rangle_{I=0,2}$. The main difficulty in this case is the relation between ME's in a (Euclidean) finite volume and the corresponding physical infinite volume ones. Even if, in principle, it is possible to take into account FSI exactly [3, 4], in practice these methods are numerically very demanding. So, with present computing power, only simulation at unphysical kinematics may be achieved, and ChPT is still needed to extrapolate to the physical point.

In this work we choose the second strategy. To extract ME's we place the local OP Q_i in the origin and we use three local interpolating fields, one at time $t_2 = 10$ which annihilates one pion in the two pion state and one at time $t_K = 54 \equiv -10$ which creates a K (both with zero momentum). t_2 (t_K) must be chosen large enough in modulus for the two pion state (the kaon state) to be asymptotic. The third one annihilates at time t_1 (not fixed) the second pion with momentum either 0 or $2\pi/L$. As shown in [4], in the limit $T/2 \gg t_1 \gg t_2 \gg 0$, $T \gg t_K \gg T/2$ we have

$$\frac{\langle 0 | T[\hat{\pi}_1(t_1) \hat{\pi}_2(t_2) Q_i(0) \hat{K}(t_K)] | 0 \rangle_V}{G_\pi(t_1) G_\pi(t_2) G_K(t_K)} Z_\pi^2 Z_K \longrightarrow \left\{ |\langle \pi\pi | Q_i | K \rangle| \cos\delta(W) + O\left(\frac{1}{L}\right) \right\} e^{-(W - E_\pi - M_\pi)t_2}$$

where $G_\pi(t) = \langle 0 | T[\hat{\pi}(t)\hat{\pi}^\dagger(0)] | 0 \rangle_V$, $Z_\pi = |\langle 0 | \hat{\pi} | \pi \rangle|$ (the definition of $G_K(t)$ and Z_K is analougous), $|\pi\pi\rangle$ is the lower two pion state with the same momentum injected by $\hat{\pi}_1$. $\Delta E = W - E_\pi - M_\pi$ is the energy shift of two pions in a finite volume (FV) while $\delta(W)$ is the strong interaction phase (which depends on the isospin and on the energy of the two pion state). In order to extract the physical ME, one needs to compute both the FV corrections for the ME (represented above by the term $O(1/L)$) and the FV energy shift. Theoretical predictions [5] show that the factor $\exp(-\Delta E t_2)$ should give relevant corrections: of order $5 \div 10\%$ in the $\Delta I = 3/2$ channel and of order 25% (or larger) in the $\Delta I = 1/2$ one. A numerical study of the FV energy shift, for both $I = 2$ and $I = 0$ two pion states, is presently under way [6]. Preliminary results for ΔE are in good agreement with theoretical predictions for $I = 2$, while they are in stricking disagreement in the $I = 0$ case, where even the dependece of the energy shift on the mass of the pion is different to the one predicted. Since we work with large unphysical pion masses ($500 - 700$ MeV), this problem coluld be due to the presence of a stable scalar particle below the two pion state. In this case the behaviour of the shift measured should have only exponentially small correction in the volume. Instead the "genuine" FV energy shift of the two pion state has a power dependence on the volume. To clarify the situation we are thus currently varying the volume of the simulations and trying to reach smaller masses. We are also computing, by using FV quenched ChPT (qChPT), the corrections to the ME indicated above with the $O(1/L)$ term.

2. $\Delta I = 3/2$ ME's at next-to-leading order in ChPT

As explained in the previous section, we extract the ME's of the OP's $Q_i \in \{Q^+, Q_7, Q_8\}$ between K^+ at rest and two pions, one (which can be interpreted either as the π^+ or as the π^0) at rest and the other with momentum $|\vec{p}| = 0$, $2\pi/L$. In the latter case, to have a

pure $I = 2$ state, we symmetrize the two pion state. In the $I = 2$ channel, $\cos \delta(W) = 1$ to a good approximation also in the case of pion with non-zero momentum. In the following analysis, corrections due to the FV energy shift are already included.

The leading term in the chiral expansion is $O(p^2)$ for Q^+ (which, under the chiral $SU(3)_L \otimes SU(3)_R$, transforms in the (27,1) representation) and $O(p^0)$ for $Q_{7,8}$ (which transform in the (8,8) representation). In both cases there is only one representative OP at this order. To include next-to-leading order (NLO) corrections ($O(p^4)$ for Q^+ and $O(p^2)$ for $Q_{7,8}$) in the extrapolation to the physical point, we have to compute the expression for the ME's in one loop ChPT, in our unphysical kinematics. This contains two parts: the chiral logarithms, which come from the loops and are proportional to the leading order coupling; the counterterms of the NLO, which cancel the divergences of the loop integrals and compensate the renormalization scale dependence coming from the logarithms, ensuring the scale indipendence order by order in the expansion. In the (27,1) representation there are 34 OP's at $O(p^4)$ [7]. For our kinematics only $\mathcal{O}_2^{(27)}$, $\mathcal{O}_4^{(27)}$, $\mathcal{O}_5^{(27)}$, $\mathcal{O}_7^{(27)}$, $\mathcal{O}_{22}^{(27)}$ and $\mathcal{O}_{24}^{(27)}$ of [7] are indipendent. In the (8,8) representation, at $O(p^2)$ seven OP's $\mathcal{O}_i^{(8,8)}$ ($i = 1, 2, \ldots, 7$) are needed [8]. Due to the energy-momentum injection in the weak OP, the calculation is much more involved than for the physical kinematics (reported in Ref. [9] for (27,1) OP's and in Ref. [8] for (8,8) OP's). For the sake of illustration we present here the case of (8,8) OP's (analogous expressions can be written for the (27,1) OP's, see [10] for a more detailed presentation) at the physical point

$$\mathcal{M}_{\text{phys}}^{(8,8)} = \gamma + \{4\delta_6 - (\delta_2 + \delta_3) + 2(\delta_4 + \delta_5)\}m_K^2 +$$
$$\{(\delta_1 + \delta_2) + 4(\delta_4 + \delta_5) + 2\delta_6\}m_\pi^2 + \gamma \times (\text{chiral logs})_{\text{phys}}^{(8,8)} \tag{2.1}$$

(where m_π and m_K are the physical masses) and, in the unphysical kinematics,

$$\mathcal{M}_{\text{unphys}}^{(8,8)} = \gamma + \{4(\delta_4 + \delta_5) + 2\delta_6\}M_\pi^2 + \{-(\delta_1 + \delta_2)\}E_\pi M_\pi + \{\frac{1}{2}(\delta_1 + \delta_2) -$$
$$(\delta_2 + \delta_3)\}(M_\pi + E_\pi)M_K + \{2(\delta_4 + \delta_5) + 4\delta_6\}M_K^2 + \gamma \times (\text{chiral logs})_{\text{unphys}}^{(8,8)} \tag{2.2}$$

This expression is function of three variables (M_K, M_π and E_π). By varying the kaon and pion masses and momenta, one may fit the coefficients of the leading order OP and some combinations of the coefficients of the NLO (usually called low energy constants (LEC's), indicated as γ and δ_i in Eq. (2.2)). It is easy to show that these combinations are enough to compute, at NLO, the expression at the physical point (Eq. 2.1).

Until now, our discussion was referred only to infinite volume full ChPT. However, to have the correct comparison with numerical results, a one loop computation in FV qChPT is presently under way. In qChPT with $m_u = m_d \neq m_s$, unphysical divergences appear in the limit $m_u, m_d \to 0$ (see for example [9]). So, it is clear that qChPT can only be used to estimate the LEC's from numerical simulation. This represent our best estimate of the LEC's of the full theory, and full ChPT should thus be used for the extrapolation to the physical point.

Since at present the computation of the chiral logarithms for our kinematics is not yet completed, we give here preliminary results obtained from a fit in which only the contribution of NLO operators is included. We report in Tab. 1 the results of a fit which includes values of M_K, M_π and E_π smaller than 1 GeV. In the case of Q^+ we started

	$C_+\langle\pi^+\pi^0|O^+|K^+\rangle$	$\langle\pi\pi|O_7(2\text{ GeV})|K^0\rangle_{I=2}$	$\langle\pi\pi|O_8(2\text{ GeV})|K^0\rangle_{I=2}$
LO	0.0107(20)	0.05(1)	0.57(6)
NLO/LO	$\approx -(0.05 \div 0.10)$	-0.4(1)	-0.07(2)
expt. value	0.01041		
from $K \to \pi$		0.11(2)	0.51(5)

Table 1: Results are in GeV^3. 340 configurations, $\beta = 6.0$ on a $24^3 \times 64$ volume. Only point with values of M_K, M_π and E_π smaller than 1 GeV are included in the fit. Results from lattice $K \to \pi$ matrix elements are taken from [11]. ME's of $Q_{7,8}$ are in $\overline{\text{MS}}$ NDR.

our analysis with the $O(p^4)$ operators classified in [12], which are the only needed at the physical point. We later realize that in our unphysical kinematics other two operators were neded (namely $\mathcal{O}_{22}^{(27)}$ and $\mathcal{O}_{24}^{(27)}$). Once included in the fit, the determination of the LO contribution remains stable while NLO contribution becomes unstable. For this reason we give here only the order of magnitude of this correction. The low value for the ME of Q_7 originates from large cancellations occurring in the renormalization of the bare lattice operator and deserves further investigation. In Fig. 1.a we show the values in lattice units of the ME's of the bare Q_8, in function of M_K^2, and the corresponding values of the fit with Eq. 2.2 (without chiral logarithms).

3. Very briefly on $\Delta I = 1/2$ ME's

$\Delta I=1/2$ OP's mix, through penguin contractions, also with lower dimensional OP's with power divergent coefficients. In this case a non-perturbative subtraction is needed (in both strategies: $K \to \pi\pi$ or $K \to \pi$). Here we will present preliminary results (obtained with the same set of 340 configurations used for $\Delta I = 3/2$ ME's) for Q^- and Q^6 with the purpose of showing that, for the first time, a signal has been observed.

Q^- enters, together with Q^+ (for their definition see Eq. (1.1)), in the computation of $\text{Re}\mathcal{A}_{I=0}$. In our simulation the charm quark is propagating. Due to the GIM mechanism, the subtraction is thus implicit in the difference of penguin diagrams with an up quark and a charm quark inside the loop. In Fig. 1.b the penguin contractions of the bare O^- are shown, in a kinematical configuration in which $M_K a = 2M_\pi a = 0.72$. It is interesting, even though only indicative, to note that the ratio of bare OP's $\langle\pi\pi|Q^-|K\rangle_{I=0}/\langle\pi\pi|Q^+|K\rangle_{I=2} \approx 9$. In fact, although affected by a huge statistical error and incomplete for the lacking of the RC's and of part of the contractions, this result shows that penguin contractions are of the right order of magnitude needed to explain the $\Delta I=1/2$ rule (remember that the ratio of Wilson coefficients $|C^-(\mu = 2\text{GeV})/C^+(\mu = 2\text{GeV})| \approx 2$). Concerning O^+, penguin contractions gives a value close to zero (but again with large errors).
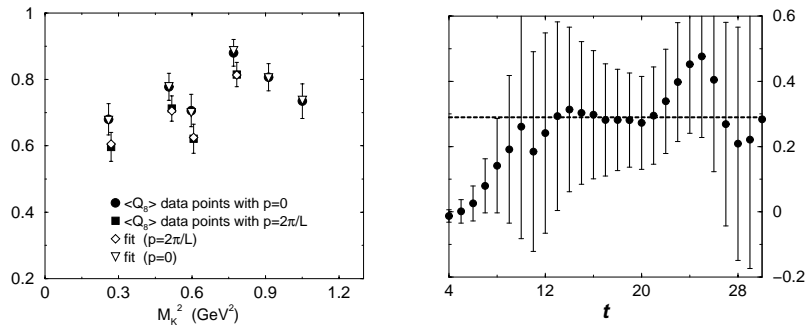

Figure 1: a. values (in lattice units) of the ME's of the bare Q_8 and corresponding fit, as function of M_K^2. b. Plateau for the ME of the bare Q^- (only penguin contractions, scales in lattice units).
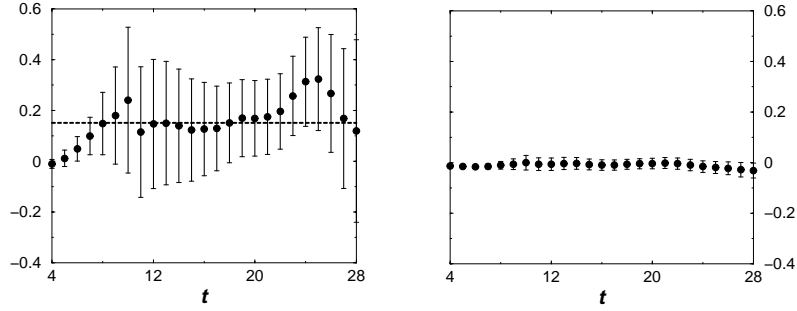
Figure 2:   a. Plateau for the ME of the bare Q^6 (only penguin contractions).   b. Plateau for $\langle\pi\pi|C_p/a^2(m_s{-}m_d)\bar{s}\gamma_5 d|K\rangle$ (all scales are in lattice units).

In Fig. 2.a we show the signal found for the ME of the bare QCD penguin Q_6 (defined in Eq. (1.1)), again with $M_K a = 2M_\pi a = 0.72$. Q^6 mixes in the following way:

$$\tilde{Q}^6 = Q^6 + \frac{C_p}{a^2}(m_s{-}m_d)\bar{s}\gamma_5 d\,. \tag{3.1}$$

The subtraction is determined by imposing that $\langle 0|\tilde{Q}^6|K\rangle = 0$. By using fully $\mathcal{O}(a)$ improved fermions this subtraction is finite. In fact we have

$$\langle\pi\pi|\bar{s}\gamma_5 d|K\rangle = \langle\pi\pi|\frac{\partial_\mu A_\mu^I}{(m_s + m_d)Z_P^I}|K\rangle + \mathcal{O}(a^2)\,,$$

where the ME of $\partial_\mu A_\mu^I$ vanishes on-shell. In Fig. 2.b we can see that the subtraction seems to be not only finite, but also much smaller than $\langle\pi\pi|Q^6|K\rangle$ and thus under control. In order to reduce the statistical errors, a study to find the best source for the two pion state is presently under way. Finally, we want to mention that, in the quenched approximation, there are special problems related to the (8,1) OP's (like Q^- and Q^6) [13] which deserves further investigation in order to extract sensible results.

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