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Discrete non-Abelian groups and asymptotically free models

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Abstract: We consider a two-dimensional σ-model with discrete icosahedral/dodecahedral symmetry. Using the perturbative renormalization group, we argue that this model has a different continuum limit with respect to the $O(3)$ σ model. Such an argument is confirmed by a high-precision numerical simulation.

Recently, there has been interest in the critical behavior of two-dimensional σ-models in which the spins take values in some discrete subset of the sphere. In particular, two groups [1, 2, 3, 4, 5] studied the nearest-neighbor σ-model

$$H = \beta \sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \tag{1}$$

in which the spins have unit length and belong to the vertices of a Platonic solid, i.e. of a tetrahedron, cube, octahedron, icosahedron, or dodecahedron. Several quantities have been computed: the renormalized two-point function, the current-current correlation function, the finite-size scaling (FSS) curve for the second-moment correlation length, and the four-point renormalized coupling. Surprisingly enough, the results for the icosahedral and the dodecahedral model are very close to the $O(3)$ ones, suggesting that these three models might have the same continuum limit. Patrascioiu and Seiler [1, 2, 3] considered these results as evidence for the $O(3)$ σ-model not being asymptotically free, since the discrete-symmetry models have a finite β phase transition, which cannot be described in

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perturbation theory. However, the overwhelming evidence we have collected in the years in favor of asymptotic freedom made Hasenfratz and Niedermayer [4, 5] suggest that, may be, the icosahedral and the dodecahedral models have an asymptotically-free continuum limit, in spite of the fact that the critical point is at a finite value of β.

Here, we wish to show that, by using some standard assumptions, the perturbative renormalization-group (RG) approach predicts that the suggestion of Hasenfratz and Niedermayer cannot be true. If the continuum limit of the $O(3)$ σ-model is correctly described by the perturbative RG, then any discrete-symmetry model cannot belong to the same universality class of the $O(3)$ σ-model.

The argument goes as follows [6]. Consider the Hamiltonian

$$H = \beta \sum_{<i,j>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - h \sum_i I_n(\boldsymbol{\sigma}_i), \tag{2}$$

where $\boldsymbol{\sigma}_i$ is an $O(3)$ unit spin and $I_n(\boldsymbol{\sigma}_i)$ is a polynomial in $\boldsymbol{\sigma}_i$ with the following properties: it has $O(3)$ spin n; the maxima (or minima) of $I_n(\boldsymbol{\sigma}_x)$ correspond to the set of vertices of a Platonic solid; it is invariant under the discrete-symmetry group of the solid. For all Platonic solids, it can be shown explicitly that such a polynomial exists. The model (2) interpolates between the $O(3)$ ($h = 0$) and the discrete-symmetry model ($|h| = +\infty$). Now, with quite standard assumptions, one can show that $I_n(\boldsymbol{\sigma})$ is a *relevant perturbation* of the $O(3)$ fixed point. In other words, any arbitrarily small perturbation with discrete symmetry of the $O(3)$ σ-model drives the system to a different fixed point.

The argument is fairly standard. Consider a p-point connected correlation function $G^{(p)}(\beta, h)$ at zero external momenta in a finite box L^2. If $hL^2 \ll 1$ and $\xi \gg L$, we can compute the correlation function in perturbation theory, obtaining

$$G^{(p)}(\beta, h) = \sum_{i,j=0}^{\infty} t^i h^j \; a_{ij}^{(p)}(L), \tag{3}$$

where $t \equiv 1/\beta$. The coefficients of the expansion diverge as $L \to \infty$, since the infinite-volume correlation function cannot be computed directly in perturbation theory because of infrared divergences. However, by using the perturbative expansion (3), one can show that in the continuum limit $G^{(p)}(\beta, h)$ satisfies the RG equation

$$\left[-a\frac{\partial}{\partial a} + W(t)\frac{\partial}{\partial t} + \gamma^{(n)}(t) h \frac{\partial}{\partial h} + \frac{p}{2}\gamma(t) \right] G^{(p)}(\beta, h) = 0, \tag{4}$$

where $W(t)$, $\gamma^{(n)}(t)$, and $\gamma(t)$ are L-independent RG functions. Then, we make the following assumption:

> The RG equation (4)—but *not* the expansion (3) we started from—is valid for all values of L, including $L = \infty$.

Such an assumption is routinely made in the perturbative analysis of the σ-model and is used, for instance, to obtain the small-t behavior of long-distance quantities, such as the susceptibility, correlation length, and so on. Solving Eq. (4), we obtain in the infinite-volume limit

$$G^{(p)}(\beta, h) = G^{(p)}(\beta, 0)\Phi^{(p)}(z), \tag{5}$$

where

$$z \equiv h t^{\rho} \exp(4\pi/t) \sim h\xi(t)^2 [\log \xi(t)]^{\sigma}, \tag{6}$$

ρ and σ are universal exponents that can be easily computed by using the perturbative results of Ref. [7], $\Phi^{(p)}(z)$ is a nonperturbative crossover function, and $\xi(t)$ is the correlation length for $h = 0$. Precisely, Eq. (5) is valid in the crossover limit $t \to 0$, $h \to 0$, keeping z fixed. Equations (5) and (6) show that $I_n(\boldsymbol{\sigma})$ is a relevant perturbation with RG eigenvalue 2, as expected on the basis of dimensional analysis. Different physical results (i.e. different results for universal quantities) are obtained by varying the variable z, as usual in the vicinity of a point perturbed by two relevant perturbations (more precisely the thermal direction is marginally relevant). Thus, within the standard perturbative approach, the discrete-symmetry model and the $O(3)$ model are expected to have different continuum limits.

The previous argument together with the numerical results of Refs. [1, 2, 3, 4, 5] puts asymptotic freedom on a dangerous ground since it shows that the conventional scenario is wrong if the icosahedral or the dodecahedral models have the some continuum limit of the $O(3)$ model. We have thus decided to extend the previous numerical work and indeed, we have found good evidence that the $O(3)$ model and the discrete-symmetry model belong to different universality classes: the conventional scenario is saved. However, the surprising fact is that these differences appear only very near to the critical point, i.e. for $\xi_\infty \gtrsim 10^5$!

In the numerical simulation we have considered the Hamiltonian (2) with

$$\begin{aligned} I_6(\boldsymbol{\sigma}) &= \sigma_z^6 - 5\sigma_z^4\left(\sigma_x^2+\sigma_y^2\right) + 5\sigma_z^2\left(\sigma_x^2+\sigma_y^2\right)^2 \\ &\quad + 2\sigma_x\sigma_z\left(\sigma_x^4 - 10\sigma_x^2\sigma_y^2 + \sigma_y^4\right), \end{aligned} \tag{7}$$

and $h = 0.1$. Such a polynomial is invariant under the rotation group of the icosahedron and of the dodecahedron. We measured the second-moment correlation length as defined in Refs. [8, 9], and the spin-n susceptibilities

$$\chi_n = \sum_x \langle P_n(\sigma_0 \cdot \sigma_x)\rangle, \tag{8}$$

where $P_n(x)$ is a Legendre polynomial, for $n = 1, 3, 4$. For each observable $\mathcal{O}(L, \beta)$, we considered the so-called step function, i.e. the ratio $\mathcal{O}(2L, \beta)/\mathcal{O}(L, \beta)$, which, in the continuum limit should become a universal function of $\xi(L, \beta)/L$, i.e.

$$\frac{\mathcal{O}(2L,\beta)}{\mathcal{O}(L,\beta)} = F_{\mathcal{O}}\left(\frac{\xi(L,\beta)}{L}\right) + O(L^{-\omega}, \xi^{-\omega}). \tag{9}$$

We measured the step function of the above-mentioned observables in the discrete-symmetry theory (i.e. keeping $h = 0.1$ fixed) and in the $O(3)$ model, thereby extending the results of Refs. [8, 9]. If the two models have the same continuum limit, the function computed for $h = 0.1$ and $h = 0$ should coincide.

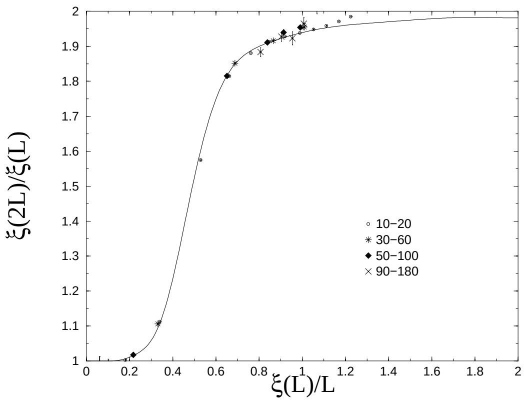

Figure 1: FSS function for the second-moment correlation length.

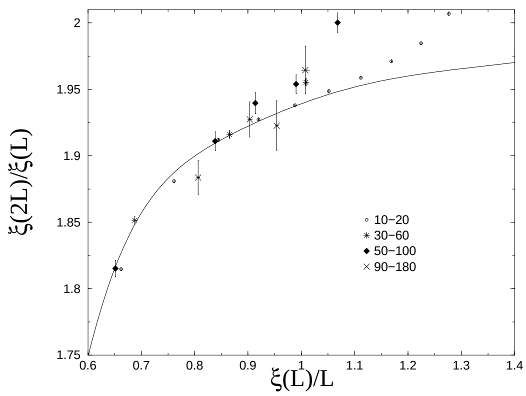

Figure 2: FSS function for the second-moment correlation length. Here, we restrict the horizontal range to $0.6 \le x \le 1.4$.

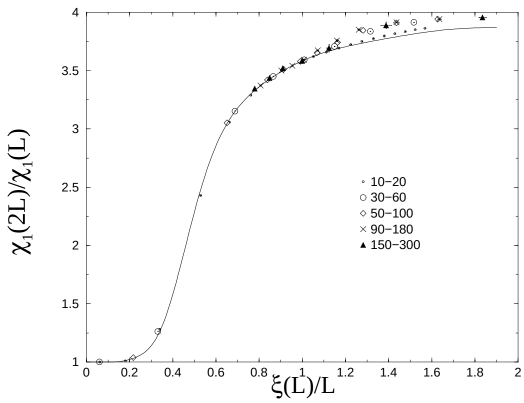

Figure 3: FSS function for the spin-1 susceptibility χ_1.

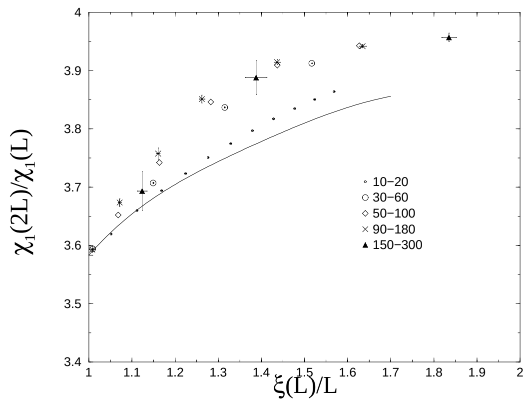

Figure 4: FSS function for the spin-1 susceptibility χ_1. Here, we restrict the horizontal range to $1 \le x \le 2$.

In Figs. 1, 2 we report the numerical results for the correlation length. The continuous line is a fit to the $O(3)$ data, while the points refer to the model with $h = 0.1$. As observed in previous work, there is indeed very good agreement between the numerical results for the two models, but such an agreement disappears for $\xi(L)/L \gtrsim 1$, where small discrepancies are observed. As it can be seen from Fig. 2, the icosahedral points tend to be above the $O(3)$ curve and, more importantly, the discrepancy tends to increase with L: the points with $L = 10$–20 are systematically below the points with $L = 50$–100.

In Figs. 1, 2 the difference in behavior between the two models is quite small and not totally convincing. Better evidence is obtained from the results for the susceptibilities, since in this case the statistical errors are smaller. In Figs. 3 and 4 we report the spin-1 susceptibility and in Figs. 5 and 6 the spin-3 and spin-4 analogues. Again, the numerical results for the icosahedral and the $O(3)$ model agree very nicely up to $\xi(L)/L \sim 0.8 - 1$, but then they indicate that the icosahedral FSS curve is steeper than the $O(3)$ one. Again, notice that the discrepancy between the two models increases with L, indicating

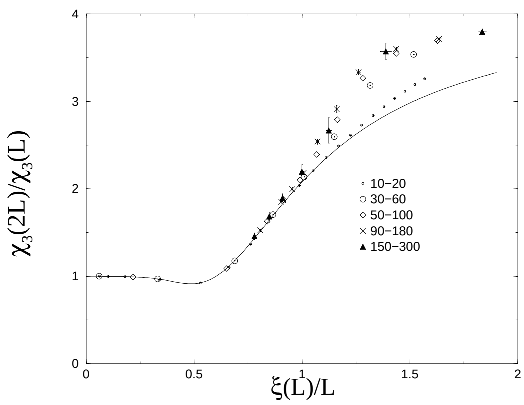

Figure 5: FSS function for the spin-3 susceptibility χ_3.

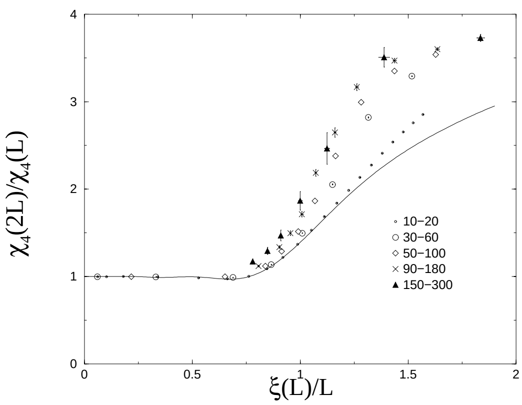

Figure 6: FSS function for the spin-4 susceptibility χ_4.

that the observed effect is not due to corrections to scaling, i.e. it is not a lattice artifact disappearing in the continuum limit.

In conclusion, the numerical results show that the icosahedral and the $O(3)$ model belong to different universality classes. Note however that discrepancies are observed only for $\xi(L)/L \sim 1$, which corresponds to very large values of the infinite-volume correlation length (see, e.g. Ref. [8]).

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