Inverse of the $\mathcal{PCT}$ Theorem

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ABSTRACT: $\mathcal{PCT}$, a fundamental symmetry of quantum field theory, is derived from the assumption of Lorentz invariance and positivity of the spectrum. What happens if we assume only Lorentz invariance and $\mathcal{PCT}$ symmetry? Hamiltonians having this property need not be Hermitian but, except when $\mathcal{PCT}$ is spontaneously broken, the energy levels of such Hamiltonians are all real and positive! In this talk I examine quantum mechanical and quantum field theoretic systems whose Hamiltonians are non-Hermitian but obey $\mathcal{PCT}$ symmetry. These systems have weird and remarkable properties. Examples of such Hamiltonians are $H = p^2 + ix^3$ and $H = p^2 - x^4$. Hamiltonians such as these are *complex deformations* of conventional Hermitian Hamiltonians. Thus, in this talk I study the analytic continuation of conventional quantum mechanics into the complex plane.

PrHEP hep2001

In a recent series of papers we showed that when properly defined, the ix^3 and $-gx^4$ potentials in quantum mechanics possess positive definite spectra. The positivity of the spectrum is apparently due to the $\mathcal{PT}$ symmetry of the Hamiltonian[1–21]. In this talk we extend these ideas to quantum field theory. We argue that in a $-x^4$ theory the expectation value $\langle x \rangle$ is *not* zero. The corresponding result for a $-g\phi^4$ quantum field theory in D-dimensional Euclidean space is that the one-point Green's function $G_1 = \langle \phi \rangle$ is also nonzero. This finding may allow us to construct new models for the Higgs boson. We also examine bound states in a $-g\phi^4$ quantum field theory.

In 1952 Dyson argued heuristically that perturbation series in quantum electrodynamics must diverge[22]. Applied to the quantum anharmonic oscillator,

$$H = p^2/2 + m^2x^2/2 + gx^4/4 \qquad (g > 0), \tag{1}$$

the argument goes as follows: If the coupling constant g is replaced by $-g$, then the potential is no longer bounded below, so the resulting theory has no ground state. Thus, the ground-state energy $E_0(g)$ has an abrupt transition at $g = 0$. If we represent $E_0(g)$ as a series in powers of g, this series must have a zero radius of convergence because $E_0(g)$ has a singularity at $g = 0$ in the complex-coupling-constant plane. Hence, the perturbation

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series must diverge for all $g \neq 0$. While the perturbation series does indeed diverge, this heuristic argument is flawed because the spectrum of the Hamiltonian

$$H = p^2/2 + m^2x^2/2 - gx^4/4 \qquad (g > 0) \tag{2}$$

is ambiguous due to the lack of well-specified boundary conditions for the eigenfunctions.

The spectrum depends crucially on how this Hamiltonian with a negative coupling constant is obtained. One way to obtain the Hamiltonian (2) is to substitute $g = |g|e^{i\theta}$ into the Hamiltonian (1) and rotate from $\theta = 0$ to $\theta = \pi$. Under this rotation, the ground-state energy $E_0(g)$ becomes complex. Evidently, $E_0(g)$ is real and positive when $g > 0$ and complex when $g < 0$.[1] One can also obtain (2) as a limit of the Hamiltonian

$$H = p^2/2 + m^2x^2/2 + gx^2(ix)^\alpha/4 \qquad (g > 0) \tag{3}$$

as $\alpha : 0 \to 2$. Having studied Hamiltonians like that in (3) in great detail, we and others have shown that for $\alpha \geq 0$ the spectra of such Hamiltonians are real, positive, and discrete. The spectrum of the limiting Hamiltonian (2) obtained in this manner is similar to that of the Hamiltonian in (1); it is entirely real, positive, and discrete. Very recently, the reality and positivity of the spectra have been established rigorously[23].

How can one Hamiltonian (2) possess two different spectra? The answer lies in the boundary conditions satisfied by the eigenfunctions $\psi_n(x)$. In the first case, in which $\theta = \arg g$ is rotated from 0 to π, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-4\pi/3 < \arg x < -\pi$. In the second case, in which α runs from 0 to 2, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-\pi < \arg x < -2\pi/3$. In this case the boundary conditions hold in wedges that are symmetric with respect to the imaginary axis; these boundary conditions enforce the $\mathcal{PT}$ symmetry of H and account for the reality of the spectrum.

Weak-Coupling Calculation of G_1: There is another striking difference between the two theories corresponding to H in (2). The one-point Green's function $G_1(g)$ is

$$G_1(g) = \langle 0|x|0\rangle / \langle 0|0\rangle \equiv \int_C dx\, x\psi_0^2(x) \Big/ \int_C dx\, \psi_0^2(x), \tag{4}$$

where C is a contour that lies in the asymptotic wedges described above. The value of $G_1(g)$ for H in (2) depends on the limiting process by which we obtain H. If we substitute $g = g_0 e^{i\theta}$ into the Hamiltonian (1) and rotate from $\theta = 0$ to $\theta = \pi$, we get $G_1(g) = 0$ for all g on the semicircle in the complex-g plane. Thus, this rotation in the complex-g plane preserves parity symmetry ($x \to -x$). However, if we define H in (2) by using the Hamiltonian in (3) and by allowing α to run from 0 to 2, we find that $G_1(g) \neq 0$. Indeed, $G_1(g) \neq 0$ for *all* values of $\alpha > 0$. Thus, in this theory $\mathcal{PT}$ symmetry (reflection about the imaginary axis, $x \to -x^*$) is preserved, but parity symmetry is permanently broken.

These two different results for $G_1(g)$ emphasize the ambiguity in Dyson's argument and show that the boundary conditions in the integrals in (4) are crucial for determining the

[1] Rotating from $\theta = 0$ to $\theta = -\pi$, we obtain the same Hamiltonian as in (2) but the spectrum is the complex conjugate of the spectrum obtained when we rotate from $\theta = 0$ to $\theta = \pi$.

PrHEP hep2001

one-point Green's function. We are concerned in this talk with the theory that preserves $\mathcal{PT}$ symmetry. In this theory the energy spectrum is real and positive and $G_1(g)$ is nonzero.

We have extended these quantum-mechanical arguments to the quantum field theory whose D-dimensional Euclidean space Lagrangian is $\mathcal{L} = (\nabla\phi)^2/2 + m^2\phi^2/2 - g\phi^4/4$. What is remarkable about this "wrong-sign" field theory is that, when it is obtained using the $\mathcal{PT}$-symmetric limit, the energy spectrum is real and positive, and the one-point Green's function is nonzero. Furthermore, the field theory is renormalizable, and in four dimensions is asymptotically free (and thus nontrivial). Based on these features of the theory, we believe that the theory may provide a useful setting to describe the Higgs particle.

The one-point Green's function G_1 is a *complex* functional integral in Euclidean space: $G_1 = \int_C \mathcal{D}\phi\,\phi(0)e^{-L[\phi]}/\int_C \mathcal{D}\phi\, e^{-L[\phi]}$. Here, $L[\phi] = \int d^Dx\,\mathcal{L}$ and C is a contour in the complex-ϕ plane defined as follows: Functional integrals are infinite products of ordinary integrals, one integral for each lattice point in Euclidean space. For these ordinary integrals the contour of integration must lie within 45° wedges that lie in the lower-half plane and are centered about -45° and -135°. In D-dimensional space we use $\epsilon = gm^{D-4}/4$ to represent the dimensionless coupling constant. The small-ϵ asymptotic behavior of G_1 is determined by a *soliton* (not an instanton). In general, G_1 has a *negative imaginary* value:

$$\begin{aligned} D = 0:&\ G_1 \sim -\frac{i}{m}2^{-1/2}\epsilon^{-1/2}e^{-1/\epsilon} \quad (\epsilon \to 0^+); \\ D = 1:&\ G_1 \sim -\frac{i}{\sqrt{m}}16\sqrt{\pi e}(2/\epsilon)^{2/3}e^{-16/(3\epsilon)}3^{-1/6}/\Gamma^2(1/3) \quad (\epsilon \to 0^+). \end{aligned} \tag{5}$$

In dimension D, $G_1 \sim e^{-4\Lambda[D]/\epsilon}$ as $\epsilon \to 0^+$, where $\Lambda[D]$ is determined by a spherically symmetric soliton. Numerical values of $\Lambda[D]$ for $0 \leq D \leq 4$ are given in Ref. [14].

Bound States: A significant difference between the Hermitian Hamiltonian (1) and the $\mathcal{PT}$-symmetric Hamiltonian (2) is that when g is sufficiently small, the latter Hamiltonian possesses bound states while the former does not. The bound states persist in the corresponding non-Hermitian $\mathcal{PT}$-symmetric $-g\phi^4$ quantum field theory for all dimensions $0 \leq D < 3$ but are not present in the conventional Hermitian $g\phi^4$ field theory.

We calculate the bound states perturbatively. For the anharmonic oscillator Hamiltonian (1), the perturbation series for the kth energy level E_k begins $E_k \sim m[k + 1/2 + 3(2k^2 + 2k + 1)\epsilon/4 + \mathrm{O}(\epsilon^2)]$ as $\epsilon \to 0^+$, where $\epsilon = g/(4m^3)$. The *renormalized mass* M is the first excitation above the ground state: $M \equiv E_1 - E_0 \sim m[1 + 3\epsilon + \mathrm{O}(\epsilon^2)]$ as $\epsilon \to 0^+$.

To determine if the two-particle state is bound, we examine the *second* excitation above the ground state. We define $B_2 \equiv E_2 - E_0 \sim m\left[2 + 9\epsilon + \mathrm{O}(\epsilon^2)\right]$ as $\epsilon \to 0^+$. If $B_2 < 2M$, then a two-particle bound state exists and the (negative) binding energy is $B_2 - 2M$. If $B_2 > 2M$, then the second excitation above the vacuum is interpreted as an unbound two-particle state. In the small-coupling regime, where perturbation theory is valid, the conventional anharmonic oscillator does not possess a bound state. Indeed, using WKB, variational methods, or numerical calculations one can show that there is no two-particle bound state for any $g > 0$. Thus, the gx^4 interaction represents a repulsive force.[2]

[2]In general, a repulsive force in a quantum field theory is represented by an energy dependence in

PrHEP hep2001

We obtain the perturbation series for H in (2) from the perturbation series for the conventional anharmonic oscillator by replacing ϵ with $-\epsilon$. Thus, while the conventional anharmonic oscillator does not possess a two-particle bound state, the $\mathcal{PT}$-symmetric oscillator does indeed possess such a state. We give the binding energy of this state in units of the renormalized mass M and we define the *dimensionless* binding energy Δ_2 by

$$\Delta_2 \equiv (B_2 - 2M)/M \sim -3\epsilon + \mathrm{O}(\epsilon^2) \qquad (\epsilon \to 0^+). \tag{6}$$

This bound state evaporates when ϵ increases beyond $\epsilon = 0.0465\ldots$. As ϵ continues to grow, Δ_2 reaches a maximum of 0.427 at $\epsilon = 0.13$ and then approaches 0.28 as $\epsilon \to \infty$.

In the $\mathcal{PT}$-symmetric anharmonic oscillator, there are not only two-particle bound states for small coupling constant but also k-particle bound states for all $k \geq 2$. The dimensionless binding energies are $\Delta_k \equiv (B_k - kM)/M \sim -3k(k-1)\epsilon/2 + \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0+$. Since the coefficient of ϵ is negative, the dimensionless binding energy becomes negative as ϵ increases from 0, and there is a k-particle bound state. The higher k-particle bound states cease to be bound for smaller values of ϵ; the binding energies Δ_3, Δ_4, Δ_5, and Δ_6 become positive as ϵ increases past 0.039, 0.034, 0.030, and 0.027.

For any value of ϵ there are always a finite number of bound states and an infinite number of unbound states. The number of bound states decreases with increasing ϵ until there are no bound states at all. Note that there is a range of ϵ for which there are only two- and three-particle bound states. This situation is analogous to the physical world in which one observes only states of two and three bound quarks. In this range of ϵ if one has an initial state containing a number of particles (renormalized masses), these particles will clump together into bound states, releasing energy in the process. Depending on the value of ϵ, the final state will consist either of two- or of three-particle bound states, whichever is energetically favored. Note also that there is a special value of ϵ for which two- and three-particle bound states can exist in thermodynamic equilibrium.

These results generalize from quantum mechanics to the D-dimensional $\mathcal{PT}$-symmetric $-g\phi^4$ quantum field theory. There exists a bound state because *to leading order in the dimensionless coupling constant* ϵ the binding energy becomes negative as ϵ increases from 0. We calculate the bound-state energy by summing all "sausage-link" graphs and identifying the bound-state pole. The dimensionless binding energy to leading order in ϵ is

$$\Delta_2 \sim -(4\pi)^{(D-1)/(D-3)}[3\Gamma(3/2 - D/2)]^{2/(3-D)}\epsilon^{2/(3-D)}, \tag{7}$$

which reduces to (6) at $D = 1$. Equation (7) holds for $0 \leq D < 3$ because we have performed mass renormalization (but not wave function or coupling-constant renormalization).

We conclude by comparing a $g\phi^3$ theory with a $g\phi^4$ theory: A $g\phi^3$ theory represents an attractive force. The bound states that arise as a consequence of this force can be found by using the Bethe-Salpeter equation. However, the $g\phi^3$ field theory is unacceptable

which the energy of a two-particle state decreases with separation. The conventional anharmonic oscillator Hamiltonian corresponds to a field theory in one space-time dimension where there cannot be any spatial dependence. The repulsive nature of the force is understood to mean that the energy B_2 needed to create two particles at a given time is more than twice the energy M needed to create one particle.

PrHEP hep2001

because the spectrum is not bounded below. If we replace g by ig, the spectrum becomes real and positive, but the force becomes repulsive and there are no bound states. The same is true for a two-scalar theory with interaction of the form $ig\phi^2\chi$. This latter theory is an acceptable model of scalar electrodynamics, but has no analog of positronium.

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PrHEP hep2001