

Polyakov conjecture and 2+1 dimensional gravity coupled to particles

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ABSTRACT: A proof is given of Polyakov conjecture about the auxiliary parameters of the $SU(1,1)$ Riemann-Hilbert problem for general elliptic singularities. Such a result is related to the uniformization of the sphere punctured by n conical defects. Its relevance to the hamiltonian structure of 2+1 dimensional gravity in the maximally slicing gauge is stressed.

1. We shall deal with the proof of a conjecture put forward by Polyakov [1] about the accessory parameters of the Riemann-Hilbert problem and with the role such a conjecture plays in 2+1 dimensional gravity coupled to particles on an open universe. Such a theory when formulated in the maximally slicing gauge (Dirac gauge)[2] shows a deep connection with the Riemann-Hilbert problem which consists in determining a fuchsian differential equation whose independent solutions transform according to a given representation of the $SL(2C)$ group when one encircles the singularities. In the second order canonical ADM formulation [3, 4] of 2+1 dimensional gravity a variant of the Riemann-Hilbert problem appears which is described in Sec. 2. Such a problem occurs in determining the conformal factor $e^{2\sigma}$ in the ADM metric

$$ds^2 = -N^2 dt^2 + e^{2\sigma} (dz + N^z dt)(d\bar{z} + N^{\bar{z}} dt). \quad (1)$$

In fact the hamiltonian constraint in the maximally slicing gauge can be written as [3]

$$4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n \delta^2(z - z_n)(-1 + \mu_n) + 4\pi \sum_B \delta^2(z - z_B) \quad (2)$$

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†Supported in part by MURST.

where

$$\phi = -2\sigma + \ln \left[\frac{1}{2\pi^2} \sum_n \frac{P_n}{(z - z_n)} \sum_n \frac{\bar{P}_n}{(\bar{z} - \bar{z}_n)} \right] \quad (3)$$

being z_n the particle positions, P_n the canonically conjugate momenta and $4\pi\mu_n$ the particle masses in Planck units. The z_B are the zeros of $\sum_n P_n/(z - z_n)$ and are known as the apparent singularities. In eq.(2) one recognizes the Liouville equation with point sources.

The particle equations of motion in the relative coordinates can be written as [4]

$$\dot{z}'_n = - \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial P'_n}, \quad \dot{P}'_n = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial z'_n} \quad (4)$$

where β_B are the accessory parameters related to the apparent singularities. One is faced with the problem of proving the hamiltonian nature of eqs.(4) and possibly to give the hamiltonian. As we shall see this problem has a straightforward solution if one assumes the validity of Polyakov's conjecture to which now we come.

2. Polyakov [1] put forward the following conjecture on the accessory parameters β_n which appear in the solution of the $SU(1, 1)$ Riemann-Hilbert problem

$$- \frac{1}{2\pi} dS_P = \sum_n \beta_n dz_n + c.c. \quad (5)$$

where S_P is the regularized Liouville action [5], $S_P = \lim_{\epsilon \rightarrow 0} S_\epsilon$ with

$$S_\epsilon[\phi] = \frac{i}{2} \int_{X_\epsilon} (\partial_z \phi \partial_{\bar{z}} \phi + \frac{e^\phi}{2}) dz \wedge d\bar{z} + \frac{i}{2} \sum_n g_n \oint_{\gamma_n} \phi \left(\frac{d\bar{z}}{\bar{z} - \bar{z}_n} - \frac{dz}{z - z_n} \right) \\ + \frac{i}{2} g_\infty \oint_{\gamma_\infty} \phi \left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right) - \pi \sum_n g_n^2 \ln \epsilon^2 - \pi g_\infty^2 \ln \epsilon^2, \quad \text{where} \quad dz \wedge d\bar{z} = -2i dx \wedge dy \quad (6)$$

and X_ϵ is the disk of radius $1/\epsilon$ in the complex plane from which disks of radius ϵ around all singularities have been removed; γ_n are the boundaries of the small disks and γ_∞ is the boundary of the large disk. In eq.(5) S_P has to be computed on the solution of the inhomogeneous Liouville equation which arises from the minimization of the action i.e.

$$4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n g_n \delta^2(z - z_n) \quad (7)$$

with behavior at infinity $\phi = -g_\infty \ln z\bar{z} + O(1)$. Such a conjecture plays an important role in the quantum Liouville theory [5, 6] and in the ADM formulation of 2 + 1 dimensional gravity [3, 4]. The conjecture is interesting in itself as it gives a new meaning to the rather elusive accessory parameters [7] of the Riemann-Hilbert problem. In particular it implies that the form $\omega = \sum_n \beta_n dz_n + c.c.$ is exact. Zograf and Takhtajan [6] provided a proof of eq.(5) for parabolic singularities. In addition they remark that the same technique can be applied when some of the singularities are elliptic of finite order. On the other hand in 2 + 1 gravity one is faced with general elliptic singularities and here the mapping technique cannot be applied. Picard [8] proved that eq.(7) for real ϕ with asymptotic behavior at

infinity $\phi(z) = -g_\infty \ln(z\bar{z}) + O(1)$, $-1 < g_n$, $1 < g_\infty$ and $\sum_n g_n + g_\infty < 0$ admits one and only one solution (see also [9]). The interest of such results is that they solve the following variant of the Riemann-Hilbert problem: at z_1, \dots, z_n we are given not with the monodromies but with the class, characterized by g_j , of the elliptic monodromies with the further request that all such monodromies belong to the group $SU(1, 1)$. The last requirement is imposed by the fact that the solution of eq.(7) has to be single valued.

From eq.(7) one can easily prove [11], that

$$e^{-\frac{\phi}{2}} = \frac{1}{\sqrt{8|w_{12}|}} [\bar{y}_2(\bar{z})y_2(z) - \bar{y}_1(\bar{z})y_1(z)] \quad (8)$$

being y_1, y_2 two independent solutions of

$$y'' + Q(z)y = 0 \quad \text{where} \quad Q(z) = \sum_n -\frac{g_n(g_n + 2)}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)}. \quad (9)$$

w_{12} is the constant wronskian and the β_n are the accessory parameters [7].

3. The result of Picard assures us that given the position of the singularities z_n and the classes of monodromies characterized by the real numbers g_n there exists a unique fuchsian equation which realizes $SU(1, 1)$ monodromies of the prescribed classes. In particular the uniqueness of the solution of Picard's equation tells us that the accessory parameters β_i are single valued functions of the parameter z_n and g_n . We shall examine in this section how such dependence arises from the viewpoint of the imposition of the $SU(1, 1)$ condition on the monodromies. Starting from the singularity in z_1 we can consider the canonical pair of solutions around z_1 , $y_1^1 = \zeta^{\frac{g_m}{2}+1} A(\zeta)$, $y_2^1 = \zeta^{-\frac{g_m}{2}} B(\zeta)$ with $A(\zeta) = 1 + O(\zeta)$, $B(\zeta) = 1 + O(\zeta)$, $\zeta = z - z_1$, i.e. those solutions which behave as a single fractional power multiplied by an analytic function with first coefficient one. Let (y_1, y_2) the solutions which realize $SU(1, 1)$ around all singularities. Obviously all conjugations with any element of $SU(1, 1)$ is still an equivalent solution in the sense that they provide the same conformal factor ϕ . The canonical pairs around different singularities are linearly related i.e. $(y_1^1, y_2^1) = (y_1^2, y_2^2)C_{21}$. We fix the conjugation class by setting $(y_1, y_2) = (y_1^1, y_2^1)K$ with $K = \text{diag}(k, k^{-1})$ being the overall constant irrelevant in determining ϕ . Moreover if the solution (y_1, y_2) realizes $SU(1, 1)$ monodromies around all singularities also $(y_1, y_2) \times \text{diag}(e^{i\alpha}, e^{-i\alpha})$ accomplishes the same purpose being $\text{diag}(e^{i\alpha}, e^{-i\alpha})$ an element of $SU(1, 1)$. Thus the phase of the number k is irrelevant and so we can consider it real and positive. This choice of the canonical pairs is always possible in our case. If D_n denote the diagonal monodromy matrices around z_n , we have that the monodromy around z_1 is D_1 and the one around z_2 is $M_2 = K^{-1}C_{12}D_2C_{21}K$, where with C_{12} we have denoted the inverse of the 2×2 matrix C_{21} . In the case of three singularities (one of them at infinity) by using the freedom on K we can reduce M_2 to the $SU(1, 1)$ form. The possibility of such a choice is assured by Picard's result and in this simple case also by the explicit solution in terms of hypergeometric functions [2, 3]. We come now to a qualitative description of the case of four singularities. We recall that the accessory parameters β_n are bound by two algebraic relations known as Fuchs relations [7]. Thus after choosing M_1 of the form $M_1 = D_1K$, in imposing

the $SU(1,1)$ nature of the remaining monodromies we have at our disposal three real parameters i.e. k , $\text{Re } \beta_3$ and $\text{Im } \beta_3$. It is sufficient to impose the $SU(1,1)$ nature of M_2 and M_3 as the $SU(1,1)$ nature of M_∞ is a consequence of them. As the matrices $M_n = K^{-1}C_{1n}D_nC_{n1}K$ satisfy identically $\det M_n = 1$ and $\text{Tr}M_n = 2 \cos \pi g_n$ we need to impose generically on M_2 only two real conditions e.g. $\text{Re } b_2 = \text{Re } c_2$ and $\text{Im } b_2 = -\text{Im } c_2$. The same for M_3 . Thus it appears that we need to satisfy four real relations when we can vary only three real parameters. The reason why we need only three and not four is that for any solution of the fuchsian problem the following relation among the monodromy matrices is identically satisfied $D_1KM_2M_3M_\infty = 1$. The above reported discussion can be put on rigorous grounds [10] for any number of singularities by exploiting the existence and uniqueness of the solution and using some basic results of the theory of analytic function of several complex variables, reaching the result that β_n are analytic functions of z_m, \bar{z}_m in the neighborhood of any point of the complex plane, except for a finite number of points.

4. Being defined through a limit procedure the action S_P is somewhat uncomfortable to work with. It is however possible, introducing a background field to rewrite S_P as a simple integral. In the global coordinate system z on C one writes $\phi = \phi_M + \phi_0 + \phi_B$ where ϕ_B is a background conformal factor which is regular and behaves at infinity like $\phi_B = -2 \ln(z\bar{z}) + c_B + O(1/|z|)$ while ϕ_0 is given by

$$\phi_0 = \sum_n g_n \ln |z - z_n|^2 - \alpha \phi_B + c_0 \quad \text{where} \quad \alpha = -(\sum_n g_n + g_\infty - 2)/2. \quad (10)$$

Then we have for ϕ_M

$$4\partial_z\partial_{\bar{z}}\phi_M = e^{\phi_0+\phi_B+\phi_M} + (\alpha - 1)4 \partial_z\partial_{\bar{z}}\phi_B. \quad (11)$$

ϕ_M is a continuous function on the Riemann sphere. The action which generates the above equation is

$$S = \int [\partial_z\phi_M\partial_{\bar{z}}\phi_M + \frac{e^\phi}{2} + 2(\alpha - 1)\phi_M\partial_z\partial_{\bar{z}}\phi_B] \frac{idz \wedge d\bar{z}}{2}. \quad (12)$$

The integral in eq.(12) converges absolutely. S computed on the solution of eq.(11) is related to the original Polyakov action S_P also computed on the solution of eq.(11) by

$$S_P = S - (\alpha - 1)^2 \int \phi_B\partial_z\partial_{\bar{z}}\phi_B \frac{idz \wedge d\bar{z}}{2} + 2\pi(\alpha - 1)^2 c_B + \pi \sum_m \sum_{n \neq m} g_m g_n \ln |z_m - z_n|^2 + 4\pi c_0(1 - \alpha). \quad (13)$$

Our aim now is to compute the derivative of S_P with respect to z_n . Again computing the derivative of the new action S is not completely trivial because one cannot take directly the derivative operation under the integral sign. In fact such unwarranted procedure would give rise to an integrand which is not absolutely summable. One can devise however a subtraction procedure [10] which allows to perform such operations. The rigorous result for the derivative, using the equation of motion (11) is [10]

$$\frac{\partial S}{\partial z_m} = \lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} [\partial_z(\frac{\partial\phi_M}{\partial z_m}\partial_{\bar{z}}\phi_M) + \partial_{\bar{z}}(\frac{\partial\phi_M}{\partial z_m}\partial_z\phi_M) + \frac{\partial\phi_0}{\partial z_m} \frac{e^\phi}{2}] \frac{idz \wedge d\bar{z}}{2} \quad (14)$$

which can be computed by using eq.(11) to obtain

$$\frac{\partial S}{\partial z_m} = -ig_m \lim_{\epsilon \rightarrow 0} \oint_{\gamma_\epsilon} \frac{1}{z - z_m} \partial_z (\phi_M - (\alpha - 1)\phi_B) dz. \quad (15)$$

Using $\phi_M - (\alpha - 1)\phi_B = \phi - \sum_n g_n \ln |z - z_n|^2$ and the expansion of $A = 1 + c_1 \zeta + \dots$ and $B = 1 + c_2 \zeta + \dots$ which are obtained by substituting into the differential equation (9) to obtain

$$c_1 = -\frac{\beta_m}{2(g_m + 2)} \quad \text{and} \quad c_2 = \frac{\beta_m}{2g_m} \quad (16)$$

finally we have

$$\frac{\partial S}{\partial z_m} = -2\pi\beta_m - 2\pi \sum_{n, n \neq m} \frac{g_m g_n}{z_m - z_n} \quad (17)$$

equivalent to Polyakov conjecture eq.(5) due to the relation (13) between S and S_P . From eqs.(4,5) we see that the hamiltonian is given by $H = \frac{1}{2\pi} \frac{\partial S_P}{\partial \mu}$, because

$$\frac{\partial H}{\partial P'_n} = -\sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial P'_n} \quad \text{and} \quad -\frac{\partial H}{\partial z'_n} = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial z'_n}. \quad (18)$$

Acknowledgments We are grateful to Mauro Carfora for pointing out to us reference [9] and for useful discussions.

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