NONCOMMUTATIVITY OF BOUNDARY CLOSED STRING COORDINATES FOR AN OPEN MEMBRANE ON p-BRANE

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Abstract: We study the dynamics of an open membrane with a cylindrical topology, in the background of a constant three form. We use the action, due to Bergshoeff, London and Townsend, to study the noncommutativity properties of the boundary string coordinates. The constrained Hamiltonian formalism due to Dirac is used to derive the noncommutativity of coordinates. The chain of constraints is found to be finite for a suitable gauge choice.

1. Introduction:

Recently, the study of noncommutative geometry, from the perspective of string theory, has attracted considerable attention. The noncommutativity of the target space coordinates becomes manifest when a constant background NS two form potential is introduced along the D-brane [1]. In the presence of the two-form potential, the end points of the open strings attached to the D-brane do not commute. It is natural, therefore, to examine the corresponding issue when an open membrane ends on a D-brane under an analogous situation. In this talk, we present the noncommutativity property of an open membrane-brane configuration from a different perspective; details of the result are given in [2]. We adopt a modified version of the action due to Bergshoeff, London and Townsend [3]. With the modified form of the action, we are able to make some head way with the computation of

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the matrix of constraints as well as the evaluation of the Dirac brackets (DB) in a systematic manner, without linearizing the action as was done at the outset in [4]. Surprisingly, however, we find that with an alternate, suitable choice of gauge, the constraint chain for the same membrane system terminates. In other words, in this alternate gauge, after a finite number of iterations, new constraints are not generated.

2. The Action

The action for a membrane, interacting with an anti-symmetric background field C_{MNP} can be described by a Nambu-Goto action

$$S = T\int_{\Sigma_3} d^3\xi \left(\sqrt{g} - \frac{1}{6}\varepsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P A_{MNP}\right) \tag{2.1}$$

where $A = C + dB$. Here $M, N, P = 0, 1, .., 10$ are indices of the 11-dimensional target space, $\xi = (\tau, \sigma_1, \sigma_2)$ are the coordinates of the world volume of the membrane with the corresponding indices taking values $i, j, k = 0, 1, 2$, $g = \det g_{ij}$, where $g_{ij} = G_{MN}\partial_i X^M \partial_j X^N$; is the induced metric on the membrane. For simplicity, we are going to choose both G_{MN} and A_{MNP} to be constants (In fact, we will choose $G_{MN} = \eta_{MN}$ from now on). An alternate description for the membrane is through the first order action of the form (due to Bergshoeff, London and Townsend) [3],[5]

$$S_{BLT} = \int_{\Sigma_3} d^3\xi \, \frac{1}{2V}(g - \widetilde{\mathcal{F}}^2) \tag{2.2}$$

Here we have defined

$$\widetilde{\mathcal{F}} \equiv \varepsilon^{ijk}\widetilde{\mathcal{F}}_{ijk} = \varepsilon^{ijk}(F_{ijk} + \frac{1}{6}A_{ijk}) \tag{2.3}$$

where

$$F_{ijk} = \partial_{[i}U_{jk]} = \partial_i U_{jk} + \text{cyclic} \tag{2.4}$$

$$A_{ijk} = \partial_i X^M \partial_j X^N \partial_k X^P A_{MNP} \tag{2.5}$$

with A_{MNP} defined earlier. Clearly, $V(\xi)$ and $U_{ij}(\xi)$ are auxiliary field variables. It can be shown that the dynamical equation for the coordinates and boundary conditions coincide with the ones following from the action in (2.1).

Due to the cylindrical topology of the membrane, in the presence of p-branes, the boundary condition reduces to

$$\sqrt{g}g^{1j}\partial_j X_\mu - \frac{1}{2}\varepsilon^{1jk}\partial_j X^\nu \partial_k X^\rho A_{\mu\nu\rho} \,|_{\sigma_1=0,\pi} = 0, \qquad \mu,\nu = 0,1,...,p \tag{2.6}$$

$$X^a = x_0^a, \qquad a = p+1, ..., 10 \tag{2.7}$$

where the coordinates, x_0^a, specify the positions of the p-branes at the two boundaries $\sigma_1 = 0, \pi$.

3. Constraint analysis:

We take the first order action (2.2) as the starting point of our canonical description of the system. Let us note that the Lagrangian density of the action (2.2) is singular. Therefore, the velocities cannot be expressed in terms of the phase space variables. We will choose a gauge condition which brings out an interesting feature of our analysis, namely, that with a suitable gauge choice, the chain of constraints can terminate. Let us look at the action (2.2) in the gauge

$$g_{0a} = 0, \qquad a = 1,2 \tag{3.1}$$

In this gauge, action (2.2) takes the form

$$S = \int_{\Sigma_3} d^3\xi \, \frac{1}{2V} \left(\overline{g} \dot{X}^M \dot{X}_M - \widetilde{\mathcal{F}}^2 \right) \tag{3.2}$$

Adding boundary constraints, it is now straightforward to determine the complete set of primary constraints:

$$\begin{aligned}
\varphi_1 &= P_V \approx 0 \\
\varphi_2^a &= \Pi^{(U)oa} \approx 0 \\
\varphi_{3\mu} &= (\overline{g}^{a1}\partial_a X_\mu \mathcal{P}^2 + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda})\delta(\sigma_1) \approx 0 \\
\varphi_{4\mu} &= (\overline{g}^{a1}\partial_a X_\mu \mathcal{P}^2 + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda})\delta(\sigma_1 - \pi) \approx 0
\end{aligned} \tag{3.3}$$

One can obtain the Hamiltonian for the system to be

$$H = \frac{V}{2\overline{g}}\mathcal{P}^2 - \frac{V}{72}(\Pi^{(U)ab}\varepsilon_{ab})^2 + 2\Pi^{(U)ab}\partial_a U_{0b} + c\varphi_1 + k_a\varphi_2^a + \lambda^\mu\varphi_{3\mu} + \widetilde{\lambda}^\mu\varphi_{4\mu} \tag{3.4}$$

where c, k_a, λ^μ, and $\widetilde{\lambda}^\mu$ are Lagrange multipliers.

The analysis for the consistency of constraints can now be carried out in a straightforward manner. The consistency conditions lead to

$$\begin{aligned}
\lambda^\mu &= \widetilde{\lambda}^\mu = 0 \\
\varphi_5 &= \frac{\mathcal{P}^2}{2\overline{g}} - \frac{1}{72}\left(\Pi^{(U)ab}\varepsilon_{ab}\right)^2 \approx 0
\end{aligned} \tag{3.5}$$

$$\varphi_6^a = \partial_b \Pi^{(U)ba} \approx 0 \tag{3.6}$$

This is the analog of Gauss' law in electrodynamics and it can be easily checked that the consistency of these constraints leads to no new constraints. Consistency of the boundary constraint $\varphi_{3\mu} \approx 0$,

$$\dot{\varphi}_{3\mu} = \left\{ \varphi_{3\mu}, \int H \right\} \approx 0$$

leads to the secondary constraint,

$$\begin{aligned}
\varphi_{7\mu} = \delta(\sigma_1) \Big[& \overline{g}^{a1}\partial_a[\frac{V}{\overline{g}}\mathcal{P}_\mu] + \varepsilon^{ab}\partial_a X_\mu \partial_{(b} X^\lambda \partial_{2)}[\frac{V}{\overline{g}}\mathcal{P}_\lambda] + \frac{V}{3\overline{g}}\Pi^{(U)a1}A_{\mu\nu\lambda}\mathcal{P}^\nu\partial_a\mathcal{P}^\lambda \\
& - \frac{1}{3}\Pi^{(U)a1}A_{\mu\nu\lambda}\partial_a X^\lambda \partial_c[V\overline{g}^{bc}\partial_b X^\nu] \Big]
\end{aligned} \tag{3.7}$$

Since the Hamiltonian (3.4) contains a term of the form cP_V and the secondary constraint $\varphi_{7\mu}$ depends on V as well as ∂V, consistency of this new constraint

$$\dot{\varphi}_{7\mu} = \left\{\varphi_{7\mu}, \int H\right\} \approx 0$$

simply determines the Lagrange multiplier c and leads to no further constraint. An identical analysis goes through for the constraint $\varphi_{4\mu}$ at the other boundary and generates only one secondary constraint $\varphi_{8\mu}$, whose structure is identical to that of $\varphi_{7\mu}$ except that it is at the other boundary.

4. Dirac brackets:

Since we have determined all the constraints of our theory, it is now straightforward, in principle, to determine the Dirac brackets [6] . However, we note that the boundary constraints are, in particular, highly nonlinear and, consequently, evaluation of the inverse of the matrix of constraints is, in general, a very difficult problem. Things, however, do simplify enormously if we use a weak field approximation for $A_{\mu\nu\lambda}$.The constraints $\varphi_{3\mu}, \varphi_{7\mu}$ are second class and, therefore, the Dirac bracket between the coordinates takes the form

$$\{X_\mu(\sigma), X_\nu(\sigma')\}_D = -\int d^2\sigma'' d^2\sigma''' \{X_\mu(\sigma), \phi_A(\sigma'')\} C^{-1AB}(\sigma'', \sigma''') \times \{\phi_B(\sigma'''), X_\nu(\sigma')\} \tag{4.1}$$

where $\phi_A \equiv (\varphi_{3\mu}, \varphi_{7\mu})$ and

$$C_{AB}(\sigma, \sigma') = \begin{pmatrix} \{\varphi_{3\mu}(\sigma), \varphi_{3\nu}(\sigma')\} & \{\varphi_{3\mu}(\sigma), \varphi_{7\nu}(\sigma')\} \\ \{\varphi_{7\nu}(\sigma'), \varphi_{3\mu}(\sigma)\} & \{\varphi_{7\mu}(\sigma), \varphi_{7\nu}(\sigma')\} \end{pmatrix} \tag{4.2}$$

We can, of course, calculate exactly all the brackets entering into the matrix, C_{AB}. However, determining the inverse matrix exactly is a technically nontrivial problem. It is here that the weak field approximation is of immense help (We want to emphasize that there is no other approximation used in our derivations.). It is shown ([2]) that in this approximation the Dirac bracket has the form

$$\begin{aligned} \{X_\mu(\sigma), X_\nu(\sigma')\}_D = \Big[& \widetilde{T}_{\mu\lambda}\left((G+F)^{-1}\right)^{\lambda\rho} S_{\rho\nu} \\ & -\widetilde{S}_{\mu\lambda}\left(\left((G+F)^{-1}\right)^{\lambda\rho} T_{\rho\nu}\right) \\ & + \widetilde{T}_{\mu\lambda}\left(((F+G)^{-1}(V+W)(F+G)^{-1})^{\lambda\rho} T_{\rho\nu}\right)\Big](\sigma, \sigma') \end{aligned} \tag{4.3}$$

where the operators T, F, G are independent on $A_{\mu\nu\lambda}$, while S, V, W are liner in $A_{\mu\nu\lambda}$. It is worth emphasizing here that because of the structure of the boundary constraints the σ_1 coordinate is fixed at the boundary (to be $0, \pi$) and, therefore, the Dirac bracket, evaluated at equal τ, effectively depends only on the world volume coordinates τ, σ_2. This shows that the boundary string coordinates indeed become noncommutative in the presence of an antisymmetric background field and what is even more interesting is that they have a structure that is quite analogous to that in the case of strings.

5. Summary

We have studied an open membrane, with cylindrical geometry, ending on p-branes. The boundary of the open membrane on the brane is a closed string. We have adopted a modified action which has some distinct advantages as discussed in the text. We have treated the boundary conditions as primary constraints and have shown that, one can carry out the Dirac formalism without restricting to the linearized approximation of the action. We have also introduced a gauge choice, different from the one adopted in ref [4], and have shown that the Dirac procedure, in this gauge, leads to a finite number constraints. As a consequence, we are able to compute the PB brackets of all the second class constraints which are necessary for the evaluation of Dirac brackets.

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