PROCEEDINGS

Workshop on Integrable Theories, Solitons and Duality

Boundary sine-Gordon model

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Abstract: We review our recent results on the on-shell description of sine-Gordon model with integrable boundary conditions. We determined the spectrum of boundary states together with their reflection factors by closing the boundary bootstrap and checked these results against WKB quantization and numerical finite volume spectra obtained from the truncated conformal space approach. The relation between a boundary resonance state and the semiclassical instability of a static classical solution is analyzed in detail.

PrHEP unesp2002

Introduction

Sine-Gordon field theory is defined by the Lagrangean

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{\beta^2}\cos(\beta\Phi) \quad , \tag{1}$$

where Φ is a real scalar field and β is the coupling constant. It is one of the most important quantum field theoretic models with numerous applications ranging from particle theoretic problems to condensed matter systems, and one which has played a central role in our understanding of $1+1$ dimensional field theories. A crucial property of the model is integrability, which permits an exact analytic determination of many of its physical properties and characteristic quantities.

Integrability can also be maintained in the presence of boundaries [1]; for sine-Gordon theory, the most general boundary potential that preserves integrability was found by Ghoshal and Zamolodchikov [2]

$$V_B = M_0 \cos\left(\frac{\beta}{2}(\Phi(0) - \varphi_0)\right) \quad . \tag{2}$$

They also introduced the notion of 'boundary crossing unitarity', and combining it with the boundary version of the Yang-Baxter equations they were able to determine soliton

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reflection factors on the ground state boundary. Later Ghoshal completed this work by determining the breather reflection factors [3] using a boundary bootstrap equation first proposed by Fring and Köberle [4].

The first (partial) results on the spectrum of the excited boundary states were obtained by Saleur and Skorik for Dirichlet boundary conditions [5]. However, they did not take into account the boundary analogue of the Coleman-Thun mechanism, the importance of which was first emphasized by Dorey et al. [6]. Using this mechanism Mattsson and Dorey were able to close the bootstrap in the Dirichlet case and determine the complete spectrum and the reflection factors on the excited boundary states [7]. Recently we used their ideas to obtain the spectrum of excited boundary states and their reflection factors for the Neumann boundary condition [8] and then for the general two-parameter family of integrable boundary conditions [9].

Another interesting problem is the relation between the ultraviolet (UV) parameters that appear in the perturbed CFT Hamiltonian and the infrared (IR) parameters in the reflection factors. This relation was first obtained by Al. B. Zamolodchikov [11] together with a formula for the boundary energy; however, his results remain unpublished. In order to have these formulae, we rederived them in our paper [12], where we used them to check the consistency of the spectrum and of the reflection factors against a boundary version of truncated conformal space approach (TCSA). Combining the TCSA results with analytic methods of the Bethe Ansatz we found strong evidence that our understanding of the spectrum of boundary sine-Gordon model is indeed correct.

Recently M. Kormos and one of us (L.P.) achieved the semiclassical quantization of the two lowest energy static solutions of the model [13]. By comparing the quantum corrected energy with the exact one the perturbative correspondence between the Lagrangean and the bootstrap parameters has been established. In the paper we extend their analysis for an unstable solution which corresponds to a boundary resonance state. We compute the decay rate and decay width of the resonance state and show how these results agree with the semiclassical considerations. We comment also on the possible changes in the finite volume spectra due to the resonance state.

The paper is organized as follows: in Section 2 we review the boostrap philosophy by applying to the bulk sine-Gordon theory. In Section 3 we give the boundary analogue of this picture, the boundary spectrum is determined and the boundary bootstrap is closed by explaining any pole in the reflection matrix either as a new boundary state or by the boundary analogue of the Coleman-Thun mechanism. In Section 4 we check the boundary spectrum and reflection factors against finite volume spectra. Finally in Section 5 we analyze the semiclassical issues and conclude in Section 6.

Bootstrap in the bulk sine-Gordon theory

The bulk sine-Gordon theory described by (1) is an integrable model since it has infinitely many conserved quantities. As a consequence, there is no particle production in the scatterings and the multiparticle S-matrix factorizes into the product of two particle S-matrices, which can be computed using the requirements of unitarity, crossing symmetry and the

Yang-Baxter equation modulo CDD type ambiguity. The most general solution for the scattering of an $O(2)$ symmetric doublet has the following form

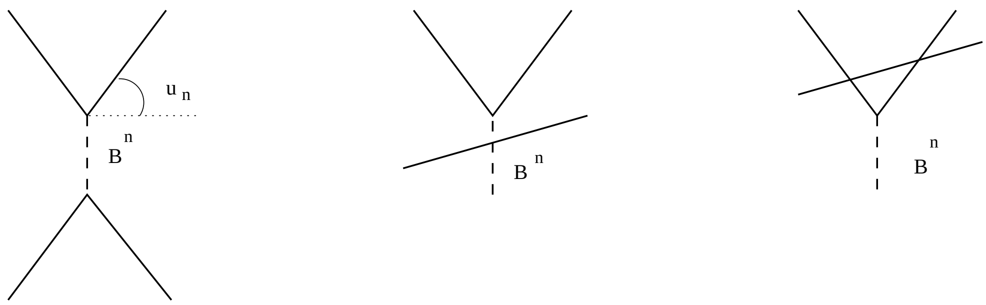
$$S(\lambda,\theta)=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{-\sin\lambda\pi}{\sin\lambda(\pi+i\theta)} & \frac{\sin i\lambda\theta}{\sin\lambda(\pi+i\theta)} & 0 \\ 0 & \frac{\sin i\lambda\theta}{\sin\lambda(\pi+i\theta)} & \frac{-\sin\lambda\pi}{\sin\lambda(\pi+i\theta)} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\times$$

$$\prod_{l=1}^{\infty}\left[\frac{\Gamma(2(l-1)\lambda+\frac{\lambda i\theta}{\pi})\Gamma(2l\lambda+1+\frac{\lambda i\theta}{\pi})}{\Gamma((2l-1)\lambda+\frac{\lambda i\theta}{\pi})\Gamma((2l-1)\lambda+1+\frac{\lambda i\theta}{\pi})}/(\theta\to-\theta)\right]\quad ,$$

where λ is a free parameter. It describes the scattering of the soliton anti-soliton doublet $(s,\bar{s})$.

The poles of the scattering matrix signal the existence of other particles appearing as bound states. For the soliton anti-soliton scattering they are located at $\theta=i(\pi/2-u_n)=i\pi/2-\frac{in\pi}{2\lambda}$. The corresponding particles are called breathers B_n and have masses $m_{B_n}=2M\sin(u_n)$.

The scattering matrix of the breathers on the soliton doublet can be computed from the bootstrap procedure, which graphically looks as follows:



Soliton (anti-soliton) lines are shown as straight lines, while the dashed ones correspond to breathers. The result turns out to be

$$S^n=\{n-1+\lambda\}\{n-3+\lambda\}\dots\quad\begin{cases}\{1+\lambda\} & \text{if } n \text{ is even}\\ -\sqrt{\{\lambda\}} & \text{if } n \text{ is odd}\end{cases}\quad ,$$

where

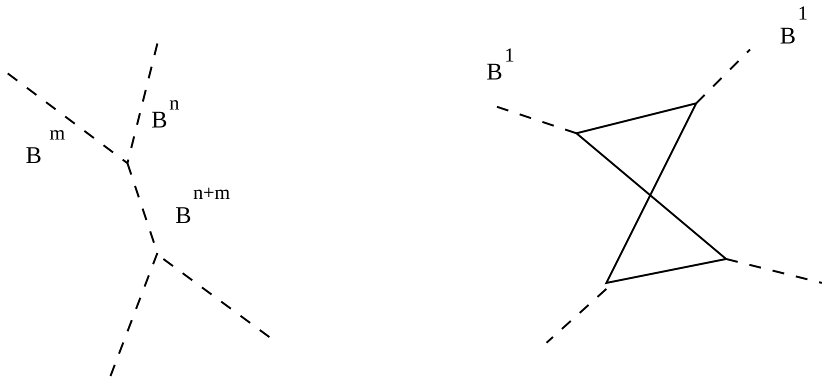
$$\{y\}=\frac{\left(\frac{y+1}{2\lambda}\right)\left(\frac{y-1}{2\lambda}\right)}{\left(\frac{y+1}{2\lambda}-1\right)\left(\frac{y-1}{2\lambda}+1\right)}\quad ;\quad (x)=\frac{\sin(i\theta/2-x\pi/2)}{\sin(i\theta/2+x\pi/2)}\quad .$$

Analysing the pole structure of the breather-soliton/anti-soliton scattering matrix we find poles, which can be explained by soliton/anti-soliton intermediate states.

The breather-breather scattering matrix can be computed again by applying the fusion procedure, but now for the breather-soliton scattering matrix. The result has the following compact form:

$$S^{n,m}=\{n+m-1\}\{n+m-3\}\dots\{n-m+3\}\{n-m+1\}\quad n\geq m\quad . \tag{3}$$

The poles of the scattering matrix (3) can be explained either by breather intermediate states or by Coleman-Thun type mechanism as illustrated on the following figure:



Since we explained all the poles of all the scattering matrices the sine-Gordon model is solved in the bootstrap sense. This solution has, however, no clear relation to the Lagrangean. In order to relate the bootstrap parameters to the parameters of the Lagrangean an alternative analysis is needed. Performing the comparison of Thermodynamic Bethe ansatz with conformal perturbation theory [14] or carrying out the semiclassical quantization of the model [15] the exact mapping can be obtained

$$\lambda = \frac{8\pi}{\beta^2} - 1 \quad ; \quad M = m^{\frac{8\pi}{8\pi-\beta^2}} \kappa(\beta) \quad ,$$

where the actual form of $\kappa(\beta)$ depends on the scheme in which the quantum theory is defined. In the conformal perturbation framework

$$\kappa(\beta) = \frac{2\Gamma\left(\frac{\beta^2}{2(8\pi-\beta^2)}\right)}{\sqrt{\pi}\Gamma\left(\frac{4\pi}{8\pi-\beta^2}\right)} \left[\frac{\pi\Gamma\left(1-\frac{\beta^2}{2\pi}\right)}{2\beta^2\Gamma\left(\frac{\beta^2}{8\pi}\right)}\right]^{\frac{4\pi}{8\pi-\beta^2}} \quad .$$

Bootstrap in the boundary sine-Gordon theory

The boundary sine-Gordon theory can be obtained by restricting the bulk sine-Gordon theory, (1), to the negative half line and imposing the most general integrability preserving boundary condition at the origin

$$\partial_x \Phi(x,t)|_{x=0} = -\frac{dV_B(\Phi)}{d\Phi} \quad .$$

Due to the integrability of the model the allowed physical processes are highly restricted. Besides the usual bulk constraints we also have factorized and elastic reflection on the boundary. Moreover, the one particle reflection matrices have to obey unitarity, boundary Yang-Baxter equations and boundary crossing. The most general solution contains two parameters, similarly to the boundary potential, and has the following form

$$R(\lambda,\eta,\Theta)=\begin{pmatrix} P^{+} & Q \\ Q & P^{-}\end{pmatrix} R_0(\theta)\frac{\sigma(\eta,\theta)}{\cos\eta}\frac{\sigma(i\Theta,\theta)}{\cosh\Theta}$$

$$P^{\pm}=\cos(i\lambda\theta)\cos\eta\cosh\Theta\pm(\cos\leftrightarrow\sin)\quad;\quad Q=\cos i\lambda\theta\sin i\lambda\theta\quad,$$

where

$$R_0(\theta)=\prod_{l=1}^{\infty}\left[\frac{\Gamma(4l\lambda+2i\lambda\theta/\pi)\Gamma(4(l-1)\lambda+1+2i\lambda\theta/\pi)}{\Gamma((4l-3)\lambda+2i\lambda\theta/\pi)\Gamma((4l-1)\lambda+1+2i\lambda\theta/\pi)}/(\theta\to-\theta)\right]$$

is the boundary condition independent part and

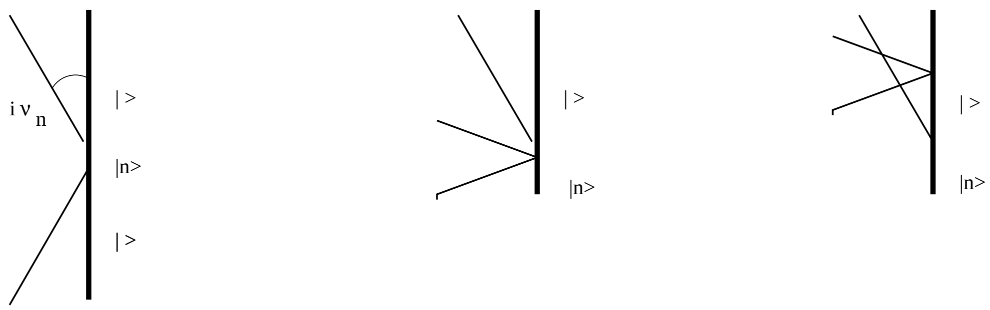
$$\sigma(x,\theta)=\frac{\cos(x)}{\cos(x-i\lambda\theta)}\prod_{l=1}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+\frac{x}{\pi}+(2l-1)\lambda+\frac{i\lambda\theta}{\pi})\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-1)\lambda+\frac{i\lambda\theta}{\pi})}{\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-2)\lambda+\frac{i\lambda\theta}{\pi})\Gamma(\frac{1}{2}+\frac{x}{\pi}+2l\lambda+\frac{i\lambda\theta}{\pi})}/(\theta\to-\theta)\right]$$

describes the boundary condition dependence.

The poles of these amplitudes signal the presence of boundary bound states. The boundary independent poles of $R_0(\theta)$ have explanations in terms of boundary Coleman-Thun mechanism [9]. The boundary dependent poles at

$$\theta=i\nu_n=i\left(\frac{\eta}{\lambda}-\frac{(2n+1)}{2\lambda}\right)$$

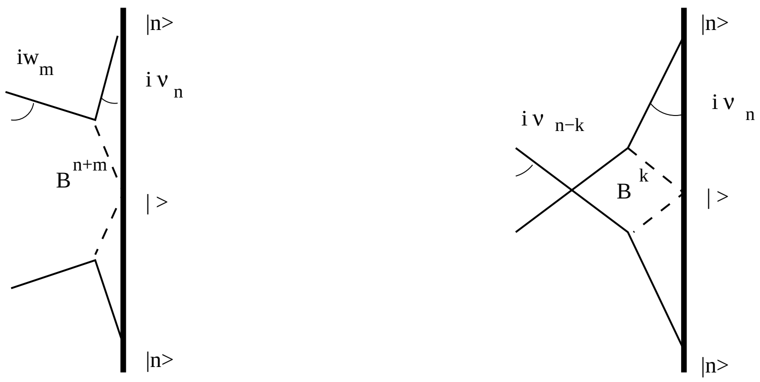
correspond to bound states, denoted by $|n\rangle$ with energy $m_{|n\rangle}=M\cos(\nu_n)$. The reflection factors on these boundary bound states can be computed from the bootstrap principle, which graphically looks as follows:



The result of the computation is

$$R_{|n\rangle}(\lambda,\eta,\Theta)=\bar{R}(\lambda,\bar{\eta},\Theta)a_n(\eta,\theta)\quad;\quad a_n(\eta,\theta)=\prod_{l=1}^{n}\left\{2\left(\frac{\eta}{\pi}-l\right)\right\}\quad,$$

where in $\bar{R}$ the solitons and the anti-solitons are changed as $\bar{P}^{\pm}(\lambda,\eta,\Theta)=P^{\mp}(\lambda,\bar{\eta},\Theta)$, $Q(\lambda,\eta,\Theta)=Q(\lambda,\bar{\eta},\Theta)$ and $\bar{\eta}=\pi(\lambda+1)-\eta$. Analyzing the pole structure of these excited reflection factors we find poles at $\theta=iw_m=i\nu_m(\bar{\eta})$ and at $\theta=i\nu_{n-k}$, with the following boundary Coleman-Thun explanations:



The first diagram, however, exists only for $w_m < \nu_n$, since B^{n+m} must travel towards the boundary. Thus for $w_m > \nu_n$ we have a new boundary boundstate denoted by $|n,m\rangle$: with energy $m_{|n,m\rangle} = M(\cos(\nu_n) + \cos(w_m))$.

Repeating these procedures one obtains the following pattern of boundary excited states and reflection factors: Ground state boundary

$$|\,\rangle \qquad R(\lambda,\eta,\Theta) \quad .$$

The excited states with one index have reflection factors and masses of the form

$$\begin{array}{c} |0\rangle \quad \bar{R}(\lambda,\bar{\eta},\Theta) \\ M\cos(\nu_0) \end{array} \dots \begin{array}{c} |n\rangle \quad \bar{R}(\lambda,\bar{\eta},\Theta)a_n(\eta,\theta) \\ M\cos(\nu_n) \end{array} \quad .$$

The excited boundary states with two indices have reflection factors and masses as

$$\begin{array}{c} |n,m\rangle \qquad R(\lambda,\eta,\Theta)a_n(\eta,\theta)a_m(\bar{\eta},\theta) \\ M\cos(\nu_n) + M\cos(w_m) \end{array} \quad .$$

The general state has any of these forms

$$\begin{array}{c} |n_1,m_1,\dots,n_k\rangle \qquad \bar{R}(\lambda,\bar{\eta},\Theta)a_{n_1}(\eta,\theta)a_{m_1}(\bar{\eta},\theta)\dots a_{n_k}(\eta,\theta) \\ M\cos(\nu_{n_1}) + M\cos(w_{m_1}) + \dots + M\cos(\nu_{n_k}) \end{array}$$

$$\begin{array}{c} |n_1,m_1,\dots,m_k\rangle \qquad R(\lambda,\eta,\Theta)a_{n_1}(\eta,\theta)a_{m_1}(\bar{\eta},\theta)\dots a_{m_k}(\bar{\eta},\theta) \\ M\cos(\nu_{n_1}) + M\cos(w_{m_1}) + \dots + M\cos(w_{m_k}) \end{array} \quad .$$

By finding these boundary excited states the bootstrap is closed in the sense, that on these boundaries with these reflection factors every pole can be explained by either a boundary Coleman-Thun diagram or a boundary bound state creation or both.

By virtue of their derivation the solution of the boostrap program contains the parameters of the ground state boundary reflection factor and has nothing to do with the parameters of the Lagrangean. Al. B. Zamolodchikov gave the relations of these parameters, which can be checked both in finite volume analysis and in semiclassical considerations.

$$\cos\left(\frac{\eta}{\lambda+1}\right)\cosh\left(\frac{\Theta}{\lambda+1}\right) = \frac{M_0}{M_{crit}(\beta)}\cos(\alpha) \quad ; \quad \alpha = \frac{\beta\varphi_0}{2}$$
$$\sin\left(\frac{\eta}{\lambda+1}\right)\sinh\left(\frac{\Theta}{\lambda+1}\right) = \frac{M_0}{M_{crit}(\beta)}\sin(\alpha) \quad ,$$

where, similarly to the bulk model, the actual form of $M_{crit}(\beta)$ depends on the renormalization scheme in which the model is defined. In the perturbed conformal field theory framework it is

$$M_{crit}(\beta) = m\sqrt{\frac{2}{\beta^2 \sin\left(\frac{\beta^2}{8}\right)}} \quad .$$

Finite volume analysis

In establishing the relation above one has to compare the exact bootstrap quantities with computations coming directly form the Lagrangean. One possible way is to put the system in a finite interval of size L and analyze the energy levels of the system as a function of the system size, imposing two different boundary conditions on the ends. The two extreme limits can be solved exactly.

The $L \to \infty$ limit is called the infrared (IR) limit. This is just the theory we solved: The energy eigenstates consists of arbitrary number of moving solitons, anti-solitons and breathers together with the boundary excited states corresponding to the two boundaries.

The $L \to 0$ limit is called the ultraviolet (UV) limit. Since all the potential terms, both the bulk and boundary, are scaled out the system looks like a free boson

$$H = \frac{1}{8\pi}\int_0^L \left(\Pi^2 + (\partial_x\Phi)^2\right) dx$$

with compactification radius r satisfying $r\beta = \sqrt{4\pi}$. That is a $c = 1$ conformal field theory with Neumann boundary condition applied on both ends. The spectrum can be read off from the Hilbert space $a_{-n_1}\dots a_{-n_k}|n\rangle$; $\Pi_0|n\rangle = \frac{n}{r}|n\rangle$ and from the Hamiltonian of the model

$$H = \frac{\pi}{L}\left(2\Pi_0^2 + \sum_{n\neq 0} n a_{-n}a_n\right) \quad ; \quad [a_n, a_m] = n\delta_{n+m}.$$

The matching of the IR and UV parameters can be achieved by introducing a finite volume analysis starting from the UV and another from the IR with overlaping regions and comparing the energy levels.

For small L we regard the boundary sine-Gordon theory as a joint bulk and boundary perturbation of the boundary conformal field theory introduced [8]. Using the vertex operators of the $c = 1$ model

$$V_n(x,t) \propto : e^{i\frac{n}{r}\Phi(x,t)} : \quad ; \quad \Psi_n(t) =: e^{i\frac{n}{r}\Phi(0,t)} :$$

the perturbation of the Hamiltonian has the form

$$H^{pert.}_{bulk} \to \frac{m^2}{2\beta^2}(V_2 + V_{-2}) \quad ; \quad H^{pert.}_{bd.} \to \frac{M_0}{2}(e^{-\frac{i}{r}\varphi_0}\Psi_1 + e^{\frac{i}{r}\varphi_0}\Psi_{-1}) \quad .$$

The computation of the matrix elements of the perturbing potential is straightforward, but tedious. Truncating the Hilbert space at certain energy levels and diagonalizing the total

Hamiltonian numericaly we arrive at the Truncated Conform Space Approach (TCSA) which provides a numerical finite volume spectra being exact for small L.

For large L we can obtain a finite volume spectrum by computing corrections to the IR spectrum. The energy levels of the moving particles

$$E(L) = \sqrt{M^2 + P(L)^2}$$

are affected by finite spatial volume. In the case of periodic boundary condition the momenta are quantized as

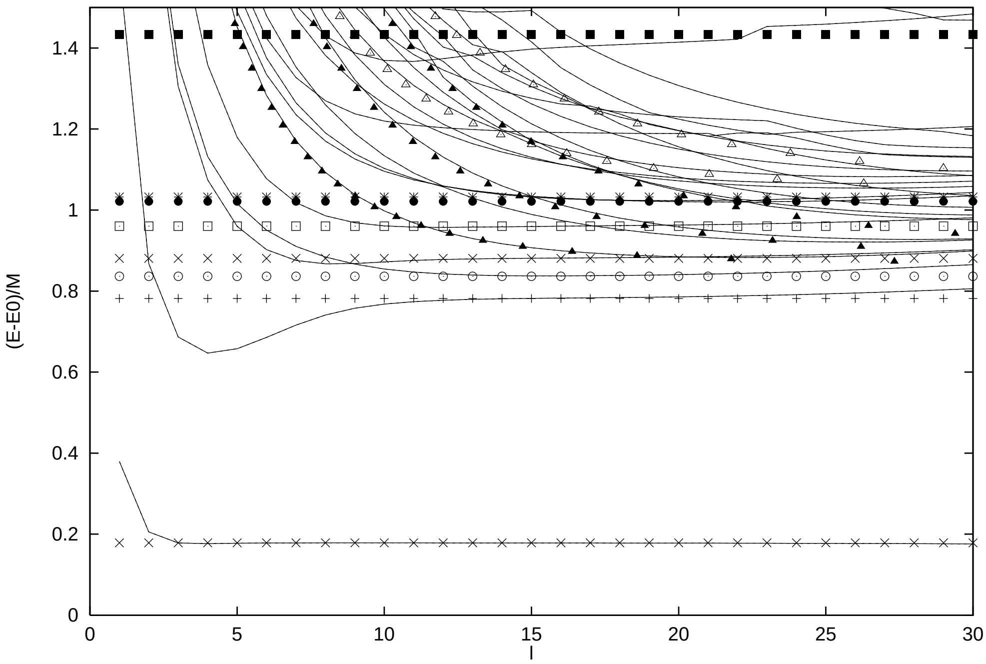
$$e^{iPL} = 1 \to P(L) = \frac{2\pi}{L} N \ .$$

If, however, we have reflection factors $R_0(P)$ and $R_L(P)$ on the two ends of the strip, then the momentum quantization for singlet one particle states changes as

$$e^{i2PL} R_0(P) R_L(P) = 1 \to P(L) \quad .$$

As a consequence the finite volume energy levels depend on the reflection factors and, in the case multiparticle states, also on the scattering matrices, depending in this way on the IR parameters.

The comparison between the small L and large L regions either gives a numerical matching between the UV and IR parameters or if this relation is already conjectured then it provides a numerical justification of its correctness. We used the second possibility with the result shown on the following figure.



On the figure continous lines come from the TCSA, while the others correspond to the various multiparticle IR lines. The observed very good aggrement shows the correctness not only of the UV-IR relation but also of the entire IR spectrum together with the reflection factors.

For completeness we mention that in order to derive the exact UV-IR relation one needs to compute the finite volume energy at least for one state, say for the ground state, exactly. The thermodynamic Bethe ansatz provides an integral equation for the ground state energy, containing the IR parameters, from which the boundary energy can be extracted [11, 10]. This quantity is related to the vacuum expectation value of the boundary vertex operator, which can also be calculated exactly in terms of the UV parameters [16]. The comparison of the two results gives the required UV-IR relation.

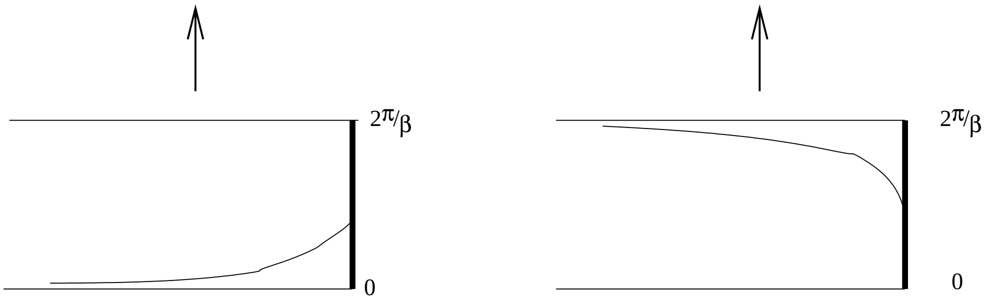
Semiclassical considerations

We have seen that the two lowest energy boundary state, the ground state and the first excited state, characterized by

$$|\rangle \quad R(\lambda,\eta,\Theta) \quad ; \quad \begin{array}{c} |0\rangle \quad \bar{R}(\lambda,\bar{\eta},\Theta) \\ M\cos(\nu_0) \end{array}$$

are related by the $s \leftrightarrow \bar{s}, \eta \leftrightarrow \bar{\eta}$ transformations. The classical analogues of these states are the two static solutions with lowest energy, given by a static bulk soliton/anti-soliton standing at the right place' : i.e. by choosing $\Phi \equiv \Phi_s(x,a^+)$ or $\Phi \equiv \Phi_{\bar{s}}(x,a^-)$ for $x \leq 0$, where

$$\Phi_s(x,a^+) = \frac{4}{\beta}\mathrm{arctg}(e^{m(x-a^+)}), \qquad \Phi_{\bar{s}}(x,a^-) = \frac{2\pi}{\beta} - \Phi_s(x,a^-),$$



and $a^{\pm}$ are determined by the boundary condition:

$$\sinh(ma^{\pm}) = \frac{A \pm \cos(\alpha)}{\sin(\alpha)} \quad ; \quad A = \frac{4m}{M_0\beta^2} \quad .$$

The energy difference of these two solutions can be written as

$$E_{\bar{s}}(M_0,\varphi_0) - E_s(M_0,\varphi_0) = M_0(R(+) - R(-)) \quad ; \quad R(\pm) = \sqrt{1 \pm 2A\cos(\alpha) + A^2} \quad . \tag{4}$$

In the process of semi-classical quantisation the oscillators associated to the linearized fluctuations around the static solutions $\Phi(x,t) = \Phi_{s,\bar{s}} + e^{i\omega t}\xi_{\pm}(x)$ are quantised. The equations of motion of these fluctuations describe how the elementary excitations of the field Φ -namely the first breather- behave in the presence of the nontrivial background. It can be written as:

$$\left[-\frac{d^2}{dx^2} + m^2 - \frac{2m^2}{\cosh^2(m[x-a^{\pm}])}\right]\xi_{\pm}(x) = \omega^2\xi_{\pm}(x); \qquad x < 0 \ , \tag{5}$$

where $\xi_\pm(x)$ must satisfy also the linearized version of the boundary condition:

$$\xi'_\pm(x)|_{x=0} = -\frac{m}{A}\frac{1 \pm A\cos\alpha}{R(\pm)}\xi_\pm(0) \quad . \tag{6}$$

These eigenvalue problems can be solved exactly by mapping eq.(5) to a hypergeometric differential equation, whose spectrum in general has a discrete and a continous part. The discrete real eigenvalues correspond to excited boundary states, while imaginary eigenvalues signals the instability of the static solution. The continous spectrum shows how the first breather reflects on the classical boundary. By summing up the zero point energies of the quantized fluctuations and eliminating the logarithmic divergencies by boundary (δm^2) and bulk parameter (δM_0) renormalization the semiclassically corrected energy difference can be computed exactly. Performing the complete analyzis we know semiclassically

- the energy of the excited boundary state,
- the reflection factor of the first breather on the ground state boundary,
- and the energy difference between the two lowest lying energy levels.

Comparing these quantities with the semiclassical limit, $\lambda \to \infty$, of their exact quantum values the semiclassical UV-IR parameter correspondence can be established. If we use the parametrization

$$\eta = \eta_{cl}(\lambda + 1) \quad ; \quad \Theta = \Theta_{cl}(\lambda + 1) \quad ,$$

then the relation is

$$\cos\eta_{cl} = \frac{R(+) - R(-)}{2A} \quad ; \quad \cosh\Theta_{cl} = \frac{R(+) + R(-)}{2A} \quad , \tag{7}$$

which also determines M_{crit} in the perturbative scheme as $M_{crit}/M_0 = A$. These correspondence can be also be confirmed by comparing the semiclassical limit of the solitonic reflection factors with the classical time delay [13].

Boundary resonance states and the stability of the classical solutions

The stability of a classical solution can be read off from the discrete spectrum in the semiclassical analyzis. It is convenient to write $\omega^2 = m^2(1-\epsilon^2)$. The normalizable solutions of eq.(5) must vanish at $x \to -\infty$, and assuming ϵ to be positive, they are given by:

$$\xi_\pm(x) = Ne^{m\epsilon(x-a^\pm)}(\epsilon - \tanh[m(x-a^\pm)]) \quad .$$

The boundary conditions determine the possible values of ϵ as

$$\epsilon^2 + \epsilon\frac{R(\pm)}{A} \pm \frac{\cos\alpha}{A} = 0 \quad .$$

It is easy to show, that for the solitonic ground state there is no positive solution of this equation, while for the anti-solitonic exited' state one of the roots, namely

$$\epsilon = \cos\eta_{cl} \quad , \tag{8}$$

is positive. In the framework of semi-classical quantisation these findings imply, that there are no boundary bound states for the ground state, described by Φ_s, while for the state, described by $\Phi_{\bar{s}}$, there is such a boundary bound state. The semiclassical energy of the boundary state $\omega_0 = m\sin\eta_{cl}$ vanishes for $\alpha \to 0$ if $A > 1$, which shows the instability of the anti-solitonic boundary solution. This is consistent with the classical picture, where the energy difference, (4), is precisely the mass of the bulk soliton, and since topological charge is not conserved in the boundary theory, the higher energy state can decay into the lower one by emitting a standing soliton.

At this point it is worth comparing the stability analysis of this $\alpha \to 0$ situation and the one when $\alpha = 0$ is set from the start, to emphasize the non smooth nature of the limit. In the latter case the two classical solutions become $\Phi_1 \equiv \frac{2\pi}{\beta}$ and $\Phi_2 \equiv 0$. The equations for the small fluctuations are

$$\left[-\frac{d^2}{dx^2} + m^2\right]\xi_\pm(x) = \omega^2\xi_\pm(x) \quad ; \quad \xi'_\pm(x)|_{x=0} = \mp\frac{m}{A}\xi_\pm(0) \quad .$$

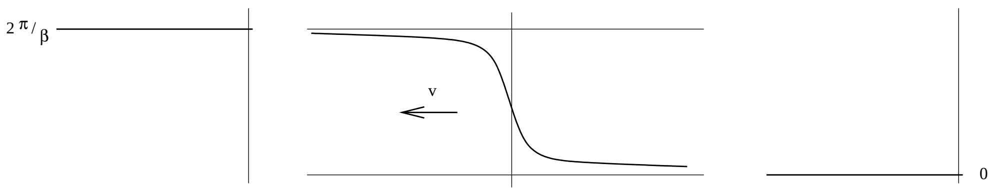
Repeating the stability analysis reveals that there are no normalizable bound state solutions for the ground state, Φ_2, while for the 'excited' state, Φ_1, there is a normalizable solution $Ne^{\frac{m}{A}x}$, with $\omega^2 = m^2(1-A^{-2})$. When $A > 1$ this solution signals the existence of a boundary state, while for $A < 1$, when this ω^2 becomes negative, it indicates the instability of Φ_1. The absolute value of the purely imaginary frequency is interpreted as the semiclassical resonance width:

$$\Gamma_{cl} = m\sqrt{\frac{1}{A^2} - 1} \quad . \tag{9}$$

Similarly to the $\alpha \to 0$ case analyzed above, we also have a nice classical interpretation. For this range of A there is a moving anti-solitonic solution of the equation of motions

$$\Phi_s(x,v) = \frac{4}{\beta}\mathrm{arctg}\left[e^{m\left(\frac{x-vt}{\sqrt{1-v^2}}-a\right)}\right] \quad , \quad v = \sqrt{1-A^2} \quad ,$$

which looks as follows



This solution for $t \to -\infty$ looks like the excited boundary state without any anti-soliton, (the anti-soliton is on the nonphysical part of the space time). For $t \to \infty$ the situation changes as follows: the boundary is in the ground state while an anti-soliton is moving far away from the boundary. So the excited boundary decayed into the ground state boundary by emitting a moving anti-soliton.

Let us focus on the quantum theory now. In a theory with bulk a resonance state the scattering matrix of the stable particles exhibits a pole singularity at $s = (M_{res} + i\Gamma/2)^2$, where M_{res} is the mass, while Γ the decay width of the resonance. This can be seen from the form of the bulk propagator $G(p)^{-1} \propto p^2 - m^2$. The boundary propagator has the form $G_B(p_0)^{-1} \propto p_0 - m$, thus a boundary resonance state shows up in the reflection factor as a pole in the energy at $p_0 = M_{res} + i\Gamma/2$, where M_{res} is the energy, while Γ is the decay width of the resonance state. In order to find the boundary resonance state we analize the solitonic reflection factors.

The semiclassical region, where we see the instable state, corresponds to the $\alpha = 0$ and $A > 1$ domain, which can be parametrized by $\eta_{cl} = 0$ and $\Theta_{cl} = \frac{\Theta}{\lambda+1}$ and can be reached as $\Theta \to \infty$, $\lambda \to \infty$. Concentrating on this asymptotic region the $\frac{\sigma(i\Theta,\theta)}{\cosh\Theta}$ term of the reflection factor has simple poles at

$$\theta_n = \frac{\Theta}{\lambda} - i\frac{(2n+1)\pi}{2\lambda} \quad ; \quad n \geq 0 \quad ,$$

from which the closest to the real axis is

$$\theta_0 = \frac{\Theta}{\lambda} - i\frac{\pi}{2\lambda} \quad .$$

The energy of this resonance state has a real and an imaginary part

$$E - E_0 = M\cosh\left(\frac{\Theta}{\lambda}\right)\cos\left(\frac{\pi}{2\lambda}\right) - iM\sinh\left(\frac{\Theta}{\lambda}\right)\sin\left(\frac{\pi}{2\lambda}\right) \quad ,$$

which in the semiclassical limit can be written as

$$E - E_0 = M\cosh\Theta_{cl} - iM\frac{\pi}{2\lambda}\sinh\Theta_{cl} \quad .$$

Using the semiclassical UV-IR relation, (7), for $\eta_{cl} = 0$ we have $\cosh(\Theta_{cl}) = A^{-1}$. Since the semiclassical soliton mass is $M = \frac{8m}{\beta^2}(1 - \frac{\beta^2}{2\pi})$ the leading order of the real part reproduces the energy difference (4), while the imaginary part the semiclassical decay width (9).

We have tried to analyze the effect of the boundary resonance state for the finite volume spectra of the model. We investigated the behaviour of the solitonic reflection factor near the resonance, but we could not tune the parameters to keep the resonance strong and obtain a believable TCSA spectrum in the same time. Thus the resonance was unobservable. P. Dorey pointed out, however, that a more significant effect might be obtained by analyzing the ground state energy of the system for small volumes similarly to the homogenous sine Gordon case talk by J. L. Miramontes.

Conclusions

We reviewed our recent results on the boundary sine-Gordon model. By closing the boundary bootstrap we determined the spectrum of boundary excitations together with the corresponding reflection factors. In order to check the results we rederived Zamolodchikov‘s UV-IR relation and used it in finite volume analyzis to confirm their correctness. We also performed a semiclassical quantization, were the correspondence between a semiclassically unstable static solution and the resonance pole of the solitonic reflection factors was analyzed in detail.

The main open problems are the calculation of off-shell quantities (e.g. correlation functions) and exact finite size spectra. While correlation functions in general present a very hard problem even in theories without boundaries, in integrable theories significant progress was made using form factors. It would be interesting to make further progress in this direction.

It would also be interesting to work out a formalism (an analogue of the Cutkosky rules of quantum field theory in the bulk) in which the rules for the boundary Coleman-Thun diagrams can be justified. Following [17] a work is in progress in this direction.

Now we are able to report that, as an important step in the supersymmetric generalisation of the model, the boundary spectrum and reflection factors have been determined [18] by closing the boundary bootstrap. In order to confirm these results we are developing a TCSA analyzis for the supersymmetric theory.

Acknowledgements

We thank M. Kormos and G. Zs. Tóth for the collaboration in [13] and [9], respectively. Z. B. thanks for the kind hospitality in Sao Paulo and the useful discussions with Marco Moriconi, Gustav Delius and Nial MacKay. Z.B. also thanks for the Hungarian National Science Fund (OTKA) for supporting the travel expences and for the Bolyai Fellowship. This research was supported in part by the Hungarian Ministry of Education under FKFP 0043/2001 and by the Hungarian National Science Fund (OTKA) T037674/02, T034299/01.

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