

Universal Ratios and Correlation Functions

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ABSTRACT: We review some recent results concerning the quantitative analysis of the universality classes of two-dimensional statistical models near their critical point. We also discuss the exact calculation of the two-point correlation functions of disorder operators in a free theory of complex bosonic and fermionic field, correlators ruled by a Painlevé differential equation.

1. Introduction

One of the most important successes of Quantum Field Theory (QFT) in recent years is the quantitative analysis of the universality classes of two-dimensional statistical mechanical models near their second order phase transition points. This has been possible, in particular, thanks to new developments in the methods of computing off-critical correlation functions. The aim of this talk is to review some of the recent results relative to these two important subjects. Our discussion is divided in two parts: in the first part, we shall see how correlation functions can be used to extract useful numbers (the so-called *universal ratios*) that can be directly compared with experiments. In the second part, we shall concentrate on a particularly interesting class of models, those of fermionic and bosonic theories with a Z_n symmetry, for which there is an elegant approach to derive differential equations satisfied by the two-point functions of their disorder operators.

2. Universal Amplitude Ratios

Consider a statistical model with n relevant fields $\varphi_i(x)$ at criticality. Near the critical point, its action can be parameterised as

$$\mathcal{A} = \mathcal{A}_{CFT} + g_i \int \varphi_i(x) d^D x , \quad (2.1)$$

where \mathcal{A}_{CFT} is the action which encodes the data of the Conformal Field Theory of the critical point. The short-distance behaviour of the two-point functions is given by

$$\langle \varphi_i(x) \varphi_i(0) \rangle \simeq \frac{1}{|x|^{4\Delta_i}}$$

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for $|x| \rightarrow 0$, where Δ_i is the conformal dimension of the operator φ_i . Hence, their conjugate coupling constants g_i behave as $g_i \sim \Lambda^{D-2\Delta_i}$, where Λ is a mass scale. Away from criticality there will be generally a finite correlation length ξ which in the thermodynamical limit scales as $\xi \sim a (K_i g_i)^{-\frac{1}{D-2\Delta_i}}$, where $a \sim \Lambda^{-1}$ may be regarded as a microscopic length scale. The dimensionless quantities K_i are non-universal metric factors specific of the system representing the universality class and depending on the units chosen for measuring the external sources g_i . If there are several deformations of the conformal action, the most general expression for the scaling form of the correlation length may be written as

$$\xi = \xi_i \equiv a (K_i g_i)^{-\frac{1}{D-2\Delta_i}} \mathcal{L}_i \left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}} \right), \quad (2.2)$$

where $\phi_{ji} \equiv \frac{D-2\Delta_j}{D-2\Delta_i}$ are the so-called *crossover exponents* whereas \mathcal{L}_i are universal homogeneous scaling functions of the ratios $\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}$. There are of course several (but equivalent) ways of expressing these scaling forms, depending on which coupling constant is selected as a prefactor. In the limit where $g_l \rightarrow 0$ ($l \neq i$) but $g_i \neq 0$, equation (2.2) becomes

$$\xi_i = a \xi_i^0 g_i^{-\frac{1}{D-2\Delta_i}}, \quad \xi_i^0 \sim K_i^{-\frac{1}{D-2\Delta_i}}. \quad (2.3)$$

Let us consider now the free-energy $\hat{f}[g_1, \dots, g_n]$, a dimensionless quantity defined by

$$e^{-\hat{f}(g_1, \dots, g_n)} = \int \mathcal{D}\phi e^{-[\mathcal{A}_{CFT} + \sum_{i=1}^n g_i \int \varphi_i(x) d^D x]}. \quad (2.4)$$

Under the hypothesis of hyperscaling, its singular part (per unit of volume) $f[g_1, \dots, g_n]$ is proportional, in the thermodynamical limit, to the D -th power of the correlation length. Depending on the scaling form adopted for the correlation length, we have correspondingly several (but equivalent) ways of parameterizing this quantity

$$f_i[g_1, \dots, g_n] \equiv (K_i g_i)^{\frac{D}{D-2\Delta_i}} \mathcal{F}_i \left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}} \right). \quad (2.5)$$

The functions \mathcal{F}_i are universal homogeneous scaling functions of the ratios $\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}$.

From the above expression we can derive the expectation values of several quantities. We will use the notation $\langle \dots \rangle_i$ to denote expectation values computed in the off-critical theory obtained by keeping (at the end) only the coupling constant g_i different from zero. Basic quantities are the vacuum expectation values (VEV) of the fields φ_j which can be parameterized as

$$\langle \varphi_j \rangle_i = - \left. \frac{\partial f_i}{\partial g_j} \right|_{g_l=0} \equiv B_{ji} g_i^{\frac{2\Delta_j}{D-2\Delta_i}}, \quad (2.6)$$

with

$$B_{ji} \sim K_j K_i^{\frac{2\Delta_j}{D-2\Delta_i}}. \quad (2.7)$$

The above relations can be equivalently expressed as

$$g_i = D_{ij} (\langle \varphi_j \rangle_i)^{\frac{D-2\Delta_i}{2\Delta_j}}, \quad (2.8)$$

with

$$D_{ij} \sim \frac{1}{K_i K_j^{\frac{D-2\Delta_i}{2\Delta_j}}} . \quad (2.9)$$

The generalized susceptibilities of the model are defined by

$$\hat{\Gamma}_{jk}^i = \frac{\partial}{\partial g_k} \langle \varphi_j \rangle_i = - \left. \frac{\partial^2 f_i}{\partial g_k \partial g_j} \right|_{g_l=0} . \quad (2.10)$$

They are obviously symmetrical in the two lower indices. By extracting their dependence on the coupling constant g_i , they can be expressed as

$$\hat{\Gamma}_{jk}^i = \Gamma_{jk}^i g_i^{\frac{2\Delta_j+2\Delta_k-D}{D-2\Delta_i}} , \quad (2.11)$$

with

$$\Gamma_{jk}^i \sim K_j K_k K_i^{\frac{2\Delta_j+2\Delta_k-D}{D-2\Delta_i}} . \quad (2.12)$$

The various quantities obtained by taking the derivatives of the free-energy obviously contain the quantities K_i which make their values not universal. However, it is easy and always possible to consider special combinations thereof in such a way to cancel out all metric factors. Universal Ratios usually considered in the literature are those given below

$$(R_c)_j^i = \frac{\Gamma_{ii}^i \Gamma_{jk}^i}{B_{ji} B_{ki}} ; \quad (2.13)$$

$$(R_\chi)_j^i = \Gamma_{jj}^i D_{jj} B_{ji}^{\frac{D-4\Delta_j}{2\Delta_j}} ; \quad (2.14)$$

$$R_\xi^i = (\Gamma_{ii}^i)^{1/D} \xi_i^0 ; \quad (2.15)$$

$$(R_A)_j^i = \Gamma_{jj}^i D_{ii}^{\frac{4\Delta_j+2\Delta_i-2D}{D-2\Delta_i}} B_{ij}^{\frac{2\Delta_j-D}{\Delta_i}} ; \quad (2.16)$$

$$(Q_2)_{jk}^i = \frac{\Gamma_{jj}^i}{\Gamma_{jj}^k} \left(\frac{\xi_k^0}{\xi_j^0} \right)^{D-4\Delta_j} . \quad (2.17)$$

From their definition, these quantities are pure numbers attached to the universality classes and therefore they can be used to characterize them. Contrary to the critical exponents (which are characteristic of the critical point) they carry information about the *scaling region*. Moreover, the amplitude ratios are numbers which typically present significant variations between different classes of universality, whereas the critical exponents usually assume small values which only vary by a small percent when changing the universality class. Hence the universal ratios may be ideal marks of the critical scaling regime [1]. It is also worth emphasizing that, from an experimental point of view, it should be simpler to measure universal amplitude ratios rather than critical exponents: to determine the former quantities one needs to perform several measurements at a single, fixed value of the coupling which drives the system away from criticality whereas to determine the latter, one needs to make measurements over several decades along the axes of the off-critical couplings. Moreover, although not all of them are independent, the universal ratios are a larger set of numbers than the critical exponents and therefore permit a more precise determination of the class of universality.

3. Quantum Field Theory Approach

It is interesting to see how in two dimensions Quantum Field Theory provides a powerful approach to compute the Universal Ratios of a given class of universality in a non-perturbative way. In the following we concentrate mostly our attention on the off-critical models obtained as deformations of the so-called *Minimal Models* of two-dimensional Conformal Field Theory [2, 3].

As we have seen before, each coupling constant g_i relative to the relevant operator $\varphi_i(x)$ of the model under investigation is a dimensional quantity which can be related to the lowest mass-gap $m_i = \xi_i^{-1}$ of the off-critical theory according to the formula

$$m_i = C_i g_i^{\frac{1}{2-2\Delta_i}} . \quad (3.1)$$

When the QFT associated to the action (2.1) is integrable, the pure number C_i can be exactly determined by means of the Thermodynamical Bethe Ansatz [4, 5]. When the theory is not integrable, the constant C_i can be nevertheless determined by a numerical method, based on the so-called Truncated Conformal Space Approach [6]. In conclusion, for all individual deformations of a given model, we are able to completely set the relationship which links the coupling constant to the mass-gap of the theory and therefore to switch freely between these two variables.

Additional quantities which can be determined by QFT are the matrix elements of the order parameters, the simplest ones being the vacuum expectation values (VEV). They are parameterised as

$$\langle \varphi_j \rangle_i = B_{ji} g_i^{\frac{\Delta_j}{1-\Delta_i}} . \quad (3.2)$$

When the theory is integrable, the constant B_{ji} can be fixed exactly, thanks to the results of a remarkable series of papers [7, 8]. When it is not integrable, the constant B_{ji} can be nevertheless estimated by means of a numerical approach, as firstly shown in [9]. Hence, we are able to determine completely also these quantities. Moreover, as shown in [11], a generalization of the numerical approach of ref. [9] often leads to a reasonable estimate of the matrix elements of the order parameters between the vacuum states and some of the excited states, as for instance $\langle 0 | \varphi_j | A_k \rangle_i$ where A_k is a one-particle state of mass M_k . These quantities turn out to be useful for obtaining sensible approximation of the large-distance behavior of several correlators.

Another important and useful piece of information on the off-critical models can be obtained by exploiting the properties of the stress-energy tensor $T_{\mu\nu}(x)$. In the presence of the perturbing field φ_i , the trace of the stress-energy tensor is given by

$$\Theta(x) = 2\pi g_i (2 - 2\Delta_i) \varphi_i . \quad (3.3)$$

It enters a useful sum rule, the so called Δ -theorem [12]

$$\Delta_j = -\frac{1}{4\pi \langle \varphi_j \rangle_i} \int d^2x \langle \Theta(x) \varphi_j(0) \rangle_i^c , \quad (3.4)$$

which relates the conformal dimension Δ_j of the field φ_j to its VEV and to the integral of its connected off-critical correlator with $\Theta(x)$. It is easy to see that the above formula simply expresses the content of the fluctuation-dissipation theorem and when the above integral

diverges, so does the VEV in the denominator, in such a way that eq. (3.4) always keeps its validity [11].

As discussed above, basic quantities entering the universal ratios are the generalized susceptibilities Γ_{jk}^i . By using the fluctuation–dissipation theorem, they are given by

$$\hat{\Gamma}_{jk}^i = \int d^2x \langle \varphi_j(x) \varphi_k(0) \rangle_i^c . \quad (3.5)$$

By extracting their dependence on the coupling constant g_i , we have $\hat{\Gamma}_{jk}^i = \Gamma_{jk}^i g_i^{\frac{\Delta_j + \Delta_k - 1}{1 - \Delta_i}}$ with

$$\Gamma_{jk}^i = C_i^{2\Delta_j + 2\Delta_k - 2} \int d\tau \frac{1}{\tau^{2\Delta_j + 2\Delta_k}} Q_{jk}(\tau) . \quad (3.6)$$

Some of the above susceptibilities can be determined exactly, such as the components Γ_{ik}^i , whose values are provided by the Δ –theorem sum rule

$$\Gamma_{ik}^i = -\frac{\Delta_k}{1 - \Delta_k} B_{ki} . \quad (3.7)$$

In all other cases, when an exact formula is not available, the strategy to evaluate the generalized susceptibilities relies on two different possible representations of the correlators. As we will see, these representations have the advantage of converging very rapidly in two distinct regions and to smoothly join in between.

The first representation is based on Conformal Perturbation Theory (with the use of the non–analytic expression of the VEV) [13]. In this approach the two–point correlators are expressed as

$$\langle \varphi_i(x) \varphi_j(0) \rangle = \sum_p C_{ij}^p(g; x) \langle A_p(0) \rangle \quad (3.8)$$

where the structure functions $C_{jk}^p(g; x)$ admit the expansion

$$C_{ij}^p(g; x) = r^{2(\Delta_p - \Delta_i - \Delta_j)} \sum_{n=0}^{\infty} C_{i,j}^{p(n)} (gr^{2-2\Delta_p})^n , \quad (3.9)$$

($r = |x|$) and can be computed perturbatively in g , the coupling conjugated to the perturbing operator φ . For instance, the first order correction to the conformal structure constant $C_{i,j}^{p(0)}$ is determined by [13]

$$C_{i,j}^{p(1)} = - \int' d^2w \langle A^p(\infty) \varphi(w) \varphi_i(1) \varphi_j(0) \rangle_{CFT} , \quad (3.10)$$

where the prime indicates a suitable infrared (large distance) regularization of the integral. This representation allows a very efficient estimate of the correlation function in its short distance regime $r \ll \xi$.

The second representation is based on the Form Factors (which will be discussed in more detail below) and allows an efficient control of the large distance behavior, i.e. when $r \gg \xi$. In this second representation, one makes use of the knowledge of the off–critical mass spectrum of the theory to express the correlators as

$$\langle \varphi_i(x) \varphi_j(0) \rangle = \sum_{n=0}^{\infty} g_{i,j}^{(n)}(r) , \quad (3.11)$$

where

$$g_{i,j}^{(n)}(r) = \int_{\theta_1 > \theta_2 \dots > \theta_n} \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} e^{-r \sum_{k=1}^n m_k \cosh \theta_k} \\ \times \langle 0 | \varphi_i(0) | \dots A_{a_n}(\theta_n) \rangle \langle \dots A_{a_n}(\theta_n) | \varphi_j(0) | 0 \rangle .$$

$|A_{a_1}(\theta_1) \dots A_{a_n}(\theta_n)\rangle$ are the multi-particle states relative to the excitations of mass m_k , with relativistic dispersion relations given by $E = m_k \cosh \theta$, $p = m_k \sinh \theta$, where θ is the rapidity variable. The spectral representation (3.12) is an expansion in the parameter $e^{-\frac{r}{\xi}}$, where $\xi^{-1} = m_1$ is the lowest mass-gap.

Basic quantities of the large distance approach are the Form Factors (FF), *i.e.* the matrix elements of the operators φ_i on the physical asymptotic states

$$F_{a_1, \dots, a_n}^{\varphi_i}(\theta_1, \dots, \theta_n) = \langle 0 | \varphi_i(0) | A_{a_1}(\theta_1) \dots A_{a_n}(\theta_n) \rangle . \quad (3.12)$$

For scalar operators, relativistic invariance requires that the FF only depend on the rapidity differences $\theta_i - \theta_j$. In integrable quantum field theories the FF are solutions of some functional equations [22, 23] which will be recalled in the next section. It is worth to stress that for most of the operators, it is sufficient to insert into the spectral representations only their one-particle and two-particle FF in order to reach a very reasonable estimate of the correlators.

In conclusion, both the expansions (3.8) and (3.11) converge very rapidly in their domain (see, for instance [13, 14, 15]) and therefore it is possible to estimate the integral (3.5) by following this procedure:

1. Express the integral in polar coordinates as

$$\hat{\Gamma}_{jk}^i = 2\pi \int_0^{+\infty} dr r \langle \varphi_j(r) \varphi_k(0) \rangle_c^i , \quad (3.13)$$

and split the radial integral into two pieces as

$$I = \int_0^{+\infty} dr r \langle \varphi_j(r) \varphi_k(0) \rangle_c^i \\ = \int_0^R dr r \langle \dots \rangle_c^i + \int_R^{+\infty} dr r \langle \dots \rangle_c^i \\ \equiv I_1(R) + I_2(R) . \quad (3.14)$$

2. Use the best available short-distance representation of the correlator to evaluate $I_1(R)$ as well as the best available estimate of its large-distance representation to evaluate $I_2(R)$.
3. Optimize the choice of the parameter R in such a way to obtain the best evaluation of the whole integral. In practice, this means looking at that value of R for which a plateau is obtained for the sum of $I_1(R)$ and $I_2(R)$. Said in another way, R belongs to that interval where there is an overlap between the short-distance and the long-distance expansion of the correlator.

The above methods have been applied to compute universal ratios of several interesting universality classes, such as the one of the Ising model [16], the Tricritical Ising Model [10, 11], the Self-Avoiding Walks [14], the q -state Potts model and percolation [17], to quote few. These field theoretical results have been confirmed through direct lattice estimates (see Refs. [18, 19, 20, 21]).

4. Disorder Operator and Fermion-Boson Correspondence

From the above discussion, we have seen that a great deal of information can be extracted for the scaling region of a model also in the absence of an exact expression of the correlation functions. It is however an important theoretical problem to see whether it is possible to obtain a closed and exact formula for these quantities. It goes without saying that, in general, this is quite a difficult problem, partially solved only for few models. In the second part of this talk, we shall discuss a particularly interesting example of such a computation, which leads to a differential equation ruling the correlators. Our presentation is based on the results presented in the paper [24].

In order to start our discussion, let us consider the (euclidean) action

$$\mathcal{A} = \int d^2x [\partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi + W(\phi^*, \phi)], \quad (4.1)$$

where the potential W is invariant under the Z_N transformation $\phi \rightarrow e^{2i\pi/N} \phi$, $\phi^* \rightarrow e^{-2i\pi/N} \phi^*$. Let $|0_j\rangle$, $j = 0, 1, \dots, N-1$, be the N vacua of the broken phase of this theory, with $\mu_k(x)$, $k = 1, \dots, N-1$, the disorder operators which create the excitations (kink states) interpolating between the vacua $|0_j\rangle$ and $|0_{j+k(\text{mod } N)}\rangle$. These operators carry k units of topological charge and satisfy the conjugation relation $\mu_k^*(x) = \mu_{N-k}(x)$. The mutual non-locality between the order and disorder operators gives rise to a phase factor when they go around each other in the euclidean plane ($z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$):

$$\begin{aligned} \phi(ze^{2i\pi}, \bar{z}e^{-2i\pi}) \mu_k(0, 0) &= e^{\frac{2i\pi k}{N}} \phi(z, \bar{z}) \mu_k(0, 0), \\ \phi^*(ze^{2i\pi}, \bar{z}e^{-2i\pi}) \mu_k(0, 0) &= e^{-\frac{2i\pi k}{N}} \phi^*(z, \bar{z}) \mu_k(0, 0). \end{aligned}$$

These relations are valid along any Z_N -invariant renormalization group trajectory flowing out of the phase transition point and characterise the operators $\mu_k(x)$ beyond their initial role of kink creation operators in the broken phase. The theory (4.1) with $W = 0$ is a particularly simple example of such a trajectory for which, as we will see, the correlation functions of disorder operators remain non-trivial due to the non-locality with respect to the boson.

The correlation functions of non-local operators in two-dimensional free massive theories were extensively studied from the point of view of the isomonodromy theory of differential equations in Ref. [26] and a series of related papers. Several works (see for instance Refs.[27, 28, 29, 30, 31]) have been devoted afterwards to dealing with this problem through more direct and general methods of quantum field theory. As we have seen above, an important approach to compute the correlation functions is the one based on the form factors, in which the correlators are expressed as sums over multiparticle asymptotic states. In our example this series can be explicitly resummed and the correlators are given in terms of solutions of non-linear differential equations. Moreover, there is a simple unified treatment of the bosonic and fermionic cases, with the final result expressed by the correspondence¹

$$\langle \tilde{\mu}_j(x) \tilde{\mu}_k(0) \rangle = \frac{1}{\langle \tilde{V}_{j/N}(x) \tilde{V}_{k/N}(0) \rangle}, \quad (4.2)$$

¹We use the notation $\tilde{\Phi}(x) \equiv \Phi(x)/\langle \Phi \rangle$.

where the correlators on the l.h.s. are computed in the theory (4.1) with $W = 0$, and those on the r.h.s. refer to the operators $V_\alpha(x) = \exp[i\sqrt{4\pi}\alpha\varphi(x)]$ in the sine-Gordon theory

$$\mathcal{A}_{sG} = \int d^2x \left[\frac{1}{2} \partial_\nu \varphi \partial^\nu \varphi - \mu \cos \beta \varphi \right], \quad (4.3)$$

with $\beta = \sqrt{4\pi}$ and μ a suitably chosen mass scale. The theory (4.3) with $\beta = \sqrt{4\pi}$ is in fact a free fermionic theory [32]. For generic values of β the elementary excitations are the solitons and antisolitons interpolating between adjacent vacua of the periodic potential. Their interaction is attractive as long as $\beta < \sqrt{4\pi}$ and they form topologically neutral bound states, the lightest one being the particle interpolated by the bosonic field in the action (4.3). These neutral particles are absent from the spectrum of asymptotic states in the repulsive region $\beta > \sqrt{4\pi}$, and at the point $\beta = \sqrt{4\pi}$ where the solitons behave as free Dirac fermions.

Since the solitons are non-local in terms of the field $\varphi(x)$, the evaluation of both sides of Eq. (4.2) amounts to computing correlation functions of operators which are non-local with respect to non-interacting particles, bosons for the l.h.s. and fermions for the r.h.s. The different statistics of the two particles is responsible for the inversion in Eq. (4.2).

Consider then a theory of free, charge conjugated particles A and \bar{A} with mass m , and denote by $\tilde{\Phi}_\alpha(x)$ a scalar operator having a non-locality phase $e^{2i\pi\alpha}$ ($e^{-2i\pi\alpha}$) with respect to (the field which interpolates) the particle A (\bar{A}). Acting on the vacuum state $|0\rangle$, such an operator produces neutral states consisting of pairs $A(\theta)\bar{A}(\beta)$. The equations satisfied by the form factors of these theories

$$\begin{aligned} f_n^\alpha(\theta_1, \dots, \theta_n, \beta_1, \dots, \beta_n) &= \\ &= \langle 0 | \tilde{\Phi}_\alpha(0) | A(\theta_1), \dots, A(\theta_n), \bar{A}(\beta_1), \dots, \bar{A}(\beta_n) \rangle \end{aligned} \quad (4.4)$$

are a particular case of those holding for generic integrable theories (see e.g. [29])

$$\begin{aligned} f_n^\alpha(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n, \beta_1, \dots, \beta_n) &= \\ &= S f_n^\alpha(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n, \beta_1, \dots, \beta_n), \\ f_n^\alpha(\theta_1 + 2i\pi, \theta_2, \dots, \theta_n, \beta_1, \dots, \beta_n) &= \\ &= S e^{2i\pi\alpha} f_n^\alpha(\theta_1, \dots, \theta_n, \beta_1, \dots, \beta_n), \\ \text{Res}_{\theta_1 - \beta_1 = i\pi} f_n^\alpha(\theta_1, \dots, \theta_n, \beta_1, \dots, \beta_n) &= \\ &= i S^{n-1} (1 - e^{2i\pi\alpha}) f_{n-1}^\alpha(\theta_2, \dots, \theta_n, \beta_2, \dots, \beta_n), \end{aligned}$$

where

$$S = \begin{cases} 1 & \text{for free bosons} \\ -1 & \text{for free fermions.} \end{cases} \quad (4.5)$$

It is easy to check that the solution of the above system of equations is given by

$$\begin{aligned} f_n^\alpha(\theta_1, \dots, \theta_n, \beta_1, \dots, \beta_n) &= S^{n(n+2)/2} (-\sin \pi\alpha)^n \\ &\times e^{(\alpha - \frac{1}{2}\delta_{S,1}) \sum_{i=1}^n (\theta_i - \beta_i)} |A_n|_{(S)}, \end{aligned} \quad (4.6)$$

where A_n is a $n \times n$ matrix ($A_0 \equiv 1$) with entries

$$A_{ij} = \frac{1}{\cosh \frac{\theta_i - \beta_j}{2}}, \quad (4.7)$$

and $|A_n|_{(S)}$ denotes the permanent² of A_n for $S = 1$ and the determinant of A_n for $S = -1$.

The two-point correlators are then given by

$$G_{\alpha,\alpha'}^{(S)}(t) = \langle \tilde{\Phi}_\alpha(x) \tilde{\Phi}_{\alpha'}(0) \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2 (2\pi)^{2n}} \int d\theta_1 \dots d\theta_n d\beta_1 \dots d\beta_n g_n^{(\alpha,\alpha')}(t | \theta_1, \dots, \beta_n),$$

where

$$\begin{aligned} g_n^{(\alpha,\alpha')}(t) &= f_n^\alpha(\theta_1, \dots, \beta_n) f_n^{\alpha'}(\beta_n, \dots, \theta_1) e^{-te_n} \\ &= (S \sin \pi\alpha \sin \pi\alpha')^n e^{(\alpha-\alpha') \sum_{i=1}^n (\theta_i - \beta_i)} \\ &\quad \times |A_n|_{(S)}^2 e^{-te_n}, \end{aligned} \quad (4.8)$$

and

$$t = m|x|, \quad e_n = \sum_{k=1}^n (\cosh \theta_k + \cosh \beta_k).$$

Analogously to the procedure followed in [31], we define a new $n \times n$ matrix M_n with entries

$$\begin{aligned} M_{ij} &\equiv M(\theta_i, \beta_j) = (\sin \pi\alpha \sin \pi\alpha')^{1/2} e^{-\frac{t}{2} \cosh \theta_i} \\ &\quad \times \frac{h(\theta_i) h^{-1}(\beta_j)}{\cosh \frac{\theta_i - \beta_j}{2}} e^{-\frac{t}{2} \cosh \beta_j}, \end{aligned} \quad (4.9)$$

where

$$h(\theta) = e^{(\alpha-\alpha')\theta/2},$$

and rewrite $g^{(\alpha,\alpha')}$ as

$$g_n^{(\alpha,\alpha')} = S^n |M_n|_{(S)}^2 = \begin{vmatrix} 0 & M_n \\ M_n^T & 0 \end{vmatrix}_{(S)}. \quad (4.10)$$

Finally we symmetrise with respect to the two sets of rapidities by introducing a charge index ε which is 1 for a particle A and -1 for \bar{A} , so that we express the correlation functions as

$$\begin{aligned} G_{\alpha,\alpha'}^{(S)}(t) &= \sum_{L=0}^{\infty} \frac{1}{L! (2\pi)^L} \sum_{\varepsilon_1 \dots \varepsilon_L} \int d\theta_1 \dots d\theta_L \\ &\quad \times |K_{\varepsilon_i \varepsilon_j}(\theta_i, \theta_j)|_{(S)}, \end{aligned} \quad (4.11)$$

where we have introduced the $L \times L$ matrices with entries

$$\begin{aligned} K_{+-}(\theta, \beta) &= M(\theta, \beta), \\ K_{-+}(\theta, \beta) &= M(\beta, \theta), \\ K_{++}(\theta, \beta) &= K_{--}(\theta, \beta) = 0. \end{aligned} \quad (4.12)$$

The last equation in (4.12) ensures that only the terms with $L = 2n$ occur in (4.11). Notice that the dependence on the statistics in (4.11) only reduces to taking the permanent or the

²The permanent of a matrix differs from the determinant by the omission of the alternating sign factors $(-1)^{i+j}$.

determinant of the same matrix. According to the theory of Fredholm integral operators (see e.g. [33]) the above expression is equal to

$$G_{\alpha,\alpha'}^{(S)}(t) = \text{Det}(1 + 2\pi \mathbf{K})^{-S}, \quad (4.13)$$

so that the correlators of operators having the same non-locality phase in the free fermion and free boson theories are the inverse of each other. This conclusion was first obtained in [26] studying the deformation theory of differential equations. The result (4.2) follows from the fact that the operators $V_\alpha(x)$ have a non-locality phase $e^{2i\pi\alpha}$ around the solitons at the sine-Gordon free fermion point (see e.g. [34]), so that $\mu_j(x)$ and $V_{j/N}(x)$ have the same non-locality with respect to the corresponding particles. Since j runs between 1 and $N - 1$, we are actually working with $0 < \alpha = j/N < 1$.

Once Eq. (4.2) has been obtained by using the large distance expansion of the correlators, it is an interesting check to see how this inversion relation between the two correlators works in the short distance limit. Both in the bosonic and fermionic case the operators $\Phi_\alpha(x)$ satisfy the OPE

$$\langle \Phi_\alpha(x) \Phi_{\alpha'}(0) \rangle \sim |x|^{-\Gamma_{\alpha,\alpha'}^{(S)}} \langle \Phi_{\alpha+\alpha'} \rangle + \dots, \quad (4.14)$$

with

$$\Gamma_{\alpha,\alpha'}^{(S)} = X_\alpha^{(S)} + X_{\alpha'}^{(S)} - X_{\alpha+\alpha'}^{(S)}, \quad (4.15)$$

$X_\alpha^{(S)}$ being the scaling dimensions. In the bosonic case the index $\alpha + \alpha'$ is taken modulo 1. The scaling dimensions can be computed through the formula [12]

$$X_\alpha^{(S)} = -\frac{1}{2\pi} \int d^2x \langle \Theta(x) \tilde{\Phi}_\alpha(0) \rangle_{\text{connected}}, \quad (4.16)$$

where $\Theta(x)$ is the trace of the energy-momentum tensor. Since the only non-zero form factor of this operator in the free theories is

$$\langle 0 | \Theta(0) | A(\theta) \bar{A}(\beta) \rangle = 2\pi m^2 \left[-i \sinh \frac{\theta - \beta}{2} \right]^{\delta_{S,-1}}, \quad (4.17)$$

one easily finds

$$X_\alpha^{(S)} = \begin{cases} \alpha(1 - \alpha), & S = 1 \\ \alpha^2, & S = -1 \end{cases} \quad (4.18)$$

in agreement with the conjugation properties $\mu_j^* = \mu_{N-j}$ and $V_\alpha^* = V_{-\alpha}$. These results coincide with those of conformal field theory with ‘twist’ fields (see [35, 36]). It follows

$$\Gamma_{\alpha,\alpha'}^{(-)} = -2\alpha\alpha', \quad (4.19)$$

$$\Gamma_{\alpha,\alpha'}^{(+)} = \begin{cases} 2\alpha\alpha', & \alpha + \alpha' < 1 \\ 2[\alpha\alpha' + 1 - (\alpha + \alpha')], & \alpha + \alpha' > 1. \end{cases} \quad (4.20)$$

The agreement for $1 < \alpha + \alpha' < 2$ is obtained by observing that in this range of $\alpha + \alpha'$, the leading short distance term in the fermionic case is not the one in (4.14) but the first off-critical contribution

$$\mu/2 \int d^2y \langle V_\alpha(x) V_\alpha(0) [V_1(y) + V_{-1}(y)] \rangle_{\mu=0}, \quad (4.21)$$

which indeed behaves as $|x|^{2[\alpha\alpha'+1-(\alpha+\alpha')]}$ when $x \rightarrow 0$.

In general, the two-point functions we are dealing with can be expressed as

$$G_{\alpha,\alpha'}^{(S)}(t) = e^S \Upsilon_{\alpha,\alpha'}(t). \quad (4.22)$$

where $\Upsilon_{\alpha,\alpha'}(t)$ is given by [26, 31]

$$\begin{aligned} \Upsilon_{\alpha,\alpha'}(t) &= \frac{1}{2} \int_{t/2}^{\infty} \rho d\rho \times \\ &\times \left[(\partial_\rho \chi)^2 - 4 \sinh^2 \chi - \frac{(\alpha - \alpha')^2}{\rho^2} \tanh \chi \right], \end{aligned} \quad (4.23)$$

with $\chi(\rho)$ satisfying the differential equation

$$\partial_\rho^2 \chi + \frac{1}{\rho} \partial_\rho \chi = 2 \sinh 2\chi + \frac{(\alpha - \alpha')^2}{\rho^2} \tanh \chi (1 - \tanh^2 \chi) \quad (4.24)$$

with suitable boundary conditions. The short distance behaviour for $\alpha + \alpha' < 1$ is given by

$$\lim_{t \rightarrow 0} G_{\alpha,\alpha'}^{(S)}(t) = \left(C_{\alpha,\alpha'} t^{2\alpha\alpha'} \right)^{-S}, \quad (4.25)$$

with an amplitude that can be deduced from the work of Ref. [7] on vacuum expectation values in the sine-Gordon model:

$$\begin{aligned} C_{\alpha,\alpha'} &= 2^{-2\alpha\alpha'} \times \\ &\exp \left\{ 2 \int_0^\infty \frac{dt}{t} \left[\frac{\sinh \alpha t \cosh(\alpha + \alpha') t \sinh \alpha' t}{\sinh^2 t} + \right. \right. \\ &\left. \left. - \alpha\alpha' e^{-2t} \right] \right\}. \end{aligned} \quad (4.26)$$

Finally, let us discuss the case in which we have a Z_2 symmetry. This is somehow special since a broken phase with two degenerate vacua can be realised in terms of a *neutral* boson. In fact, a disorder operator with non-locality factor -1 is present also in the theory of a neutral bosonic free particle. It is easy to check that, for $S = 1$ and $\alpha = 1/2$, the form factors (4.6) imply the factorisation

$$\mu_1 = \mu_{(1)} \times \mu_{(2)}, \quad N = 2 \quad (4.27)$$

where $\mu_{(j)}$ is the disorder operator with scaling dimension $X_{1/2}^{(+)} / 2 = 1/8$ associated to the neutral boson A_j entering the decomposition $A = (A_1 + iA_2) / \sqrt{2}$, $\bar{A} = (A_1 - iA_2) / \sqrt{2}$.

The fermion-boson correspondence observed above for charged particles has an analogue in the neutral case, and the correlation function

$$G(t) = \langle \tilde{\mu}_{(j)}(x) \tilde{\mu}_{(j)}(0) \rangle = \left[G_{1/2,1/2}^{(+)}(t) \right]^{1/2} \quad (4.28)$$

can be related to correlators computed in the theory of a free neutral fermion, i.e. in the Ising field theory without magnetic field. To see this, let us recall that the correlators of the spin and disorder operators in the unbroken phase of the scaling Ising model can be written as [25]

$$\begin{aligned} \tau_\pm(t) &\equiv \langle \tilde{\mu}(x) \tilde{\mu}(0) \rangle \pm \langle \sigma(x) \sigma(0) \rangle = \\ &= \exp \left[\pm \frac{1}{2} \chi(t/2) - \frac{1}{2} \Upsilon_{1/2,1/2}(t) \right], \end{aligned} \quad (4.29)$$

where the functions χ and Υ are those of Eqs. (4.23), (4.24). Hence, it follows from (4.22) and (4.28) that

$$G(t) = [\tau_+(t)\tau_-(t)]^{-1/2}. \quad (4.30)$$

Concerning the short distance behaviour of this correlator, the power law (4.25) acquires in this case a logarithmic correction due to the ‘resonance’ with the leading off-critical contribution (4.21). Such a contribution to the correlator $\langle \tilde{V}_\alpha(x)\tilde{V}_\alpha(0) \rangle$ behaves as $-C_{\alpha,\alpha}t^{1/2}[1+(2-4\alpha)\ln t]$ in the limit $\alpha \rightarrow 1/2$, so that

$$\lim_{t \rightarrow 0} G_{1/2,1/2}^{(-)}(t) = \lim_{t \rightarrow 0} G^{-2}(t) = \mathcal{B}t^{1/2}\ln(1/t), \quad (4.31)$$

with

$$\mathcal{B} = -4 \operatorname{Res}_{\alpha=1/2} C_{\alpha,\alpha} = 0.588353\dots \quad (4.32)$$

and this amplitude coincides with that of the product $\tau_+(t)\tau_-(t)$ in the Ising model.

The considerations of this section have recently been extended to the case of massive ghost theories [37]. In all these cases the correlation functions of disorder operators can be computed exactly for free theories and are related to the same differential equation (4.24). In absence of interaction, the correlators are non-trivial due to the non-locality between the operators and the particles. The resummation of the spectral series for interacting theories remains a challenging task even in presence of integrability.

5. Non-integrable Deformations

The free bosonic and fermionic theories discussed above can also be regarded as describing phases of spontaneously broken Z_N symmetry. In this dual vision, the excitations are free kinks $|K_{j,j\pm 1}(\theta)\rangle$ interpolating between two adjacent vacua $|0_j\rangle$ and $|0_{j\pm 1}\rangle$ (the indices are taken modulo N). Let us denote by $\Phi_{k/N}$, $k = 0, 1, \dots, N-1$, the operators we are interested in. They correspond to the exponential operators $V_{k/N}$ in the fermionic case³, and to the operators σ_k , dual to the disorder operators μ_k , in the bosonic case. These operators create multikink excitations with zero topologic charge, i.e. starting and ending in the same vacuum state. Form factors on kink states were discussed in [17]. Consider for the sake of simplicity the two-kink matrix elements

$$F_{j,k}^\pm(\theta_1 - \theta_2) \equiv \langle 0_j | \Phi_{k/N}(0) | K_{j,j\pm 1}(\theta_1) K_{j\pm 1,j}(\theta_2) \rangle, \quad (5.1)$$

satisfying the equations

$$\begin{aligned} F_{j,k}^\pm(\theta) &= S F_{j,k}^\mp(-\theta), \\ F_{j,k}^\pm(\theta + 2i\pi) &= F_{j\pm 1,k}^\mp(-\theta), \\ \operatorname{Res}_{\theta=i\pi} F_{j,k}^\pm(\theta) &= i \left[\langle 0_j | \Phi_{k/N} | 0_j \rangle + \right. \\ &\quad \left. - \langle 0_{j\pm 1} | \Phi_{k/N} | 0_{j\pm 1} \rangle \right]. \end{aligned}$$

³In this case the N degenerate vacua can be identified with those of the periodic potential in (4.3) identified modulo N .

Since the generator Ω of Z_N transformations ($\Omega^N = 1$) acts on states and operators as ($\omega \equiv e^{2i\pi/N}$)

$$\begin{aligned} \Omega |K_{j,j+1}(\theta_1)K_{j+1,j+2}(\theta_2)\dots\rangle &= \\ &= |K_{j+1,j+2}(\theta_1)K_{j+2,j+3}(\theta_2)\dots\rangle, \\ \Omega^{-1} \Phi_{k/N}(x) \Omega &= \omega^k \Phi_{k/N}(x), \end{aligned}$$

the above form factor equations can be rewritten as

$$\begin{aligned} F_{j,k}^{\pm}(\theta + 2i\pi) &= S \omega^{\pm k} F_{j,k}^{\pm}(\theta), \\ \text{Res}_{\theta=i\pi} F_{j,k}^{\pm}(\theta) &= i(1 - \omega^{\pm k}) \langle 0_j | \Phi_{k/N} | 0_j \rangle. \end{aligned}$$

Once the identifications $A \longleftrightarrow K_{i,i+1}$, $\bar{A} \longleftrightarrow K_{i,i-1}$ are made, these relations are equivalent to the previous equations for the Form Factors with $n = 1$. This correspondence is easily extended to all values of n and leads to the same form factors and correlation functions discussed for the unbroken phase.

The introduction of a ‘magnetic field’ pointing in the k direction and breaking explicitly the Z_N symmetry corresponds to adding to the free action a term

$$h \int d^2x \Psi_k(x), \quad (5.2)$$

where

$$\Psi_k(x) = \frac{1}{N} \sum_{l=0}^{N-1} \omega^{-kl} \Phi_{l/N}(x). \quad (5.3)$$

The first order corrections to the energy density ε_j of the vacuum state $|0_j\rangle$ and to the mass $m_{j,j\pm 1}$ of the kink $K_{j,j\pm 1}$ can be computed according to the Form Factor Perturbation Theory proposed in [38]

$$\begin{aligned} \delta\varepsilon_j &\sim h \langle 0_j | \Psi_k | 0_j \rangle = h v \delta_{j,k}, \\ \delta m_{j,j\pm 1}^2 &\sim h \langle 0_j | \Psi_k(0) | K_{j,j\pm 1}(i\pi) K_{j\pm 1,j}(0) \rangle. \end{aligned}$$

It follows from the residue equation written above that

$$\begin{aligned} \text{Res}_{\theta=i\pi} \langle 0_j | \Psi_k(0) | K_{j,j\pm 1}(\theta) K_{j\pm 1,j}(0) \rangle &= \\ \frac{iv}{N} (\delta_{j,k} - \delta_{j,k\mp 1}), \end{aligned} \quad (5.4)$$

This formula implies that the correction to the mass of the kinks interpolating between the vacua $|0_k\rangle$ and $|0_{k\pm 1}\rangle$ is infinite. This divergence simply reflects the fact that these kinks become unstable because the magnetic field removes the degeneracy of the vacuum $|0_k\rangle$ with the two adjacent vacua. These kinks are then confined in pairs $K_{k,k\pm 1}K_{k\pm 1,k}$ and the confinement gives rise to a string of bound states with zero topologic charge, as discussed in more details in [39].

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