Duality and Lorentzian Kac-Moody Algebra

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ABSTRACT: A brief review is given concerning the occurrence of solitons in certain quantum field theories, leading up to the idea of a quantum realisation of electromagnetic duality in suitable spontaneously broken gauge theories. Related ideas in maximal supergravity theories are described, motivating the study of infinite-dimensional Lie algebras of Lorentzian type. Some recent results are outlined.

I would like to thank the organisers for the invitation here to celebrate the fiftieth anniversary of IFT, UNESP, and for the splendid organisation and technical facilities. I would also like to congratulate them for their coup in organising as opening event the victory for the Brazilian football team in the World Cup.

This is a conference devoted to two related topics, integrability and duality, with the emphasis on the former, judging by the programme. I shall start by trying to explain how, in my own mind, these ideas seem to be related before briefly mentioning some of my recent work on Lorentzian Kac-Moody algebras which was motivated by these synergies.

Most people here are connected to the theoretical physics community and appreciate very well that the search for a comprehensive theory of interacting particles is a powerful motivation.

Thirty five years ago this motivation led me into the embryonic string theory and I soon joined those impressed by the extent that novel algebraic structures underlay the consistency of this theory of particles and their interactions. I am thinking of the Virasoro algebra of gauge conditions, the vertex operators "creating" particles and the "no ghost theorem", all involving infinite dimensional structures. This theory eventually spread worldwide and has continued its relentless development until now, albeit in fits and starts.

Thirty years ago, Nielsen and Olesen sought to realise the string as a soliton in a conventional, local field theory and found that the appropriate scenario was a spontaneously broken gauge theory [1]. Their string was made of magnetic lines of force and so could only end on a magnetic charge. The search for suitable end points prompted 't Hooft and Polyakov [2] to discover that slightly different (and more realistic) versions of spontaneously

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broken gauge theories could produce solitons mimicking particles with finitely extended but localised structure. Stability against decay was guaranteed by a topological quantum number which could be interpreted as magnetic charge. So unified gauge theories, being spontaneously broken, can easily exhibit the tendency to spawn magnetic monopoles, those objects whose quantum consistency was first characterised by Dirac in 1931 [3]. He found that, contrary to previous belief, the presence of magnetic charge could be consistent with quantum mechanics. The proviso was that electric charges had to be quantised, perfectly in line with what had been observed and had been hitherto inexplicable. It is ironic that one of the original motivations for unified theories was to find an alternative explanation of the quantisation of electric charge.

Since these solitonic monopoles have to be interpreted as particles there must be a quantum field theoretic interpretation of them rather than just the original classical field theory one. This means there must be a quantum field operator "creating" them and that it ought to obey equations of motion following from some action principle. There is quite an old prototype for this, found by Skyrme in 1961, when he considered the solitons of sine-Gordon theory, non-linear equations for a real scalar field in space-time of two dimensions [4]. The quantum version of the soliton was created by an exponential of the scalar field, later called a vertex operator because of its structural resemblance to the previously mentioned vertex operators found (later) in string theory. Thus the quantum soliton theory too possesses rich and unexpected algebraic structure.

In the sine-Gordon example the soliton quantum field operator obeyed equations of motion quite unlike the original sine-Gordon one. But when the gauge group is $SO(3)$ in the monopole situation, the known properties of the spectrum of particle states suggest that the putative monopole quantum field operator ought to satisfy equations of motion rather similar to that of electrically charged gauge fields [5]. Some consequences of this idea could be successfully checked without constructing the analogue of the vertex operator. Since that construction remains unrealised even now, the idea remains a conjecture that will be referred to as the electromagnetic duality conjecture for the following reasons.

The interesting new feature of these gauge theory solitons in space-time $R^{3,1}$ is the similarity, or even symmetry, between the electrically and magnetically charged particle states, quantum excitations and quantum solitons respectively according to the initial formulation. This is evidently related to an old idea of electromagnetic duality referring to the similarity between the behaviour of the electric and magnetic fields revealed by Maxwell's discovery of his equations. In fact it can be regarded as a quantum version of it.

Because of the difficulty in proceeding further with these ideas referring to space-time $R^{3,1}$ it seemed sensible to study further relativistically invariant soliton theories in two-dimensional space-time, $R^{1,1}$ and possible underlying algebraic structures. Such theories are definitely more tractable and it was discovered that part of the reason for this was that they constituted special sorts of deformation of conformally invariant theories (familiar from the world-sheet formulation of string theories) according to an important insight of Zamolodchikov [6]. The fact that the conformal group of $R^{1,1}$ has infinite dimension means that the related integrable field theories inherit an infinite number of local conservation laws that can be regarded as extra sorts of momenta carrying spins of magnitude higher

than unity.

$$[T, P_k] = kP_k, \quad [P_k, P_m] = 0; \qquad k, m \quad \subset \quad \text{exponents.}$$

With this interpretation these generate an infinite extension of the space-time symmetry algebra (the Poincaré algebra with boost T and light-cone momenta $P_{\pm 1}$) and provide the basis of many of the integrability properties and special properties of the soliton solutions. In the case of the affine Toda field theories this extended space-time algebra is actually the principal Heisenberg subalgebra of the centre-free version of the underlying affine (untwisted) Kac-Moody algebra $\hat{g}$. The spectrum of exponents is then given by integers equal to what are known as the exponents of the Lie algebra g, modulo its Coxeter number. Thus for the sine-Gordon equation, for which $g = su(2)$, the exponents are given by all odd integers, positive or negative.

In the meantime there were advances in the understanding of monopole solitons in $R^{3,1}$, some due to physicists and some due to pure mathematicians. One was the role of supersymmetry (which came for free) in promoting certain classical properties to quantum validity [7] Supersymmetry constitutes an enlargement of space-time symmetry as did the extra conservation laws in $R^{1,1}$ but is, of course, different. Likewise the monopoles that have been dubbed solitons do not possess such distinctive properties as those in $R^{1,1}$ but they do possess a sort of weaker integrability property that made it possible to establish the existence of solutions describing any configuration of N static monopole solitons of like magnetic charge [8]. For each number N, the moduli space of solutions constituted a $4N$-dimensional manifold endowed with a metric with a hyper-Kähler structure [9]. This development culminated in 1994 when A Sen saw how to synthesise these facts [10] to confirm the validity of some quite intricate predictions of the electromagnetic duality conjecture. This greatly enhanced its credibility and many of you have already heard me explain this at greater length [11] and will be aware of the many subsequent developments.

Now I want to mention a parallel and independent development in which ideas of electromagnetic duality and the enlargement of space-time symmetry also played a role. This arose in the context of what are called "extended supergravity theories". These are gravitational theories in curved space-time $(\mathcal{M}_{n,1})$ with such a large degree of local supersymmetry that the superpartners of the graviton include both gauge particles and scalar particles. Coming with the high degree of supersymmetry is a compact symmetry group called the $\mathcal{R}$-symmetry group and denoted $G(\mathcal{R})$. It was found that $G(\mathcal{R})$ could be enlarged to a noncompact group, $G(\text{Duality})$ by adding symmetries of the equations of motion and the energy momentum tensor. This was done by including duality transformations of the gauge fields (which are abelian). The scalar fields partnered with the graviton then served as coordinates on the non-compact Riemannian symmetric space that is formed by the non-compact group $G(\text{Duality})$ quotiented by the $\mathcal{R}$-symmetry group (its maximal compact subgroup).

It is this non-compact symmetry group that has become a centre of attention because of its connection with the duality that is becoming an ever more important organising principle, particularly in superstring theory. The first remarkable result [12] concerned the so-called "maximal" supergravity theories, namely those with the highest degree of

supersymmetry consistent with the graviton being the particle carrying the highest spin in space-time of given dimension. Then calculation for n running from 2 to 10 shows that the non-compact symmetry group belongs to the exceptional series of Lie algebras, with E_{10-n} occurring in $R^{n,1}$ (or in a curved space-time $\mathcal{M}_{n,1}$ with metric of the same signature). Thus

$$G(\mathcal{R}) \subset G(\text{Duality}) \equiv E_{10-n} \quad \text{on} \quad \mathcal{M}_{n,1}.$$

The exceptional Lie algebras E_r constitute a nested sequence of subalgebras:

$$E_3(\equiv A_1 \oplus A_2) \subset E_4(\equiv A_4) \subset E_5(\equiv D_5) \subset E_6 \subset E_7 \subset E_8 \subset E_9(\equiv \hat{E}_8) \subset E_{10} \subset E_{11} \dots$$

The Dynkin diagram of E_r consists of r nodes linked by single bonds in such a way that there is a central node to which are attached three legs containing 1, 2 and $r-4$ nodes respectively. The above sequence is obtained by lengthening or shortening the third of these legs.

The first step had been the discovery [13] by Cremmer and Julia of non-compact E_7 in $R^{3,1}$ (with $\mathcal{R}$-symmetry group $SU(8)$). In $R^{10,1}$ the noncompact symmetry disappears and one can expect to find the simplest possible supergravity theory, as Nahm predicted [14]. Indeed Cremmer, Julia and Scherk found that the only field besides those of the graviton and gravitino was an abelian 3-form field potential that enjoys a Chern Simons self-interaction and the ability to couple naturally and geometrically to the world 3-volume of a membrane [15].

Also, as Nahm had predicted, this result means that there are no supergravity theories with $n > 10$. At the other extreme, in $R^{1,1}$ the algebra E_9 is not a finite-dimensional Lie algebra but an affine Kac-Moody algebra, the affinisation of E_8 (often denoted $\hat{E}_8$) exactly in line with the sorts of infinite algebraic structure already met in the context of conformal and integrable field theories in $R^{1,1}$. The analogous result for Einstein's theory viewed in $R^{1,1}$ involves an infinite dimensional structure related to $\hat{A}_1 = S\hat{U}(2)$ and known to relativists as the Geroch group.

The most extreme case of all, $R^{0,1}$, is particularly intriguing. With no space coordinate the theory must be some sort of particle dynamics related to E_{10} if the pattern is to persist. E_{10} itself is an interesting Lie algebra, being neither of finite-dimensional nor affine Kac-Moody type but in fact the largest (simply-laced) hyperbolic algebra, as will be explained. Its Cartan matrix has ten rows and columns and determinant equal to -1. There is good reason to think that the particle mechanics refers to massless motions within the 10-dimensional Weyl chamber of E_{10}, confined to it by reflections off the walls and displaying chaotic behaviour, maybe a limiting case of E_{10} Toda mechanics [16].

There is an apparently heretical proposal, but nevertheless one that is very intriguing, put forward by Peter West [17]. He has arguments to support the idea that M-theory, the putative theory of quantum super-membranes on $\mathcal{M}_{10,1}$, actually does possess a hidden E_{11} symmetry, exactly the symmetry realised only in $R^{1,1}$ according to the previous ideas. It contains the conformal group of $R^{10,1}$, namely $SO(11,2)$ realised in some non-linear way. Of course these conformal groups did not play any role in the structure previously discussed. Actually he also argues that the symmetry structure may be even larger, at

least E_{11}. There may also be an analogous symmetry algebra for the bosonic string theory in $R^{25,1}$, which he calls K_{27}.

Neither of these can be hyperbolic as 10 is the highest possible rank for them but they are both Lorentzian. I now explain what this means and use West's conjecture as a motivation for investigating properties of a class of these algebras.

To do this I shall briefly recap the notions underlying what are called Kac-Moody algebras as explained in more length in the classic book of Victor Kac [18]. There is a connected Dynkin diagram encoding the Cartan matrix K. The latter is assumed, for simplicity, to be symmetric. It has r rows and columns, entries equal to 2 on the diagonal and equal to integers less than or equal to zero off the diagonal. The simple roots α_1, $\alpha_2 \ldots \alpha_r$, are a set of r linearly independent vectors whose scalar products are given by the elements of the Cartan matrix:

$$\alpha_i.\alpha_j = K_{ij}.$$

From this data may be deduced, in principle, the system of all the roots, which may very well be infinite. Then it is possible to specify, in principle, commutation relations for the whole Kac-Moody algebra, denoted $g(K)$ which is well defined according to a theorem of Serre.

Out of the infinitely many possiblities for K I shall consider only the following three classes:

F : K is positive definite: then $g(K)$ is a finite dimensional Lie algebra.

A : K is positive semi-definite: then $g(K)$ is an affine Kac-Moody algebra.

L : K is nonsingular with one eigenvalue negative so that the remainder are positive: then $g(K)$ is said to be a Lorentzian algebra.

Of course, classes **F** and **A** are completely classified, as is a subclass of **L**, the hyperbolic algebras. The paper [19] by M Gaberdiel, P West and myself discusses a particularly amenable class that includes all of **F**, **A** and a subclass of **L** that includes the hyperbolics, as well as the aforementioned E_{11} and K_{27} thought to be relevant to M-theory and bosonic string theory.

This class is characterised by possessing a connected Dynkin diagram in which it is possible to choose a node, called the central node, whose deletion breaks the original diagram into a finite number of disconnected pieces which are each of type **F** or **A** with no more than one piece of type **A**.

It is possible to consider the Cartan Matrix K for such a diagram as being bordered by the row and column corresponding to the central node. This feature can be exploited to provide a formula for the determinant, $\det K$, of the Cartan matrix of this diagram in terms of the Cartan matrices of the pieces. Similar constructions are given for the simple roots and the fundamental weights $\lambda_1, \ldots \lambda_r$ defined as forming the basis reciprocal to that provided by the simple roots:

$$\lambda_i.\alpha_j = \delta_{ij} \quad \text{so} \quad \lambda_i.\lambda_j = (K^{-1})_{ij}.$$

Then it is proven that K is of type **F**, **A** or **L** according as $\det K$ is positive, zero or negative, as K cannot have more than one negative eigenvalue when it has the structure described.

A useful and much simplified special case occurs when each disconnected component is linked to the central node by just one link. Then if the disconnected pieces have Cartan matrices $K(\hat{g}_0)$, $K(g_1)$, $K(g_2) \ldots K(g_n)$

$$\det K = -\det K(g_0)\det K(g_1)\ldots\det K(g_n)$$

and so is negative. Thus K is Lorentzian. The factors on the right-hand side are all determinants of Cartan matrices of type **F**. It is known to follow from the above definition of fundamental weights that then

$$\det K(g) = |Z(g)|$$

where $Z(g)$ is the centre of the simply connected Lie group with finite dimensional Lie algebra g. These quantities can then be easily evaluated by application of the previous identity. For example it easily follows that for any $r \geq 3$

$$\det K(E_r) = 9 - r, \quad r \geq 3.$$

So E_r is type **F**, **A** or **L** according as $r \leq 8, r = 9$ or $r \geq 10$ respectively. In particular $\det K(E_9) = 0$, $\det K(E_{10}) = -1$ and $\det K(E_{11}) = -2$.

E_{11} is an example of a type of Lorentzian algebra within the special subclass that may be called "very extended". For such algebras the central node has two incident lines, one linking it to the affine node of the Dynkin diagram of $\hat{g}_0$ and the other to the single node of the Dynkin diagram for A_1. So

$$\det K = -\det K(g_0)\det K(A_1) = -2\det K(g_0).$$

E_{11} and K_{27} are both of this type with g_0 equal to E_8 and D_{24} respectively. Hence $\det K(K_{27}) = -8$.

The occurrence of even unimodular lattices is often related to interesting situations in mathematics and physics (for example, in the theory of modular forms, in heterotic string theory etc). The root lattice of a simply laced Kac-Moody algebra constitutes such a lattice if $\det K = \pm 1$. When the above formula applies this is satisfied if and only if $g_0 = g_1 = \ldots g_n = E_8$, since E_8 is the only simply laced Lie algebra whose Cartan matrix has determinant equal to unity. Since the manner of linking the central node to each E_8 Dynkin diagram is unspecified this represents a large number of possibilities and there are even more of this type if the more general formula for $\det K$ (not written down here) is considered. All these examples yield lattices of dimension $8n + 10$. Thus there are very many different Lorentzian Kac-Moody algebras with the same root lattice, namely $II_{8(n+1),1}$, which is the unique even, unimodular, Lorentzian lattice of that dimension.

I now say something about the geometrical arrangement of the simple roots and fundamental weights. For type **F** all these lie in a Euclidean space. Each simple root has length $\sqrt{2}$. Simple roots of type **L** also have length $\sqrt{2}$. But they lie in a space with "mainly positive" Lorentzian metric and so are space-like, that is, lying outside both light cones. For the fundamental weights the situation is unclear. The distinctive feature of hyperbolic

algebras is that each fundamental weight is either time or light like. In fact they all live in or on the same light cone. So the Weyl vector,

$$\rho = \sum_{j=1}^{r} \lambda_j,$$

whose scalar product with each of the simple roots equals unity, lies strictly inside this common light cone.

For type **F** algebras this Weyl vector has an interesting relation, discovered by the mathematician Kostant [20], with the exponents of g mentioned in connection with the Lorentz spins of the local conserved charges in integrable field theories in $R^{1,1}$. ρ can be expressed as a linear combination of simple roots with strictly positive coefficients p_i, $\rho = \sum_i p_i \alpha_i$, and as a consequence it is possible to construct a *real $SO(3)$* subalgebra of g (the principal $SO(3)$ subalgebra):

$$[T^3, T^{\pm}] = \pm T^{\pm}, \quad [T^+, T^-] = T^3,$$

by

$$T^3 = \rho.H, \quad T^+ = \sum_{j=1}^{r} \sqrt{p_j} E^{\alpha_j}, \quad T^- = (T^+)^{\dagger}.$$

When the adjoint representation of g is decomposed into irreducible representations with respect to this subalgebra exactly $r = \operatorname{rank} g$ multiplets are obtained. Their spins are precisely the exponents of g. The mathematical interest of this is that there is a beautiful formula for computing the Betti numbers of the group manifold from these exponents.

It is natural to ask whether anything similar can be done for algebras of type **L**. The answer is that for the class considered there is no such *real $SO(3)$* subalgebra but that there is sometimes a *real $SO(2,1)$* subalgebra. For example this is so for all the hyperbolics and for E_{11} but not for K_{27}.

The point is that the coefficient of the j'th simple root in the expansion of the Weyl vector has the form $p_j = \rho.\lambda_j$ and so is always negative in the hyperbolic case as it is a scalar product of two vectors in the same light cone. This coefficient is also always negative for the central node of the class of Lorentzian algebras considered. The above construction of the *real* three dimensional subalgebra only works if the coefficients p_j in the expansion of the Weyl vector all have the same sign, and yields $SO(3)$ if the common sign is positive and $SO(2,1)$ if it is negative. As $\rho^2 = \sum_0^r p_j$, the Weyl vector ρ is time-like in the latter case. For some subclasses of Lorentzian algebra this condition is quite restrictive. Thus it requires the rank of "overextended" algebras to be less than 26, an intriguing result in view of the importance of that dimension in bosonic string theory [21]. These are algebras whose diagram consists of an affine diagram whose affine node is linked to the central node. But other subclasses have now been found with indefinitely high rank and time-like Weyl vector [19].

Since the group is not compact, the unitary representations of $SO(2,1)$ are of infinite dimension. It was shown by Nicolai and myself [22] that for the hyperbolics, once the trivial three dimensional non-unitary representation is taken out, the the adjoint can be

decomposed into unitary representations of the $SO(2,1)$. Only $r-1$ of these possess states with $T^3 = 0$ and their "spins" can be used to define $r-1$ analogues of the exponents in addition to the value 1 associated with the triplet. These are complex numbers with $-1/2$ as real part and examples have been calculated. But unfortunately it is not clear what the significance of these numbers is.

Let me conclude by saying that there remain many open questions in all directions concerning both integrability and duality but that I believe there is some real advantage in considering the topics together.

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