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Workshop on Integrable Field Theories, Solitons and Duality

Integrable spin ladder systems

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Abstract: We present an integrable spin ladder model which possesses a free parameter besides the rung and leg couplings. The model is exactly solvable by means of the Bethe ansatz method. We determine the dependence on the anisotropy parameter of the phase transition between gapped and gapless spin excitations and present the phase diagram. Finally, we show that the model is a special case of a more general Hamiltonian with six free parameters. We also investigate the thermodynamics of the model. Specifically, the magnetic susceptibility as a function of the temperature is obtained numerically. The influence of this anisotropy parameter on these physical curves is determined explicitly. A comparsion between the spin gap obtained from the susceptibility curve and that one obtained from the Bethe ansatz equations is performed and a good agreement is found.

1. Introduction

Since the discovery of high temperature superconductivity in doped copper oxide (or cuprate) materials [1], a tremendous effort has been made to understand the physics underlying this phenomenon. High temperature superconductivity is one of the principal examples of a *quantum* mechanical phenomenon which has no analogue in the realm of classical (or macroscopic) physics. The geometry of the copper-oxygen structure in these compounds is that of a two-dimensional square lattice. However, from the theoretical side, obtaining results regarding physical properties is much more accessible in a one-dimensional model. In order to maximize the interaction between theory and experiment, much work is now focused on quasi-two-dimensional models known as ladders. A ladder is essentially still a two-dimensional lattice however the length of the "legs" is assumed to be much greater than that of the "rungs". (Equivalently a ladder has few legs and many rungs.) The introduction of these ladder systems has brought about a significantly increased understanding of the physics of the cuprate compounds. The cuprates have the property of being *antiferromagnetic* where at each site of the lattice in the undoped state an electron

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occupies one of two spin states. As such they are reasonably approximated by the Heisenberg model, or some suitable generalization describing spin exchange type interactions. In one dimension the Heisenberg model is exactly solvable via Bethe ansatz methods and from this solution it is well known that in this model the elementary spin excitations are *gapless*. On the other hand, the existence of a spin gap is critical for the observed phenomenon of superconductivity to occur under doping, whereby the charge carriers (the holes introduced through the doping) undergo Bose-Einstein condensation. By introducing the concept of the ladder model this apparent contradiction is resolved, since the ladder allows for the formation of singlet states along the rungs which are responsible for the formation of the spin gap. (Strictly speaking the singlet states can only form when there is an even number of legs). Here we would like to point out that due to the progress presently being made in nano-engineering, many different species of ladder compounds (not only of the cuprate type) have been investigated in this context. This theory is now supported by a substantial body of experimental evidence. An excellent review of these aspects may be found in [3].

However, in contrast to its one-dimensional analogue, the usual Heisenberg ladder model cannot be solved exactly. In order to gain some results in the theory of spin ladder systems, many authors have considered generalized models, which incorporate additional interaction terms which guarantee exact solvability. Remarkably, such generalized models still exhibit realistic physical properties such as the existence of a spin gap [4, 5] and the magnetization plateaus at fractional values of the total magnetization [6].

This approach has been used to derive quasi-one-dimensional systems using the well established theories from the one-dimensional case [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In all cases cited above, no free parameters are present, other than the rung interaction coupling, due to the strict conditions of integrability. With the presence of free parameters it is reasonable to expect that the solution may provide better test models for describing the various behaviors associated with ladder systems.

The purpose of this paper is to discuss a generalized integrable spin ladder with one extra parameter, characterizing anisotropy, without violating integrability. This model is exactly solvable by the Bethe ansatz and it reduces to the model introduced by Wang [4, 5] for a special limit of this extra parameter. The situation here is akin to the generalization of the XXX chain to the anisotropic XXZ version. The introduction of the additional free parameter in the present case allows for an example of a model with a critical line varying continuously with the anisotropy. More specifically, the size of the gap in the massive region depends explicitly on the anisotropy parameter, which in turn shows dependence of the anisotropy parameter for the points at which the gap closes that define the phase transition. The thermodynamics of the model is investigated and the magnetic susceptibility curve as a function of the temperature is obtained. The influence of this anisotropy parameter on these physical curves is determined explicitly. A comparison between the spin gap obtained from the susceptibility curve and that one obtained from the Bethe ansatz equations is performed and a good agreement is found.

2. Importance of the study of spin ladder systems

Let us begin by reviewing why it is important to study spin ladders from both experimental and theoretical points of view.

With the rapid progress presently being made in nano-engineering, several compounds have been experimentally realized with a ladder structure, such as $La_{1-x}Sr_xCuO_{2.5}$, $Sr_{14-x}Ca_xCu_{24}O_{41}$, $Cu_2(C_5H_{12}N_2)_2Cl_4$, CaV_2O_5, $KCuCl_3$ (see for example [2, 3]) and many others. To illustrate, a schematic representation of a two and a three leg ladders is presented here (see [3] and references therein for details)

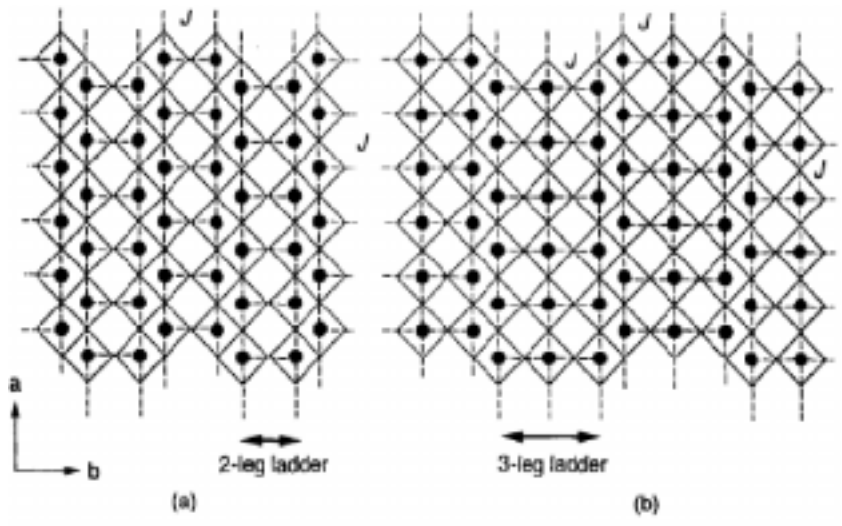


Figure 1: Schematic representation of the Cu_2O_3 sheets of : (a) the two-leg ladder $SrCu_2O_3$ and (b) the three-leg ladder $Sr_2Cu_3O_5$. The filled circles are Cu^{2+} ions, and O^{2-} ions are located at the corners of the squares drawn with solid lines.

In addition, experiments using different techniques such as the magnetic susceptibility measurements [17] or nuclear magnetic resonance [18] report the existence of a spin gap in the spectrum of elementary excitations for even leg ladders. For example, two typical curves of the magnetic susceptibility against the temperature obtained for these compounds are shown in figure 2 [3].

Notice that the first(second) compound exhibits an exponential decay (linear behavior) of the magnetic susceptibility for small temperatures reflecting the existence(absence) of the spin gap. Similar curves have been obtained for all the other ladders. It is in fact a well established point that even leg ladders are gaped while odd leg ladders are gapless, being the size of the gap the only reason for controversy among the different groups.

From the theoretical point of view, the existence of a spin gap in the spectrum of elementary excitations for even leg ladders was also predicted. Most of the predictions were obtained using the Heisenberg ladder, which consists of two one-dimensional chains put together to form a ladder with Heisenberg type interactions along the legs and the rungs, according to the following Hamiltonian

(a) (b)

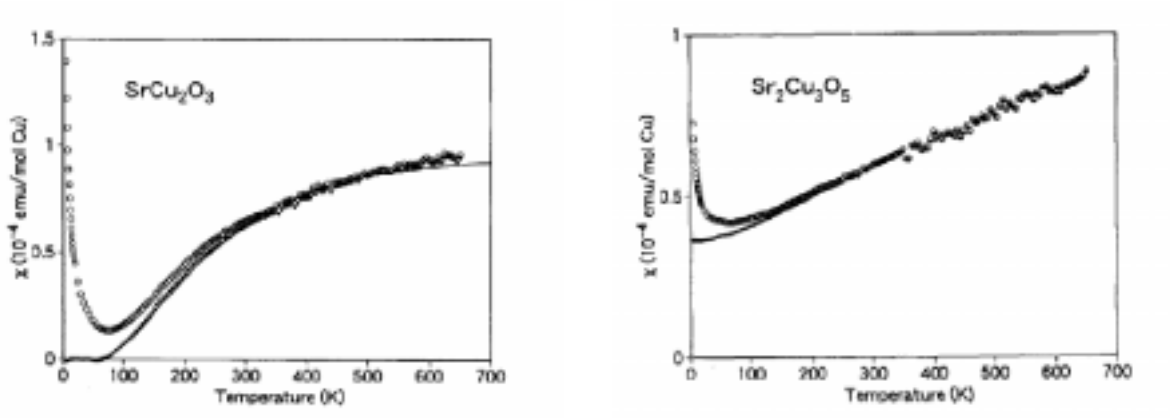


Figure 2: Magnetic susceptibility against temperature for (a) $SrCu_2O_3$ and (b) $Sr_2Cu_3O_5$. Details can be found in [3] and [17].

$$H = \frac{J_l}{4}\sum_{i=1}^{L}\{\vec{\sigma}_i.\vec{\sigma}_{i+1} + \vec{\tau}_i.\vec{\tau}_{i+1}\} + \frac{J_r}{2}\sum_{i=1}^{L}\vec{\sigma}_i.\vec{\tau}_i$$

Above $\vec{\sigma}_i$ and $\vec{\tau}_i$ are Pauli matrices acting on site i of the upper and lower legs, respectively, $J_r(J_l)$ is the strength of the rung (leg) coupling. Although very simple, this model can not be solved exactly. Just numerical techniques, such as the exact diagonalization, density matrix renormalization group, among others, or perturbation theory may be applied.

In order to get an insight in the theory of ladder systems, some integrable ladder chains have been proposed. Here we will focus in one which has the important property of having a free parameter.

3. The spin ladder Hamiltonian: ground state and elementary excitations

Let us begin by presenting the generalized spin ladder model, whose Hamiltonian reads

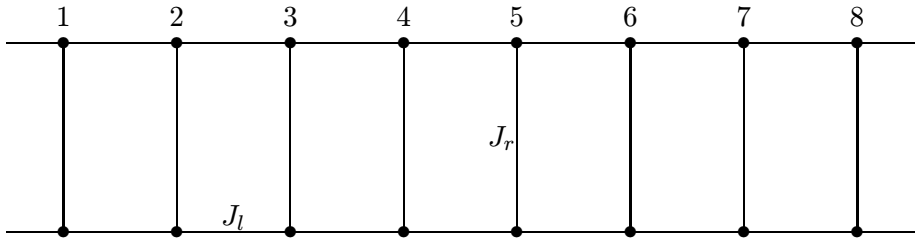
$$H^{(1)} = \sum_{j=1}^{L}\left[J_l\, h_{j,j+1} + \frac{1}{2}J_r\,(\vec{\sigma}_j.\vec{\tau}_j - 1)\right] \tag{3.1}$$

where

$$\begin{aligned} h_{j,j+1} = {} & \frac{1}{4}(1+\sigma_j^z\sigma_{j+1}^z)(1+\tau_j^z\tau_{j+1}^z) + (\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+)(\tau_j^+\tau_{j+1}^- + \tau_j^-\tau_{j+1}^+) \\ & + \frac{1}{2}(1+\sigma_j^z\sigma_{j+1}^z)(t^{-1}\tau_j^+\tau_{j+1}^- + t\tau_j^-\tau_{j+1}^+) + \frac{1}{2}(t^{-1}\sigma_j^+\sigma_{j+1}^- + t\sigma_j^-\sigma_{j+1}^+)(1+\tau_j^z\tau_{j+1}^z) \end{aligned}$$

Above $\vec{\sigma}_j$ and $\vec{\tau}_j$ are Pauli matrices acting on site j of the upper and lower legs, respectively, $J_r(J_l)$ is the strength of the rung (leg) coupling (we will consider only the case $J_l, J_r > 0$ in the subsequent analysis corresponding to *antiferromagnetic* coupling) and t is a free

parameter representing an anisotropy in the legs and inter-chain interaction. Throughout, L is the number of rungs (equivalently, the length of the ladder) and periodic boundary conditions are imposed. By setting $t \to 1$ in equation (3.1), Wang's model based on the $SU(4)$ symmetry [4, 5] can be recovered. (Strictly speaking, it is $SU(4)$ invariant in the absence of the rung interactions.) The Hamiltonian is invariant under interchange of the legs; i.e. $\vec{\sigma_j} \leftrightarrow \vec{\tau_j}$. Moreover, under spin inversion for both leg spaces the Hamiltonian is invariant with the interchange $t \leftrightarrow t^{-1}$. For this reason we see that the parameter t plays the role of spin anisotropy. The figure below shows a schematic representation for a spin ladder model with $L = 8$.



The energy eigenvalues of the Hamiltonian are given by

$$E = -\sum_{j=1}^{M_1}\left(\frac{J_l}{\lambda_j^2 + 1/4} - 2J_r\right) + (J_l - 2J_r)\, L \tag{3.2}$$

where λ_j are solutions to the Bethe ansatz equations (3.3) below. The Bethe ansatz equations arise from the exact solution of the model through the nested algebraic Bethe ansatz method and read

$$\begin{aligned}
t^{(L-2M_3)} \left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L &= \prod_{l\neq j}^{M_1} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i} \prod_{\alpha=1}^{M_2} \frac{\lambda_j - \mu_\alpha + i/2}{\lambda_j - \mu_\alpha - i/2} \\
t^{(L-2M_3)} \prod_{\beta\neq\alpha}^{M_2} \frac{\mu_\alpha - \mu_\beta - i}{\mu_\alpha - \mu_\beta + i} &= \prod_{j=1}^{M_1} \frac{\mu_\alpha - \lambda_j - i/2}{\mu_\alpha - \lambda_j + i/2} \prod_{\delta=1}^{M_3} \frac{\mu_\alpha - \nu_\delta - i/2}{\mu_\alpha - \nu_\delta + i/2} \\
t^{(L+2M_2-2M_1)} \prod_{\gamma\neq\delta}^{M_3} \frac{\nu_\delta - \nu_\gamma - i}{\nu_\delta - \nu_\gamma + i} &= \prod_{\alpha=1}^{M_2} \frac{\nu_\delta - \mu_\alpha - i/2}{\nu_\delta - \mu_\alpha + i/2}
\end{aligned} \tag{3.3}$$

We remark that although the anisotropy parameter t does not appear explicitly in the energy expression (3.2), the solutions λ_j for the Bethe ansatz equations do depend on t as will be illustrated later.

The exact diagonalization of the two-site Hamiltonian shows that for $J_r > J_l + \frac{J_l}{2}(t + \frac{1}{t})$ the (unique) ground state assumes the form of the product of the singlets with energy $E_0 = 2(J_l - 2J_r)$ and the energies of the excitations are given by:

$$E_1 = -2J_r - J_l(t+1/t) \qquad E_2 = -2J_r + J_l(t+1/t) \qquad E_3 = J_l(t+1/t) \qquad E_4 = -J_l(t+1/t)$$

$$E_5 = -2(J_r - J_l) \qquad E_6 = -2(J_r + J_l) \qquad E_7 = 2J_l \qquad E_8 = -2J_l$$

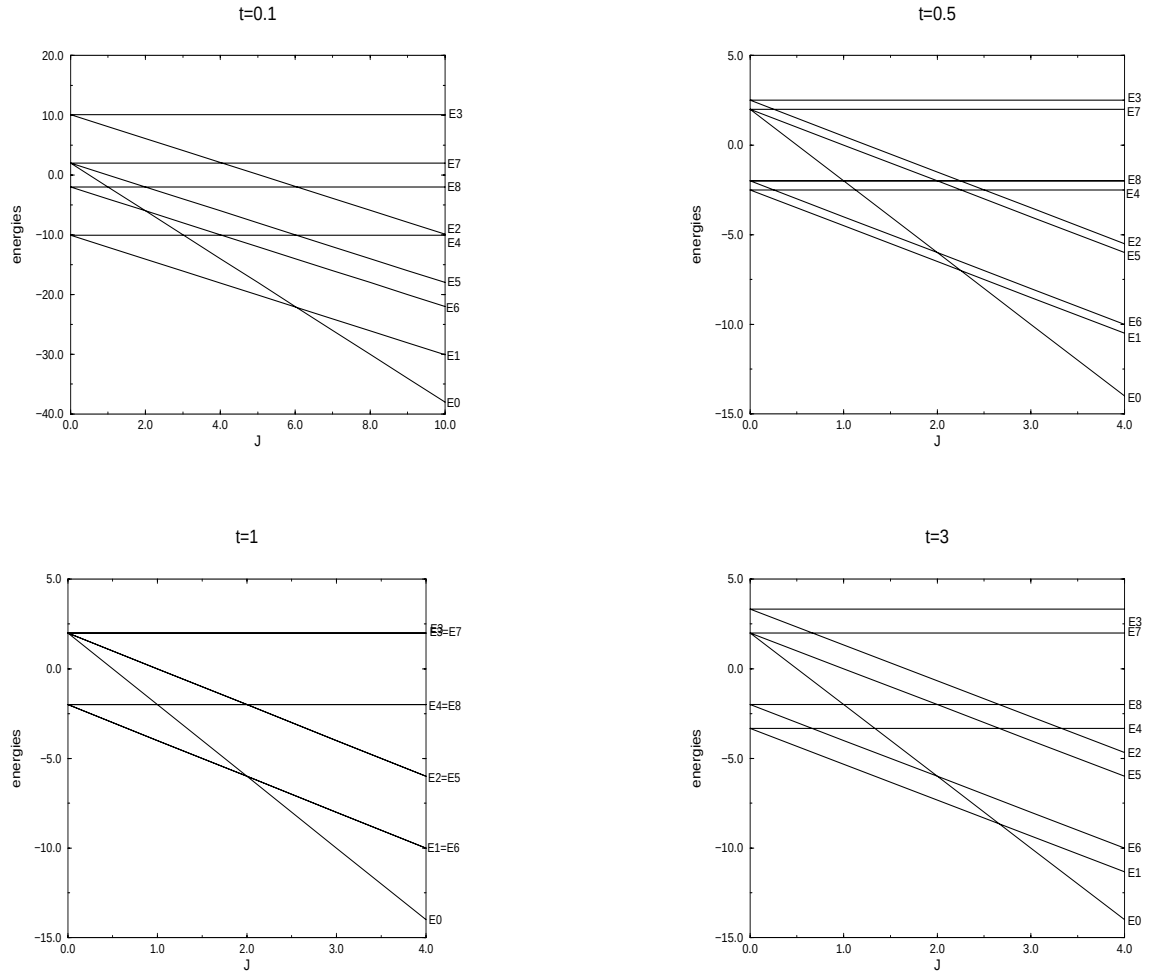


Figure 3: Energies E_i $(i = 0, ..., 8)$ versus rung coupling J for different values of the anisotropy t. Notice that there is mainly competition between E_0 and E_1 to be the lowest energy level of the model. In addition, the critical value of J above which E_0 is the ground state energy varies with t reaching its minimum value when $t = 1$.

A sample of these numerical results are presented in figure 3 above.[1]

For L sites it follows that the ground state is still given by a product of rung singlets when $J_r > J_l + \frac{J_l}{2}(t + \frac{1}{t})$ and the energy is $(J_l - 2J_r)L$. This is in fact the reference state used in the Bethe ansatz calculation and corresponds to the case $M_1 = M_2 = M_3 = 0$ for the Bethe ansatz equation (3.3). To describe an elementary spin 1 excitation, we take $M_1 = 1$ and $M_2 = M_3 = 0$ in the eq. (3.3) which leads to the imaginary solution for the variable λ (strictly, the lattice length L is assumed even)

$$\lambda = \frac{i}{2}\frac{t-1}{t+1} \tag{3.4}$$

[1]Here we have used $J_l = 1$ and $J_r = J$, for simplicity.

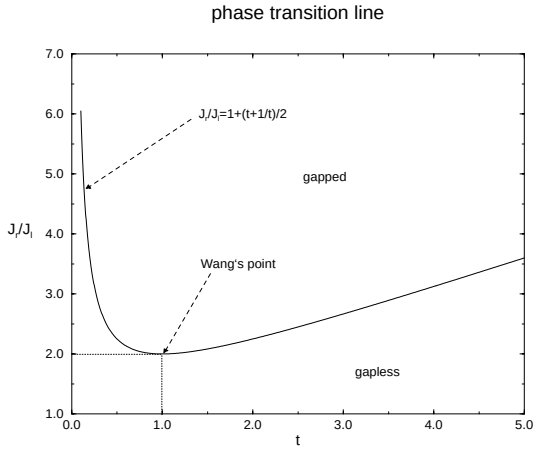


Figure 4: Rung coupling J_r/J_l versus anisotropy t. This graphic represents the phase diagram and the dotted line shows Wang's point. The curve ($J_r = J_l + J_l(t+1/t)/2$) divides the gaped and gapless phases.

giving the minimal excited state energy. The energy gap can easily be calculated using the exact Bethe ansatz solution and has the form

$$\Delta = 2\bigg(J_r - J_l - \frac{J_l}{2}(t+\frac{1}{t})\bigg). \tag{3.5}$$

By solving $\Delta = 0$ for J_r/J_l we find the critical value $(J_r/J_l)^c = 1 + \frac{1}{2}(t+\frac{1}{t})$, indicating the critical line at which the quantum phase transition from the dimerized phase to the gapless phase occurs, as shown in figure 4 (see [26] for details).

4. Integrability

The integrability of this model can be shown by the fact that it can be mapped (see eq. (4.2) below) to the following Hamiltonian, which can be derived from an $R-$matrix obeying the Yang-Baxter algebra for $J_r = 0$, while for $J_r \neq 0$ the rung interactions take the form of a chemical potential term.

$$\hat{H}^{(1)} = \sum_{j=1}^{L}\bigg[J_l \;\; \hat{h}_{j,j+1} - 2J_r\, X_j^{00}\bigg] \tag{4.1}$$

where

$$\begin{aligned}\hat{h}_{j,j+1} = &\sum_{\alpha=0}^{3} X_j^{\alpha\alpha}X_{j+1}^{\alpha\alpha} + X_j^{20}X_{j+1}^{02} + X_j^{02}X_{j+1}^{20} + X_j^{13}X_{j+1}^{31} + X_j^{31}X_{j+1}^{13}\\ &+t\bigg(X_j^{10}X_{j+1}^{01} + X_j^{12}X_{j+1}^{21} + X_j^{03}X_{j+1}^{30} + X_j^{23}X_{j+1}^{32}\bigg)\\ &+t^{-1}\bigg(X_j^{01}X_{j+1}^{10} + X_j^{21}X_{j+1}^{12} + X_j^{30}X_{j+1}^{03} + X_j^{32}X_{j+1}^{23}\bigg).\end{aligned}$$

Above $X_j^{\alpha\beta} = |\alpha_j >< \beta_j|$ are the Hubbard operators with $|\alpha_j>$ the orthogonalized eigenstates of the local operator $\vec{\sigma_j}.\vec{\tau_j}$, as in Wang's case [4, 5]. The local Hamiltonians (3.1) and (4.1) are related through the following basis transformation

$$\begin{aligned}
&|\uparrow,\uparrow\rangle \rightarrow 1/\sqrt{2}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle),\\
&|\uparrow,\downarrow\rangle \rightarrow |\uparrow,\uparrow\rangle,\\
&|\downarrow,\uparrow\rangle \rightarrow 1/\sqrt{2}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle),\\
&|\downarrow,\downarrow\rangle \rightarrow |\downarrow,\downarrow\rangle.
\end{aligned} \tag{4.2}$$

The following R-matrix

$$R = \left(\begin{array}{cccc|cccc|cccc|cccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & t^{-1}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0\\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & -\\
0 & c & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0\\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & -\\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0\\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & -\\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-1}b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array}\right), \tag{4.3}$$

with

$$a = x + 1 \quad , b = x, \quad c = 1,$$

obeys the Yang-Baxter algebra

$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y) \tag{4.4}$$

and originates the Hamiltonian (4.1) for $J_r = 0$ by the standard procedure [19]

$$\hat{h}_{j,j+1} = P\frac{d}{dx}R(x)|_{x=0},$$

where P is the permutation operator.

Actually, this model studied above represents one particular case of a more general Hamiltonian that has six free parameters $(t_i, i = 1, \ldots, 6)$ and reads

$$H^g = \sum_{j=1}^{L}\left[J_l\, h_{i,j}^g + \tfrac{1}{2}J_r\,(\vec{\sigma_i}.\vec{\tau_i} - 1)\right] \tag{4.5}$$

where

$$
\begin{aligned}
4\ h^g_{ij} &= 1+\sigma^z_i\sigma^z_j\tau^z_i\tau^z_j+q_1\{\sigma^z_i\sigma^z_j+\tau^z_i\tau^z_j\}+q_2\{\sigma^z_j\tau^z_i+\sigma^z_i\tau^z_j\}\\
&+k^+_1\tau^-_i\tau^+_j\{1+\sigma^z_i\sigma^z_j\}+k^-_1\tau^-_i\tau^+_j\{\sigma^z_i+\sigma^z_j\}\\
&+k^+_2\tau^+_i\tau^-_j\{1+\sigma^z_i\sigma^z_j\}+k^-_2\tau^+_i\tau^-_j\{\sigma^z_i+\sigma^z_j\}\\
&+k^+_3\sigma^+_j\tau^-_i\{1+\sigma^z_i\tau^z_j\}+k^-_3\sigma^+_j\tau^-_j\{\sigma^z_i+\tau^z_j\}\\
&+k^+_4\sigma^+_i\tau^-_j\{1+\sigma^z_j\tau^z_i\}+k^-_4\sigma^+_i\tau^-_j\{\sigma^z_j+\tau^z_i\}\\
&+k^+_5\sigma^-_j\tau^+_i\{1+\sigma^z_i\tau^z_j\}+k^-_5\sigma^-_j\tau^+_i\{\sigma^z_i+\tau^z_j\}\\
&+k^+_6\sigma^+_i\sigma^-_j\{1+\tau^z_i\tau^z_j\}+k^-_6\sigma^+_i\sigma^-_j\{\tau^z_i+\tau^z_j\}\\
&+k^+_7\sigma^-_i\tau^+_i\{1+\sigma^z_j\tau^z_j\}+k^-_7\sigma^-_i\tau^+_i\{\sigma^z_j+\tau^z_j\}\\
&+k^+_8\sigma^-_i\sigma^+_j\{1+\tau^z_i\tau^z_j\}+k^-_8\sigma^-_i\sigma^+_j\{\tau^z_i+\tau^z_j\}\\
&+q_3\{\sigma^+_j\tau^-_j\{\sigma^z_i-\tau^z_i\}-\sigma^+_i\tau^-_i\{\sigma^z_j-\tau^z_j\}\}\\
&+q_4\{\sigma^-_j\tau^+_j\{\sigma^z_i-\tau^z_i\}-\sigma^-_i\tau^+_i\{\sigma^z_j-\tau^z_j\}\}\\
&+q_5\sigma^+_i\sigma^+_j\tau^-_i\tau^-_j+q_6\sigma^-_i\sigma^+_j\tau^-_i\tau^+_j+q_7\sigma^+_i\sigma^-_j\tau^+_i\tau^-_j\\
&+q_8\sigma^-_i\sigma^-_j\tau^+_i\tau^+_j+q_9\{\sigma^-_i\sigma^+_j\tau^+_i\tau^-_j+\sigma^+_i\sigma^-_j\tau^-_i\tau^+_j\}
\end{aligned}
\tag{4.6}
$$

with the coefficients given by

$$
\begin{aligned}
&k^\pm_1=C_1\pm C_{33},\quad k^\pm_2=C_2\pm C_{34},\quad k^\pm_3=C_4\pm C_{26},\quad k^\pm_4=C_6\pm C_{17},\\
&k^\pm_5=C_9\pm C_{31},\quad k^\pm_6=C_{12}\pm C_{15},\quad k^\pm_7=C_{18}\pm C_{29},\quad k^\pm_8=C_{20}\pm C_{23},\\
&q_1=\frac{(1+C_3-C_{11})}{2}=\frac{(1+t_4)^2}{4t_4},\quad q_2=\frac{(1-C_3+C_{11})}{2}=-\frac{(1-t_4)^2}{4t_4},\\
&q_3=2C_5,\quad q_4=2C_{10},\quad q_5=4C_8,\quad q_6=4C_{22},\quad q_7=4C_{14},\\
&q_8=4C_{27},\quad q_9=4C_{13},
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
&C_1=\frac{t_1^{-1}+t_2^{-1}}{2},\quad C_2=\frac{t_1+t_2}{2},\quad C_3=C_{32}=\frac{(1+t_4)^2}{4t_4},\quad C_4=\frac{(t_1^{-1}-t_2^{-1})}{2},\\
&C_5=-C_{25}=-C_7=C_{16}=\frac{(-1+t_4^2)}{4t_4},\quad C_6=\frac{(t_1-t_2)}{2},\quad C_8=-\frac{(-1+t_4)^2}{4t_4},\\
&C_9=\frac{t_1-t_2}{2},\quad C_{10}=-C_{30}=-C_{19}=C_{28}=-\frac{(-1+t_4^2)}{4t_4},\quad C_{11}=C_{24}=-\frac{(-1+t_4)^2}{4t_4},\\
&C_{13}=C_{21}=\frac{(1+t_4)^2}{4t_4},\quad C_{14}=t_3,\quad C_{15}=\frac{t_5+t_6}{2},\quad C_{17}=\frac{(t_5-t_6)}{2},\\
&C_{18}=\frac{t_1^{-1}-t_2^{-1}}{2},\quad C_{20}=\frac{t_1^{-1}+t_2^{-1}}{2},\quad C_{22}=t_3^{-1},\quad C_{12}=\frac{(t_1+t_2)}{2}\\
&C_{23}=\frac{t_5^{-1}+t_6^{-1}}{2},\quad C_{26}=\frac{(t_5^{-1}-t_6^{-1})}{2},\quad C_{27}=-\frac{(-1+t_4)^2}{4t_4},\quad C_{29}=\frac{t_5^{-1}-t_6^{-1}}{2},
\end{aligned}
$$

$$C_{31} = \frac{t_5 - t_6}{2}, \quad C_{33} = \frac{t_5^{-1} + t_6^{-1}}{2}, \quad C_{34} = \frac{t_5 + t_6}{2}, C_{12} = \frac{t_1 + t_2}{2} \tag{4.8}$$

This model is obtained in a similar way from a multiparametic R-matrix associated with the $SU(4)$ algebra [16].

The physics of this integrable model is expected to be much richer, since the presence of these extra parameters will certainly influence the phase diagram of the model. However, its very difficult to extract some physics from it.

5. Thermodynamics

The partition function is the key ingredient necessary to the computation of the thermodynamics properties of the system. It is given by

$$Z = \sum_{conf} e^{-\beta E_i} \tag{5.1}$$

where $\beta = 1/T$ with T the temperature of the system and E_i is the energy of each possible configuration. In what follows we will compute numerically the magnetic susceptibility as a function of the temperature for our spin ladder Hamiltonian (3.1).

Basically, the magnetic susceptibility (χ) is defined as the derivative of the magnetization with respect to external field

$$\chi = \frac{\partial}{\partial h} M|_{h=0} \tag{5.2}$$

where M is the magnetization,which reads

$$M = \frac{1}{2} \sum_{i=1}^{L} < \sigma_i^z + \tau_i^z >= \frac{1}{2L\beta} \frac{\partial}{\partial h} Ln(Z). \tag{5.3}$$

By employing the exact diagonalization for our ladder Hamiltonian (3.1) up to 12 spins we find the energies from which we obtain the curve of the magnetic susceptibility (χ) as a function of the temperature (T) for different values of the ratio of the couplings $X \equiv \frac{J_r}{J_l}$ and the anisotropy parameter t.

These results are depicted in figure 5(a) for $X > X^c$ and they exhibit a qualitative agreement with the experimental results (see fig.2a and [3]). Some interesting features can be observed from these curves: (i) for a fixed anisotropy parameter $''t''$ there is a "smoothing" of the susceptibility curve by increasing the ratio of the coupling X; (ii) for a fixed X, the magnetic susceptibility increases by decreasing the anisotropy parameter $''t''$.

Another important property, present in all cases is that the susceptibility exhibits an exponential decay for low temperature ($T << \Delta^*$), where Δ^* is the spin gap of the system

$$\chi \propto \frac{e^{-\frac{\Delta^*}{T}}}{\sqrt{T}} \tag{5.4}$$

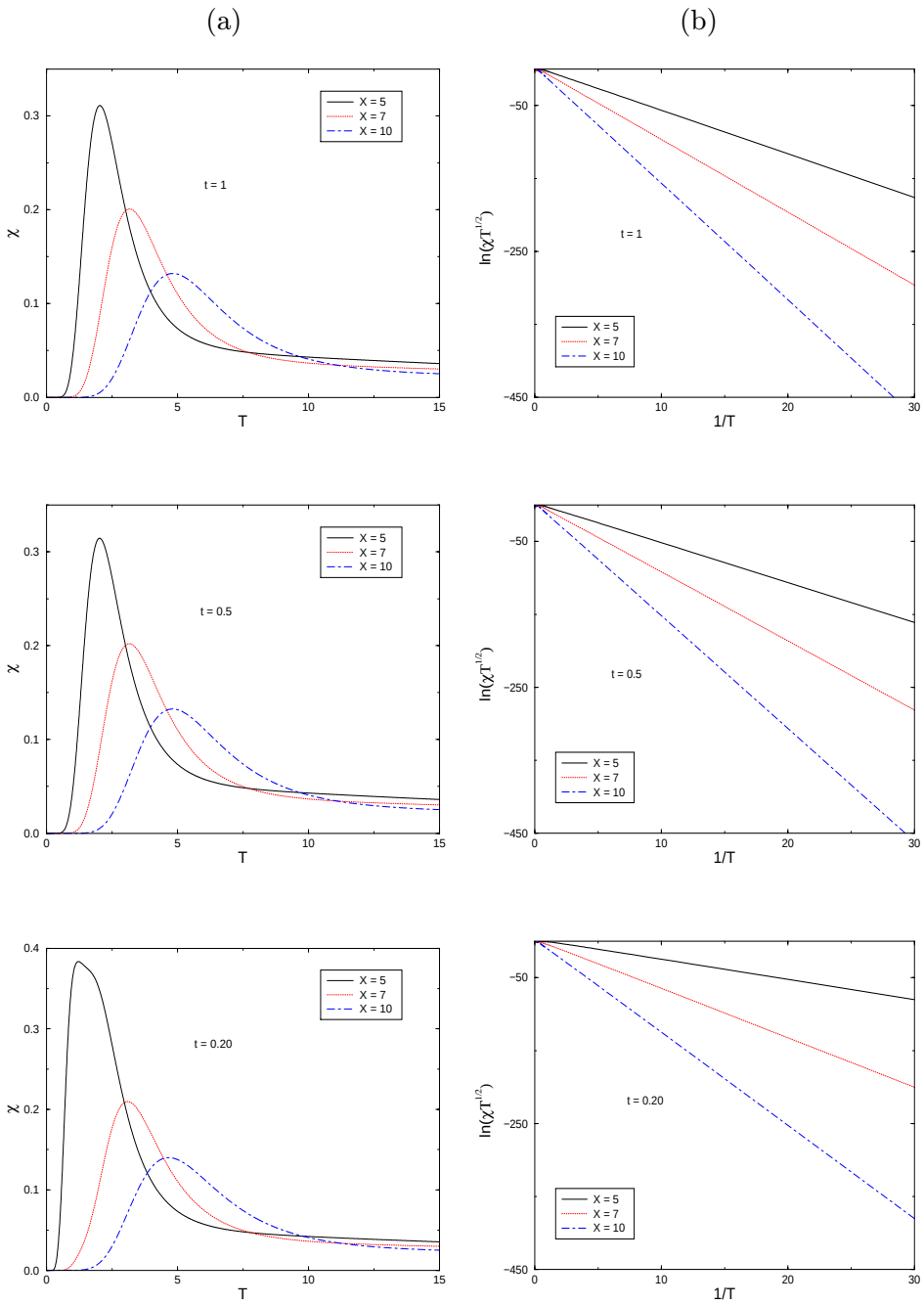


Figure 5: a) The magnetic susceptibility (χ) as a function of the temperature (T) for different values of the ratio of the couplings (X) and the anisotropy parameter (t). (b) A logarithmic plot of the susceptibility (χ) as a function of the inverse of the temperature ($1/T$), from which the spin gap (Δ^*) can be obtained.

which is exactly that found by Troyer and et al [20] for the Heisenberg spin ladder model.

In this context, let us point out that by linearizing this equation (5.4) properly a numerical value for the spin gap (Δ^*) can be found. This result can be compared with the

exact expression of the spin gap obtained from the BAE (Δ) (3.5) at $T = 0$ and we found an excellent agreement. The linearized curves are presented in figure 5(b) while Bethe ansatz and the the numerical gaps are presented on Table 1.

J_r/J_l	t	Δ/J_l	Δ^*/J_l
5	1.00	6.00	5.56
	0.50	5.00	5.56
	0.20	2.80	2.61
7	1.00	10.00	9.76
	0.50	9.50	9.29
	0.20	6.80	6.56
10	1.00	16.00	15.52
	0.50	15.00	15.16
	0.20	12.80	12.50

We remark that the behavior of the magnetic susceptibility for the critical value of the couplings (X^c) is of the form

$$\chi \sim \frac{1}{\sqrt{T}} \tag{5.5}$$

which indicates a typical quantum critical behavior. This behavior was already predicted for the isotropic case in [4, 5] and is illustrated for different values of the anisotropy parameter (t) on figure 6 .

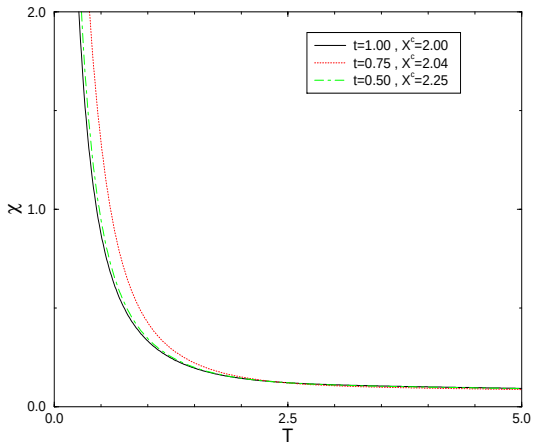


Figure 6: Magnetic susceptibility (χ) versus the temperature (T) for different values of the critical value of the ratio of the couplings X^c. These graphics indicate a typical quantum critical behavior.

We have applied this procedure to obtain the spin gap from some ladder compounds, such like $KCuCl_3$ [21, 22], $Cu_2(C_5H_{12}N_2)_2Cl_4$ [23, 24], $(C_5H_{12}N)_2CuBr_4$ [25] among others, previously studied by different authors with different methods. Our results agree well with the experimental gap (Δ^{exp})[2] and are depicted on the table below. More details will be presented somewhere [27].

compound	J_r/J_l	t	Δ	Δ^*	Δ^{exp}
$KCuCl_3$	4.0	0.3254	32.00	33.35	31.10
$Cu_2(C_5H_{12}N_2)_2Cl_4$	5.5	0.2344	10.80	13.24	10.80
$(C_5H_{12}N)_2CuBr_4$	4.0	0.3100	9.85	9.71	9.50

6. Conclusions

To summarize, we have discussed a generalized spin ladder model based on the $SU(4)$ symmetry. This was achieved by introducing one extra parameter into the system without violating integrability. The Bethe ansatz equations as well as the energy expression of the model were presented. We have also that the model exhibits a spin gap that depends on the free parameter and the critical point as well as the phase diagram were obtained. The magnetic susceptibility was also investigated and a comparsion between the gap obtained from the Bethe ansatz equations and the gap obtained from magnetic susceptibility curve was performed. Finally a connection with some strong coupling compounds was discussed.

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