

Dynamical R -Matrices for the Calogero Models

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ABSTRACT: We report on our recent systematic study of the integrability of the Calogero models using the traditional Lax representation due to Moser and, in its general form, to Olshanetsky and Perelomov, based on (semi-) simple Lie algebras or on symmetric pairs, but concentrating on the construction of a dynamical R -matrix within this context. Introducing a new family of generators in the Cartan subalgebra of the corresponding simple Lie algebra and assuming the usual functional identities satisfied by the functions that appear in the interaction potential, we show that the conditions guaranteeing integrability can be reduced to a set of algebraic constraints on these generators. As it turns out, these algebraic constraints impose strong restrictions on the (semi-) simple Lie algebra or symmetric pair underlying the construction.

KEYWORDS: Calogero models, Lax pairs, Dynamical R -matrices.

1. Introduction

In recent years, the concept of a “dynamical” R -matrix – as opposed to that of a “numerical” R -matrix which has over the last two decades given rise to an entire new branch of mathematics (see, e.g., the books by Ma [1] and by Chari and Pressley [2]) – has received increasing attention. Dynamical R -matrices first appeared in certain non-ultralocal models of $1+1$ -dimensional field theory [3], in particular the non-linear sigma models [4]. Somewhat later, they were also found to arise in mechanics, more specifically in the Calogero-Moser-Sutherland models [5] or Calogero models, as we shall call them for short [6]. Their mathematical status, however, is still far from being fully understood. For example, it has been repeatedly conjectured that they might be no more than simple “gauge transforms” of standard numerical R -matrices. Thus the Calogero models, which

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are Hamiltonian systems with finitely many degrees of freedom, provide an ideal theoretical laboratory for conceptual investigations concerning the mathematical nature of dynamical R -matrices – unlike the non-linear sigma models where such studies are hampered by the jump singularities that plague the Poisson brackets between the entries of the monodromy matrix and require regularization.

In classical mechanics, which is the context of our analysis, the standard tool for handling integrable Hamiltonian systems is the reformulation of their equations of motion as a *Lax equation*

$$\dot{L} = [L, M], \quad (1.1)$$

where L and M are mappings $L, M : P \rightarrow \mathfrak{g}$ from the phase space P of the model into some Lie algebra \mathfrak{g} . Once this has been achieved, it is obvious that the ad-invariant functions on \mathfrak{g} furnish conserved quantities. In the simplest case they are given by the trace polynomials in some irreducible representation,

$$F_k(L) = \text{tr}(L^k), \quad (1.2)$$

and it is required that the Hamiltonian of the model be one of them: in the simplest case, it will be the quadratic invariant:

$$H = \frac{1}{2} \text{tr}(L^2). \quad (1.3)$$

Moreover, these conserved quantities will be in involution if the entries of L satisfy the so-called *fundamental Poisson bracket relation*

$$\{L_1, L_2\} = [R_{12}, L_1] - [R_{21}, L_2], \quad (1.4)$$

where R is a mapping $R : P \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ from the phase space P of the model into the second tensor power of the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} and we employ the usual tensor notation, where $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$, R_{12} is R and R_{21} is the transpose of R [7]. The fact that R may exhibit non-trivial dependence on the phase space variables is indicated by saying that the R -matrix is “dynamical”. When R does not depend on the phase space variables, one says that the R -matrix is “constant” or “numerical”; an example of this situation is provided by the Toda models.

2. Calogero models

The Hamiltonian of the Calogero models describing n particles on a line with pairwise interaction is given by the expression

$$H(q, p) = \frac{1}{2} \left(\sum_{j=1}^n p_j^2 + g^2 \sum_{\substack{k, l=1 \\ k \neq l}}^n w(q_k - q_l)^2 \right), \quad (2.1)$$

where g is a coupling constant and w is defined by

$$w(t) = \begin{cases} \frac{1}{t} & \text{for the rational model} \\ \frac{1}{\sin(t)} & \text{for the trigonometric model} \\ \frac{1}{\sinh(t)} & \text{for the hyperbolic model} \end{cases} \quad (2.2)$$

For the sake of completeness, we would like to mention that the Calogero models with any one of these three types of potential are collectively referred to as the degenerate Calogero models, as opposed to the elliptic Calogero models where the potential is given by the Weierstrass \wp -function.

All these potentials are singular at the origin, so the relative ordering between the particle positions is a constant of motion. Therefore, we may assume without loss of generality that the configuration space of the model is an open subset of \mathbb{R}^n defined by some fixed ordering such as, for example,

$$q_1 < \dots < q_n . \quad (2.3)$$

A Lax pair for this model was found by Moser in the degenerate case and by Krichever in the elliptic case [5], whereas the corresponding dynamical R -matrix was first constructed by Avan & Talon in the degenerate case and by Sklyanin in the elliptic case [6]; explicitly, they are in the degenerate case given by

$$L = \sum_{j=1}^n p_j E_{jj} + i g \sum_{\substack{k,l=1 \\ k \neq l}}^n w(q_k - q_l) E_{kl} , \quad (2.4)$$

$$M = i g \sum_{\substack{k,l=1 \\ k \neq l}}^n \frac{w''(q_k - q_l)}{2w(q_k - q_l)} (E_{kk} - \frac{1}{n} \mathbf{1}) + i g \sum_{\substack{k,l=1 \\ k \neq l}}^n w'(q_k - q_l) E_{kl} , \quad (2.5)$$

and

$$R = - \sum_{\substack{k,l=1 \\ k \neq l}}^n w(q_k - q_l) \left(E_{kk} - \frac{1}{n} \mathbf{1} \right) \otimes E_{kl} + \sum_{\substack{k,l=1 \\ k \neq l}}^n \frac{w'(q_k - q_l)}{w(q_k - q_l)} E_{kl} \otimes E_{lk} , \quad (2.6)$$

where the symbol E_{kl} denotes the elementary matrix with 1 in the k^{th} row and l^{th} column and 0 everywhere else, explicitly given by $(E_{kl})_{ij} = \delta_{ki} \delta_{lj}$.

2.1 Generalization to simple Lie algebras

Consider a simple complex Lie algebra \mathfrak{g} of rank r with Cartan subalgebra \mathfrak{h} and root system Δ . Let $\mathfrak{g}_{\mathbb{R}}$ be the normal real form of \mathfrak{g} and write $\mathfrak{h}_{\mathbb{R}}$ for the intersection of $\mathfrak{g}_{\mathbb{R}}$ with \mathfrak{h} . The algebraic setting encountered above corresponds to the situation where $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(n, \mathbb{R})$. In general, the configuration space Q of the model will be (the interior of) a Weyl chamber within $\mathfrak{h}_{\mathbb{R}}$, and the phase space P is its cotangent bundle

$T^*Q = Q \times \mathfrak{h}_{\mathbb{R}}$ which we may identify with its tangent bundle $TQ = Q \times \mathfrak{h}_{\mathbb{R}}$. Choosing an orthonormal basis $\{H_1, \dots, H_r\}$ of $\mathfrak{h}_{\mathbb{R}}$, extended to a Cartan-Weyl basis of \mathfrak{g} with appropriately normalized root generators E_{α} , and introducing the vectors $q = \sum_{j=1}^r q_j H_j$ in Q and $p = \sum_{j=1}^r p_j H_j$ in $\mathfrak{h}_{\mathbb{R}}$, we obtain the Hamiltonian

$$H(q, p) = \frac{1}{2} \left(\sum_{j=1}^r p_j^2 + \sum_{\alpha \in \Delta} g_{\alpha}^2 w(\alpha(q))^2 \right), \quad (2.7)$$

where the coupling constants g_{α} are supposed to be invariant under the Weyl group $W(\mathfrak{g})$. A Lax pair for this more general context was first proposed by Olshanetsky and Peregomov [8]. In Ref. [9], their construction has been simplified through the introduction of a new set of generators F_{α} in $\mathfrak{h}_{\mathbb{R}}$, over and above the standard H_{α} , which allow to write down explicit expressions for the Cartan part of the matrix M as well as for the dynamical R -matrix, namely

$$L = \sum_{j=1}^r p_j H_j + \sum_{\alpha \in \Delta} i g_{\alpha} w(\alpha(q)) E_{\alpha}, \quad (2.8)$$

$$M = - \sum_{\alpha \in \Delta} i g_{\alpha} \frac{w''(\alpha(q))}{2w(\alpha(q))} F_{\alpha} + \sum_{\alpha \in \Delta} i g_{\alpha} w'(\alpha(q)) E_{\alpha}, \quad (2.9)$$

and

$$R = \sum_{\alpha \in \Delta} w(\alpha(q)) F_{\alpha} \otimes E_{\alpha} + \sum_{\alpha \in \Delta} \frac{w'(\alpha(q))}{w(\alpha(q))} E_{\alpha} \otimes E_{-\alpha}. \quad (2.10)$$

As shown in Ref. [9], the required equivalence between the equations of motion derived from the Hamiltonian (2.7) and the Lax equation (1.1) as well as the validity of the fundamental Poisson relation (1.4) are then guaranteed if and only if one imposes the following set of algebraic constraints on the generators F_{α} :

$$g_{\alpha} \alpha(F_{\beta}) - g_{\beta} \beta(F_{\alpha}) - g_{\alpha+\beta} N_{\alpha,\beta} = 0, \quad (2.11)$$

where the $N_{\alpha,\beta}$ are the structure constants of \mathfrak{g} , defined by $[E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha+\beta}$. Decomposing F_{α} into its even and odd part $F_{\alpha}^{\pm} = \frac{1}{2}(F_{\alpha} \pm F_{-\alpha})$, these constraints split into two independent sets, namely

$$g_{\alpha} \alpha(F_{\beta}^{+}) - g_{\beta} \beta(F_{\alpha}^{+}) - g_{\alpha+\beta} N_{\alpha,\beta} = 0, \quad (2.12)$$

$$g_{\alpha} \alpha(F_{\beta}^{-}) - g_{\beta} \beta(F_{\alpha}^{-}) = 0, \quad (2.13)$$

the second of which is easily solved by putting $F_{\alpha}^{-} = \lambda g_{\alpha} H_{\alpha}$ where λ is an arbitrary constant. The same algebraic constraints are obtained in the case of the elliptic Calogero models.

Explicit calculations show that the only simple complex Lie algebras that admit a solution of these constraints are those of the A -series. More precisely, we can state the following

Theorem 1. *For the simple complex Lie algebras $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of the A-series, the algebraic constraint (2.12) has the non-trivial solution*

$$F_{ij}^+ = -\frac{1}{2} (E_{ii} + E_{jj}) + \frac{1}{n} 1 \quad (2.14)$$

for $1 \leq i \neq j \leq n$ (where $F_{\alpha_{ij}}$ and $F_{\alpha_{ij}}^\pm$ have been abbreviated to F_{ij} and F_{ij}^\pm , respectively). In all other cases, the algebraic constraint (2.11) has no solution.

The last statement constitutes a no-go theorem that for a long time has been conjectured but has for the first time been proven explicitly in Ref. [9].

2.2 Generalization to irreducible symmetric pairs

Consider an irreducible symmetric pair (\mathfrak{g}, θ) of rank r , that is, a simple complex Lie algebra \mathfrak{g} together with an involutive automorphism θ , leading to the orthogonal direct decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad (2.15)$$

of \mathfrak{g} into the $\theta = +1$ -eigenspace \mathfrak{k} which is a subalgebra and the $\theta = -1$ -eigenspace \mathfrak{m} , with the standard commutation relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. We choose a maximally noncompact θ -invariant Cartan subalgebra \mathfrak{h} of \mathfrak{g} , leading to an analogous orthogonal direct decomposition

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a} \quad (2.16)$$

where \mathfrak{a} is a maximal abelian subspace of \mathfrak{m} [10, 11] and divide the corresponding root system Δ into two subsystems – the system Δ_0 of roots that vanish on \mathfrak{a} and the system $\tilde{\Delta}$ of all other roots. Let \mathfrak{g}_0 be the unique real form of \mathfrak{g} such that the restriction of θ to \mathfrak{g}_0 is the Cartan involution of \mathfrak{g}_0 , and write \mathfrak{k}_0 , \mathfrak{m}_0 , \mathfrak{h}_0 , \mathfrak{b}_0 and \mathfrak{a}_0 for the intersection of \mathfrak{g}_0 with \mathfrak{k} , \mathfrak{m} , \mathfrak{h} , \mathfrak{b} and \mathfrak{a} , respectively; then $\mathfrak{h}_0 = \mathfrak{b}_0 \oplus \mathfrak{a}_0$ whereas $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{b}_0 \oplus \mathfrak{a}_0$. The configuration space Q of the model is now (the interior of) a Weyl chamber in \mathfrak{a}_0 and hence the phase space P is its cotangent bundle $T^*Q = Q \times \mathfrak{a}_0^*$ which we may identify with its tangent bundle $TQ = Q \times \mathfrak{a}_0$. Choosing an orthonormal basis $\{H_1, \dots, H_r\}$ of \mathfrak{a}_0 , complemented by an orthonormal basis $\{H_{r+1}, \dots, H_{r+s}\}$ of $i\mathfrak{b}_0$ and extended to a Cartan-Weyl basis of \mathfrak{g} with appropriately normalized root generators E_α satisfying the condition¹ $\theta E_\alpha = E_{\theta\alpha}$, and introducing the vectors $q = \sum_{j=1}^r q_j H_j$ in Q and $p = \sum_{j=1}^r p_j H_j$ in \mathfrak{a}_0 as before, we obtain the Hamiltonian

$$H(q, p) = \frac{1}{2} \left(\sum_{j=1}^r p_j^2 + \sum_{\alpha \in \tilde{\Delta}} g_\alpha^2 w(\alpha(q))^2 \right), \quad (2.17)$$

which is formally the same as before, provided we assume the coupling constants g_α to vanish for $\alpha \in \Delta_0$; moreover, we suppose them to be invariant under the automorphism θ ($g_{\theta\alpha} = g_\alpha$) as well as under the Weyl group $W(\mathfrak{g}, \theta)$ associated with the symmetric pair

¹In the Appendix of Ref. [9], it is (erroneously) claimed that the root generators E_α can always be chosen to satisfy this condition, while it is in general only possible to achieve this up to signs. See the erratum to Ref. [9].

(\mathfrak{g}, θ) , which is defined to be the centralizer of θ in the full Weyl group $W(\mathfrak{g})$ of \mathfrak{g} . Again, a Lax pair for this case was first proposed by Olshanetsky and Perelomov [8], and again, their construction has been simplified in Ref. [9] through the introduction of a new set of generators in \mathfrak{ib}_0 , denoted here by K_α rather than F_α , invariant under the automorphism θ ($K_{\theta\alpha} = K_\alpha$), which allow to write down explicit expressions for the Cartan part of the matrix M as well as for the dynamical R -matrix, namely

$$L = \sum_{j=1}^r p_j H_j + \sum_{\alpha \in \tilde{\Delta}} i g_\alpha w(\alpha(q)) E_\alpha, \quad (2.18)$$

$$M = - \sum_{\alpha \in \tilde{\Delta}} i g_\alpha \frac{w''(\alpha(q))}{2w(\alpha(q))} K_\alpha + \sum_{\alpha \in \tilde{\Delta}} i g_\alpha w'(\alpha(q)) E_\alpha, \quad (2.19)$$

and

$$R = \sum_{\alpha \in \tilde{\Delta}} w(\alpha(q)) K_\alpha \otimes E_\alpha + \frac{1}{2} \sum_{\alpha \in \tilde{\Delta}} \frac{w'(\alpha(q))}{w(\alpha(q))} (E_{\theta\alpha} \otimes E_{-\alpha} + E_\alpha \otimes E_{-\alpha}). \quad (2.20)$$

As before, the required equivalence between the equations of motion derived from the Hamiltonian (2.17) and the Lax equation (1.1) as well as the validity of the fundamental Poisson relation (1.4) can be shown [9] to hold if and only if one imposes the following set of algebraic constraints on the generators K_α :

$$g_\alpha \alpha(K_\beta) - g_\beta \beta(K_\alpha) - \frac{1}{2} (g_{\alpha+\beta} N_{\alpha,\beta} + g_{\theta\alpha+\beta} N_{\theta\alpha,\beta}) = 0, \quad (2.21)$$

where the $N_{\alpha,\beta}$ are as before but are required to be θ -invariant ($N_{\theta\alpha,\theta\beta} = N_{\alpha,\beta}$). Again, decomposing K_α into its even and odd part $K_\alpha^\pm = \frac{1}{2} (K_\alpha \pm K_{-\alpha})$, these constraints split into two independent sets, namely

$$g_\alpha \alpha(K_\beta^+) - g_\beta \beta(K_\alpha^+) - \frac{1}{2} (g_{\alpha+\beta} N_{\alpha,\beta} + g_{\theta\alpha+\beta} N_{\theta\alpha,\beta}) = 0, \quad (2.22)$$

$$g_\alpha \alpha(K_\beta^-) - g_\beta \beta(K_\alpha^-) = 0, \quad (2.23)$$

the second of which is easily solved by putting $K_\alpha^- = \lambda g_\alpha (H_\alpha)_\mathfrak{b} = \frac{1}{2} \lambda g_\alpha (H_\alpha + \theta H_\alpha)$ where λ is an arbitrary constant. The same algebraic constraints are obtained in the case of the elliptic Calogero models.

The problem of solving these constraints or else of proving that there is no solution has not yet been tackled in full generality. Complete results are however available for the symmetric pairs of the *AIII*-series:

Theorem 2. *For the symmetric pairs of the AIII-series corresponding to the complex Grassmannians, that is, the symmetric spaces $SU(p+q)/S(U(p) \times U(q))$ (compact type) or $SU(p,q)/S(U(p) \times U(q))$ (noncompact type), the algebraic constraint (2.21) admits a solution if and only if*

$$|q - p| \leq 1,$$

which in terms of the explicit matrix representation employed in Ref. [9] is given as follows. Assuming that $p \leq q$, let $n = p + q$ and, for $1 \leq i \leq n$, define $\theta(i)$ by $\theta(i) = i + q$ for $1 \leq i \leq p$, $\theta(i) = i$ for $p + 1 \leq i \leq q$ (applicable only when $p < q$) and $\theta(i) = i - q$ for $q + 1 \leq i \leq n$. Then abbreviating $K_{\alpha_{ij}}$ and $K_{\alpha_{ij}}^\pm$ to K_{ij} and K_{ij}^\pm , respectively, we have

- if $q = p$, $n = 2p$, then

$$K_{ij}^+ = -\frac{1}{4} \left(E_{ii} + E_{jj} + E_{\theta(i)\theta(i)} + E_{\theta(j)\theta(j)} \right) + \frac{1}{n} 1 \quad (2.24)$$

for $1 \leq i \neq j \leq n$, independently of the values of the two free coupling constants g and g_2 allowed by Weyl group invariance;

- if $q = p + 1$, $n = 2p + 1$, then

$$K_{ij}^+ = -\frac{1}{4} \left(E_{ii} + E_{jj} + E_{\theta(i)\theta(i)} + E_{\theta(j)\theta(j)} \right) + \frac{1}{n} 1 \quad (2.25)$$

for $1 \leq i \neq j \leq n$ such that $i, j \neq p + 1$, while

$$K_{i,p+1}^+ = -\frac{g_1}{4g} (E_{ii} + E_{\theta(i)\theta(i)}) - \frac{g}{2g_1} E_{p+1,p+1} + \frac{1}{n} \left(\frac{g_1}{2g} + \frac{g}{2g_1} \right) 1 \quad (2.26)$$

for $1 \leq i \leq n$ such that $i \neq p + 1$, provided the three free coupling constants g , g_1 and g_2 allowed by Weyl group invariance satisfy the relations $g \neq 0$, $g_1 \neq 0$ and

$$g_1^2 - 2g^2 + gg_2 = 0. \quad (2.27)$$

This result allows to construct a Lax representation with a dynamical R -matrix for the Calogero models, with appropriately chosen coupling constants, based on any of the classical root systems not covered by Theorem 1: B_n , C_n , D_n and BC_n . For more details, see Ref. [9].

3. Dynamical versus numerical R -matrices

Within the general context presented in the previous section, we have recently investigated whether and to what extent the dynamical R -matrices encountered in the Calogero models can, by means of appropriate gauge transformations, be reduced to numerical R -matrices, whose mathematical interpretation in terms of Lie bialgebras is well known [2]. This analysis has first been carried out for the original Calogero models based on the A -series of simple Lie algebras, initially by Hou and Yang in the elliptic case [12] and then, using a somewhat different method, by Fehér and Pusztai in the degenerate case [13]. In what follows, we shall briefly summarize our main results, which extend those of Fehér and Pusztai; details can be found in Ref. [14].

3.1 Gauge transformations

Given a Lax pair and an R -matrix as in Sect. 1, we are free to submit them to a gauge transformation by an arbitrary function $g : P \rightarrow G$ on the phase space P of the model with values in some connected Lie group G with Lie algebra \mathfrak{g} , under which the Lax matrix and the R -matrix transform according to $L \rightarrow L'$ and $R \rightarrow R'$, with

$$L' = gLg^{-1}, \quad (3.1)$$

$$R'_{12} = g_1 g_2 \left(R_{12} + g_1^{-1} \{g_1, L_2\} + \frac{1}{2} \left[g_1^{-1} g_2^{-1} \{g_1, g_2\}, L_2 \right] \right) g_1^{-1} g_2^{-1}, \quad (3.2)$$

where the transformation law for R is determined by the requirement that the fundamental Poisson bracket relation (1.4) should be preserved. In the case of the Calogero models where the R -matrix depends only on the position but not on the momentum variables, it is natural to make the same assumption for the function g ; then using that the Lax matrix is an affine function of the momentum variables of the form

$$L = \sum_{j=1}^r p_j H_j + \sum_{\alpha \in \Delta} L_\alpha E_\alpha, \quad (3.3)$$

where the functions L_α depend only on the position but not on the momentum variables, the transformation law for the R -matrix simplifies to

$$R' = (g \otimes g) \left(R - \sum_{j=1}^r A_j \otimes H_j \right) (g^{-1} \otimes g^{-1}), \quad (3.4)$$

where $A_j = g^{-1} \partial_j g$, so that

$$\partial_k A_l - \partial_l A_k + [A_k, A_l] = 0. \quad (3.5)$$

For R' to be numerical, all partial derivatives $\partial_k R'$ have to vanish and we obtain

$$\partial_k R - \sum_{j=1}^r \partial_j A_k \otimes H_j - \left[R, A_k \otimes 1 + 1 \otimes A_k \right] + \sum_{j=1}^r A_j \otimes [H_j, A_k] = 0. \quad (3.6)$$

The last two equations are further evaluated by decomposing the gauge potentials A_j into their Cartan part and root part:

$$A_j = A_j^{\mathfrak{h}} + \sum_{\alpha \in \Delta} A_j^\alpha E_\alpha. \quad (3.7)$$

As it turns out, the standard functional equations satisfied by the function w in eq. (2.2) allow to reduce eqs (3.5) and (3.6) to a set of purely algebraic identities.

3.2 Results for simple Lie algebras

In the case of simple complex Lie algebras, there are two important algebraic identities that must be satisfied by the generators F_α appearing in the expression (2.10) for the R -matrix: one states that (except for a normalization and/or sign factor), its odd part F_α^- equals the standard Cartan generator H_α while its even part F_α^+ is orthogonal to H_α , whereas the second can be written as a vector constraint in $\mathfrak{h}_{\mathbb{R}}$:

$$\alpha(F_\beta) F_\alpha - \beta(F_\alpha) F_\beta = N_{\alpha,\beta} F_{\alpha+\beta} \quad (3.8)$$

for $\alpha, \beta \in \Delta$ such that $\beta \neq \pm\alpha$.

At first sight, eq. (3.8) seems to be more restrictive than the scalar constraint (2.11). Nevertheless, detailed calculations show that it is not: both constraints have the same solutions, forcing \mathfrak{g} to belong to the A -series. Summarizing, we can state the following

Theorem 3. *The degenerate Calogero model associated with a simple complex Lie algebra \mathfrak{g} admits a gauge transformation g from the standard Lax pair of Olshanetsky and Perelomov and the dynamical R matrix given by eq. (2.10) to a new Lax pair with a numerical R -matrix if and only if \mathfrak{g} belongs to the A -series. In this case, the solutions of eqs (2.11) and (3.8) coincide, and the gauge transformation potential takes the form (3.7) with*

$$A_k^\alpha(q) = w(\alpha(q)) (H_k, F_\alpha) , \quad (3.9)$$

and

$$A_k^{\mathfrak{h}}(q) = \sum_{\alpha \in \Delta} \frac{w'(\alpha(q))}{w(\alpha(q))} (H_k, F_{-\alpha}) F_\alpha . \quad (3.10)$$

The same statement holds for the elliptic Calogero models, despite the fact that in this case, eqs (3.5) and (3.6) lead to additional algebraic constraints which however turn out not to impose any further restrictions.

3.3 Results for irreducible symmetric pairs

In the case of irreducible symmetric pairs, the situation is somewhat more complicated. First of all, we must introduce two sets of generators in the Cartan subalgebra: one is the set of generators K_α belonging to \mathfrak{ib}_0 that appear in the formula (2.20) for the R -matrix and the other is a set of generators M_α belonging to \mathfrak{a}_0 that appear in the explicit form of the gauge potential (3.7) to be given below. There are then two important algebraic identities that must be satisfied by the generators K_α and M_α : one states that (except for a normalization and/or sign factor), their odd parts K_α^- and M_α^- equal the projection of the standard Cartan generator H_α onto \mathfrak{ib}_0 and onto \mathfrak{a}_0 , respectively, while their even parts K_α^+ and M_α^+ are orthogonal to H_α , whereas the second can be written as two vector constraints, one in \mathfrak{ib}_0 and one in \mathfrak{a}_0 :

$$\begin{aligned} \alpha(K_\beta) K_\alpha - \beta(K_\alpha) K_\beta &= \frac{1}{2} \left(N_{\alpha,\beta} K_{\alpha+\beta} + N_{\theta\alpha,\beta} K_{\theta\alpha+\beta} \right) \\ \alpha(K_\beta) M_\alpha - \beta(M_\alpha) M_\beta &= \frac{1}{2} \left(N_{\alpha,\beta} M_{\alpha+\beta} - N_{\theta\alpha,\beta} M_{\theta\alpha+\beta} \right) \end{aligned} \quad (3.11)$$

for $\alpha, \beta \in \Delta$ such that $\beta \neq \pm\alpha$, $\beta \neq \pm\theta\alpha$.

Moreover, it turns out that the possibility to solve the pertinent set of algebraic identities depends on two additional constraints on the structure of the underlying root system. To describe them, we introduce the following terminology [11]: a root α is called *imaginary* if $\theta\alpha = \alpha$, *real* if $\theta\alpha = -\alpha$ and *complex* if $\theta\alpha$ and α are linearly independent. Two roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. Then the additional constraints are

- there are no imaginary roots, i.e., $\Delta_0 = \emptyset$ and $\tilde{\Delta} = \Delta$,
- for every complex root α in Δ , $\theta\alpha$ and α are strongly orthogonal, i.e., $\theta\alpha \pm \alpha$ does not belong to Δ .

Setting

$$F_\alpha = K_\alpha + M_\alpha , \quad (3.12)$$

we then find that eq. (3.11), although apparently weaker than eq. (3.8), has the same solutions, forcing \mathfrak{g} to belong to the A -series. Moreover, the first of the two additional constraints above (together with the requirement that it should be possible to choose the root generators E_α so as to satisfy $\theta E_\alpha = E_{\theta\alpha}$, which is a necessary condition for integrability) exclude the symmetric pairs of the AI -series, of the AII -series and of the $AIII$ -series when $|p - q| > 1$. Finally, the second of the two additional constraints above excludes the symmetric pairs of the $AIII$ -series when $|p - q| = 1$. Summarizing, we can state the following

Theorem 4. *The degenerate Calogero model associated with an irreducible symmetric pair (\mathfrak{g}, θ) admits a gauge transformation g from the standard Lax pair of Olshanetsky and Perelomov and the dynamical R -matrix given by eq. (2.20) to a new Lax pair with a numerical R -matrix if and only if (\mathfrak{g}, θ) corresponds to the complex Grassmannian $SU(2n)/S(U(n) \times U(n))$ (compact type) or $SU(n, n)/S(U(n) \times U(n))$ (noncompact type). In this case, the gauge transformation potential takes the form (3.7) with*

$$A_k^\alpha(q) = w(\alpha(q)) (H_k, M_\alpha) , \quad (3.13)$$

and

$$A_k^{\mathfrak{h}}(q) = \sum_{\alpha \in \Delta} \frac{w'(\alpha(q))}{w(\alpha(q))} (H_k, M_{-\alpha}) K_\alpha . \quad (3.14)$$

The same statement holds for the elliptic Calogero models, despite the fact that in this case, eqs (3.5) and (3.6) lead to additional algebraic constraints which however turn out not to impose any further restrictions.

4. Conclusions and outlook

The main conclusions to be drawn from the results described above are the following.

- In Ref. [9], the question of integrability of the Calogero models associated with irreducible symmetric pairs has been definitely settled for the $AIII$ -series, corresponding to the complex Grassmannians $SU(p, q)/S(U(p) \times U(q))$: as shown there, integrability holds if and only if $|p - q| \leq 1$. This is sufficient to cover all classical root systems that cannot be handled in a direct Lie algebraic approach, including the non-reduced BC_n -system as well as the reduced B_n -, C_n - and D_n -systems, provided the coupling constants are appropriately chosen. A classification of integrability for the remaining irreducible symmetric pairs (see Table V in [10, p. 518]) is still missing. However, we strongly suspect that the $AIII$ -series in fact provide the only case which allows for a non-trivial solution.
- In Ref. [14], it is shown that requiring the dynamical R -matrix for an integrable Calogero model associated with some irreducible symmetric pair to be the gauge

transform of a numerical R -matrix imposes additional restrictions on the underlying root system. These additional restrictions are strong enough to rule out all Calogero models except those based on the complex Grassmannians $SU(p, q)/S(U(p) \times U(q))$ of the $AIII$ -series and, more importantly, also rule out the case $|p - q| = 1$, leaving only the case $p = q$: this includes the models of type C_n and D_n but excludes those of type B_n and BC_n .

- The results of Ref. [14] definitely confirm the position that we adopted at the very beginning of our investigation, namely that the concept of a dynamical R -matrix cannot be entirely reduced to the more traditional one of a numerical R -matrix that has only been concealed by “twisting” it with some gauge transformation. Moreover, it should be noted that even when this is the case, this gauge transformation may itself assume a dynamical role of its own – just like the transformation to action-angle variables in an integrable Hamiltonian system (in the sense of Liouville), which absorbs much of its dynamics.

In view of the picture that emerges from this analysis, various problems of mathematical nature gain renewed importance. Among these are questions such as the following.

- What are the algebro-differential constraints to be satisfied by a dynamical R -matrix? These constraints are usually referred to as the “dynamical Yang-Baxter equation”, which extends and generalizes the usual Yang-Baxter equation for numerical R -matrices. However, it seems to us that the precise form of this equation and its appropriate mathematical interpretation are not entirely clear. In the context of the Calogero models studied here, a natural candidate is the following:

$$\{L_2, R_{13}\} - \{L_3, R_{12}\} + [R_{12}, R_{23}] + [R_{12}, R_{13}] + [R_{32}, R_{13}] = 0 .$$

It has the advantage of being invariant under gauge transformations [15], but it does not seem to coincide with the equation proposed by other authors [16]. At the present stage of investigation, we do not know what are the relations between the various kinds of dynamical Yang-Baxter equations.

- What is the origin of the constraints that restrict integrability of Calogero models? We hope to find an answer to the problem of characterizing the mathematical structure behind the dynamical R -matrices of the Calogero models by studying their relation with the geodesic flow on certain symmetric spaces and also with the recently introduced spin Calogero models, a relation that is a typical application of the Marsden-Weinstein phase space reduction method [17, 18].

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