PROCEEDINGS

Workshop on Integrable Theories, Solitons and Duality

Topics in Nonlinear Sigma Models in $D = 3$

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Abstract: Nonlinear sigma models (NLSM) in $d = 3$ have many interesting and non-trivial features, which were explored poorly in contrast with NLSM in $d = 2$ and $d = 4$. We present a few results from our study of the perturbative and non-pertubative properties of three-dimensional (3D) NLSM.
i) We have shown that cancellation of ultra-violet (UV) divergences takes place in 3D extended ($N = 2, 4$) supersymmetric NLSM in low orders of the $1/n$ expansion.
ii) We consider noncommutative extension of the 3D CP^n model, and study low-energy dynamics of BPS solitons in this model. We also discuss briefly dynamics of non-BPS solutions.

1. Introduction

Nonlinear sigma models (NLSM) in various dimensions d have applications in many branches of physics. In mathematical physics, they provide a class of integrable field theories in $d = 2$. * Superstrings and super-membranes are formulated as field theories in the form of NLSM in $d = 2$ and $d = 3$, respectively. In condensed matter physics, quantum two-dimensional Heisenberg model can be described (approximately) by three-dimensional (3D) $O(3)$ NLSM [1].

*Speaker.

*Integrability does not hold at quantum level for some of the integrable field theories obtained this way , e. g., $CP^n (n \geq 2)$. This remark is due to some of the particpants of this conference.

Low-dimensional (supersymmetric) NLSM share many of the important properties with four-dimensional (4D) (supersymmetric) Yang-Mills theories, in both perturbative and non-perturbative aspects, e.g., the property of β-function and the existence of instantons/solitons. They provide toy field theories of 4D Yang-Mills theories.

One recent development in the theories of string and branes is Yang-Mills theories on noncommutative space (or space-time), which arise as low-energy description of string theories [2, 3, 4]. Noncommutative field theories have also been studied, in both their perturbative and non-perturbative aspects, for their own right and in a more general context. See, for instance, [5, 6].

In this talk we present a few results from our recent studies of NLSM in $d = 3$:
i) Cancellation of ultra-violet (UV) divergences in the 3D extended-supersymmetric NLSM [7, 8] (sec.2).
ii) The properties of solitons in noncommutative extension of 3D NLSM [9, 10] (sec.5 and sec.6).
iii) In addition, we briefly mention our attempt at noncommutative extension of integrable models in $d = 2$ ([11] and unpublished work) (sec.3.3).

2. UV properties of supersymmetric nonlinear sigma models in $d = 3$

NLSM in different space-time dimensions d have distinct ultra-violet (UV) properties. NLSM are renormalizable in $d = 2$, while they are non-renormalizable for $d \geq 4$. In between, i.e., at $d = 3$, (non)renormalizability of NLSM is a subtle question. They are apparently non-renormalizable in perturbation (in the coupling constant). It was shown some time ago that 3D NLSM (e.g., $O(n), CP^n$ models) are renormalizable if the theory is defined in the $1/n$ expansion [12, 13].

It is known that supersymmetric (SUSY) field theories have weaker UV divergences than their bosonic counterparts. Good examples are 2D NLSM and 4D super-Yang Mills theories. 2D Bosonic NLSM are renormalizable, while those with $N = 2$ extended SUSY are finite up to four loops (in the case of Ricci flat target space) [14] and those with $N = 4$ extended SUSY are all-loop finite. The β-function of 4D super-Yang-Mills theories is one-loop exact in the $N = 2$ SUSY case and vanishes in the $N = 4$ case. The question naturally arises whether there is some class of SUSY field theories in $d = 3$ in which similar cancellation of UV divergences takes place.

In view of this we have studied the UV divergence properties of 3D NLSM with extended supersymmetry and found that this class of field theories has UV divergence properties similar to those of 2D NLSM and 4D Yang-Mills theories. UV divergences manifest themselves in β-functions. We present the β-functions of 3D NLSM obtained in [7, 8].
$N = 0,\ N = 1\ O(n)$ [13, 15]

$$\beta(g) = g(1 - g/2\pi) + next{-}to{-}leading\ correc. \tag{2.1}$$

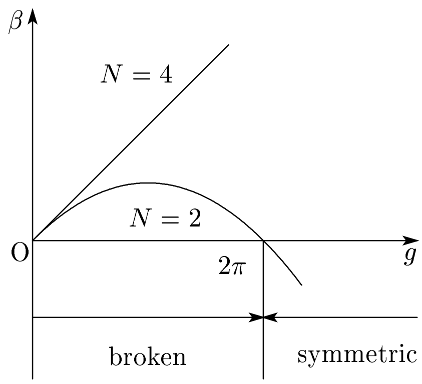

Figure 1: β-function of $N = 2$ and $N = 4$ SUSY NLSM. The critical point is at $g = g_C = 2\pi$ in the $N = 2$ case, as indicated in the figure. There are two phases, "broken" ($g < g_C$) and "symmetric" ($g > g_C$), in the $N = 0, 1$, and $N = 2$ cases.

$N = 2$ CP^n [7, 16]

$$\beta(g) = g(1 - g/2\pi) + \cancel{next\!-\!to\!-\!leading\ correc.} \tag{2.2}$$

$N = 4$ [8]

$$\beta(g) = g + \cancel{leading\ correc} + \cancel{next\!-\!to\!-\!leading\ correc.} \tag{2.3}$$

Here, ~~term~~ denotes that the term vanishes. The β-function in the $N = 2$ and $N = 4$ cases are shown in fig.2.

Remarkably, next-to-leading order corrections to the β-function vanish in $N = 2$ SUSY NLSM, while both leading and next-to-leading order corrections vanish in $N = 4$ SUSY NLSM. These low order results are consistent with the possibility that $\beta(g)$ is leading-order exact in the $N = 2$ case and has no quantum corrections in the $N = 4$ case, which resembles $\beta(g)$ of 4D super-Yang-Mills theories. It is very interesting to see whether this property of the 3D SUSY NLSM holds true in all orders (in the $1/n$ expansion). We would then have 3D finite field theories in addition to 2D and 4D finite field theories, $N = 4$ SUSY NLSM and Yang-Mills theories, respectively.

3. Noncommutative field theories

3.1 Noncommutative field theories

Noncommutative $(d-1)$-dimensional space (d-dimensional space-time) is defined by introducing the commutation relation among the coordinates $\hat{x}^\mu$,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{3.1}$$

where

$$\mu = 1, 2, \cdots, d-1 \ (\mu = 0, 1, \cdots, d-1). \tag{3.2}$$

Field theories can be constructed on this space (space-time). These noncommutative field theories have a few distinct properties.

A. Perturbative

i) Noncommutative field theories are non-local field theories in the sense that higher derivative terms are contained in the Lagrangian. In spite of this apparently undesirable nature, the theories are possibly renormalizable. 4D Wess-Zumino model is an example of consistent renormalizable noncommutative field theories [17].

ii) There appear new poles in amplitudes at zero momentum originating from UV divergences of loop integrals. This phenomenon is called UV/IR-mixing [6]. At higher orders in perturbation, it is difficult to separate these divergences from the usual UV divergences.

B. Non-perturbative

Many of noncommutative field theories have solitons and instantons. There are two kinds of solitons (instantons), those which exist in the commutative counterparts and those which do not exist in the commutative counterparts. We give two examples of the second kind.

i) GMS solitons Noncommutative scalar field theories in $d = 2+1$ have solitons—GMS solitons [18]. There cannot exist solitons in commutative scalar field theories in $d \geq 3$, as known as Derrick's theorem. This theorem is evaded in the noncommutative case because of higher derivative terms in the interaction Lagrangian.

ii) Instantons Instantons can be constructed in noncommutative non-Abelian gauge theories in parallel with those in commutative gauge theories [19]. There are some differences. There exist instantons even in $U(1)$ gauge theory in the noncommutative case. Singularities in the moduli space of instantons in commutative gauge theories are resolved in the noncommutative gauge theories [19, 20]. This will be discussed in sec.5.

3.2 Moyal product versus Weyl order product

Non-commutaive field theories can be formulated in two different ways.

i) The space (space-time) coordinates $\hat{x}^\mu$ are treated as non-commuting operators. Fields $\hat{\phi}(\hat{x}^\mu)$ are then non-commuting operators, and their product is an operator product.

ii) The coordinates x^μ are treated as real numbers. Fields $\phi(x^\mu)$ are functions but their product is defined as Moyal product. See (3.5) below.

The two schemes are equivalent and can be translated from one to the other. The scheme ii) is more appropriate for perturbative computations, and i) for studies of non-perturbative problems.

We summarise the translation rule (Moyal-Weyl correspondence). We do this in the case of scalar field theory in $d = 3$ as an illustration and for later use. We set

$$(t, x, y) = (t, \vec{x}), \tag{3.3}$$

$$[\hat{x}, \hat{y}] = i\theta \quad (\theta > 0). \tag{3.4}$$

The Weyl-Moyel correspondence is as follows.

Space coordinates are operators $\hat{x}, \hat{y}$ $\Longleftrightarrow$ Real numbers x, y

$$[\hat{x}, \hat{y}] = i\theta \qquad\qquad x \star y - y \star x = i\theta$$

Fields are operators. $\Longleftrightarrow$ Fields are functions.

$$\hat{\phi}(\hat{x}, \hat{y}) = \int \frac{d^2k}{(2\pi)^2} \tilde{\phi}(\vec{k}) e^{-i\vec{k}\cdot\hat{\vec{x}}} \qquad\qquad \phi(x, y) = \int \frac{d^2k}{(2\pi)^2} \tilde{\phi}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

Operator product of fields $\Longleftrightarrow$ $\star$-product of fields

$$\hat{\phi}_1 \, \hat{\phi}_2 \qquad\qquad \phi_1 \star \phi_2$$

Trace of (composite) operators $\Longleftrightarrow$ Volume integral of functions

$$\mathrm{Tr}_H \hat{f}(\hat{x}, \hat{y}) \qquad\qquad \int d^2x f(x, y)$$

Differentiation $\Longleftrightarrow$ Differentiation

$$\hat{\partial}_x \hat{\phi} = i\theta^{-1}[\hat{y}, \hat{\phi}] \qquad\qquad \partial_x \phi(x, y)$$

In the above formulae, $\star$ is the Moyal product defined as

$$(f \star g)(x, y) = \exp(\frac{1}{2} i\theta(\partial_{x_1}\partial_{y_2} - \partial_{y_1}\partial_{x_2})) f(x_1, y_1) g(x_2, y_2)|_{x_1 = x_2 = x,\, y_1 = y_2 = y}. \tag{3.5}$$

There is a useful formula relating the operator and the function:

$$\phi(x, y) = \int du \mathrm{e}^{-i\theta^{-1}yu} \langle x + \frac{u}{2} | \hat{\phi}(\hat{x}, \hat{y}) | x - \frac{u}{2} \rangle, \tag{3.6}$$

where $\langle x | x' \rangle = \delta(x - x')$.

Note the analogy of the commutation relation $[\hat{x}, \hat{y}] = i\theta$ with the Heisenberg's commutation relation in quantum mechanics,

$$[\hat{x}, \hat{p}] = i\hbar. \tag{3.7}$$

The above formula of differentiation can be understood from this analogy.

3.3 Noncommutative extension of integrable field theories in $d = 2$

Field theories on noncommutative space-*time* apparently have difficulty regarding unitarity and causality [21]. Noncommutative extension of 2D integrable field theories, if constructed, would be useful in clarifying this problem by solving the model explicitly.

Noncommutative extension of the Wess-Zumino-Witten (WZW) model has been constructed [11, 22], and its UV property has been studied at one-loop order. The β-function of the noncommutative $U(n)$ WZW model ($n > 1$) resembles that of the commutative WZW model at this order. The noncommutative $U(1)$ WZW model also has an infrared fixed point. We refer the reader to [11] for the details of the computation.

We have also made an attempt at noncommutative extension of massive integrable models, e.g., sine-Gordon model. It is not difficult to construct infinitely many conservation laws modifying those of the commutative sine-Gordon model. Proof of the integrability of the model will be completed by showing that they are involutive. This task is more difficult and we have not succeeded in this.[†] Integrability of noncommutative extension of sine-Gordon and principal chiral models has been studied in a different approach and the result has been presented at this workshop [23].

4. Noncommutative CP^n model in $d = 3$ and BPS soliton solutions

In commutaive space, 2D CP^n model is known to be integrable.[‡] Finite-action solutions of this model were constructed long time ago [24]. These solutions provide static solutions of the CP^n model in $d = 3$. In the CP^1 model, all static solutions saturate the BPS condition. In the CP^n model with $n \geq 2$, there exist non-BPS solutions [24]. In this section, we review the noncommutative extension of the CP^n model and it's BPS solutions [25, 9].[§]

4.1 Noncommutative CP^n model in $d = 3$ and BPS equation

Two-dimensional noncommutative space is defined by the commutation relation given previously (eq.(3.4)). We will use the complex coordinate $z = (x + iy)/\sqrt{2}$, and write

$$[\hat{z}, \hat{\bar{z}}] = \theta. \tag{4.1}$$

We use the operator formalism and introduce the creation and annihilation operators of the harmonic oscilator,

$$\hat{a} = \sqrt{\theta}\hat{z}, \tag{4.2}$$

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{4.3}$$

[†]Unpublished note of Furuta and Inami.

[‡]For the integrability of this model at quantum level, see the footnote in Sec.1.

[§]Solutions of 2D noncommutative CP^n model have recently been constructed in [26].

The basis of the Fock space is then given by

$$\begin{aligned} \hat{a}|0\rangle &= 0, \\ |n\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle. \end{aligned} \tag{4.4}$$

The differentiation and the integration are as explained in subsec.3.2.

$$\hat{\partial}_z\hat{\phi} \;=\; -\theta^{-1}[\hat{\bar{z}},\hat{\phi}], \quad \hat{\partial}_{\bar{z}}\hat{\phi} \;=\; \theta^{-1}[\hat{z},\hat{\phi}], \tag{4.5}$$

$$\int d^2x\mathcal{O} \to \mathrm{Tr}\hat{\mathcal{O}} = 2\pi\theta\sum_{\mathrm{n}\geq 0}\left\langle \mathrm{n}|\hat{\mathcal{O}}|\mathrm{n}\right\rangle. \tag{4.6}$$

It is tedious to use the hat symbol each time to denote an operator. Hereafter we will omit the hat symbol, and write, e.g., z instead of $\hat{z}$, unless confusion occurs. The hat symbol will be restored in sec.6.

To define the noncommutative CP^n model, we take the $(n+1)$-component vector field, which is an operator in the sense of subsec.3.2,

$$\Phi = {}^t(\phi_1,\phi_2,\ldots,\phi_{n+1}). \tag{4.7}$$

The noncommutative CP^n model is defined by the Lagrangian

$$L = \mathrm{Tr}[D_\mu\Phi^\dagger D^\mu\Phi + \lambda(\Phi^\dagger\Phi - 1)], \tag{4.8}$$

where D_μ is a covariant derivative defined by

$$D_\mu = \partial_\mu\Phi - i\Phi A_\mu, \quad A_\mu \;=\; -i\Phi^\dagger\partial_\mu\Phi, \tag{4.9}$$

and λ is the multiplier field imposing the constraint

$$\Phi^\dagger\Phi = 1. \tag{4.10}$$

This model has the global $SU(n+1)$ symmetry and $U(1)$ gauge symmetry,

$$\Phi(x) \to \Phi(x)g(x), \tag{4.11}$$

where $g(x) \in U(1)$.

The equation of motion reads

$$D_\mu D^\mu\Phi + \Phi(D_\mu\Phi^\dagger D^\mu\Phi) = 0. \tag{4.12}$$

The energy functional is given by

$$E = \mathrm{Tr}(|D_0\Phi|^2 + |D_z\Phi|^2 + |D_{\bar{z}}\Phi|^2). \tag{4.13}$$

We have the Bogomolnyi bound,

$$E \geq \mathrm{Tr}(|D_0\Phi|^2) + 2\pi|Q|, \tag{4.14}$$

where Q is the topological charge and it is given by

$$Q = \frac{1}{2\pi}\mathrm{Tr}(|D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2). \tag{4.15}$$

We look for static solutions satisfying the BPS equation

$$D_{\bar{z}}\Phi = 0 \quad \text{(self-dual solution)}, \tag{4.16}$$

$$D_z\Phi = 0 \quad \text{(anti-self-dual solution)}. \tag{4.17}$$

Self-dual (anti-self dual) solutions solve the equation of motion (4.12). We also have non-BPS solutions which satisfy the equation of motion (4.12) but do not satisfy the BPS equation. We postpone the discussion of the non-BPS solutions to sect.6.

4.2 BPS solitons

The solution of the BPS equation (4.16) can be cast into the form (see for instance [27])

$$\Phi = W(W^\dagger W)^{-1/2}, \tag{4.18}$$

where W is an $(n+1)$-component vector. We assume that $(W^\dagger W)^{1/2}$ is invertible. It is useful to define the projection operator,

$$P = W(W^\dagger W)^{-1}W^\dagger \tag{4.19}$$

with the properties

$$\begin{aligned} P^2 &= P, \quad P^\dagger = P, \\ & PW = W. \end{aligned} \tag{4.20}$$

The Lagrangian and the topological charge are expressed in terms of W as

$$L = \mathrm{Tr}\left[\frac{1}{\sqrt{W^\dagger W}}\partial_\mu W^\dagger(1-P)\partial^\mu W\frac{1}{\sqrt{W^\dagger W}}\right], \tag{4.21}$$

$$Q = \frac{1}{2\pi}\mathrm{Tr}\left[\frac{1}{\sqrt{W^\dagger W}}(\partial_{\bar{z}}W^\dagger(1-P)\partial_z W - \partial_z W^\dagger(1-P)\partial_{\bar{z}}W)\frac{1}{\sqrt{W^\dagger W}}\right]. \tag{4.22}$$

The gauge transformation takes the form

$$W \to Wg(z,\bar{z}), \tag{4.23}$$

where $g(z,\bar{z})$ is an arbitrary function of z and $\bar{z}$ which is assumed to be invertible.

The (self-dual) BPS equation (4.16) becomes

$$D_{\bar{z}}\Phi = (1-P)(\partial_{\bar{z}}W)(W^\dagger W)^{-1/2} = 0. \tag{4.24}$$

This equation is equivalent to

$$\partial_{\bar{z}}W = WV, \tag{4.25}$$

where V is an arbitrary scalar. The general solution of (4.25) is written as

$$W = W_0(z)\Delta(z,\bar{z}), \tag{4.26}$$

where $\Delta(z,\bar{z})$ is a scalar. $W_0(z)$ is an $(n+1)$-vector with all it's components being holomorphic polynomials. The degree k of the polynomials gives the topological charge, $Q = k$.

In the commutative case, we may set $W = {}^t(w,1)$ by a gauge transformation, w being an n-vector whose components are rational functions. We will be mainly concernd with one- and two-soliton solutions of the CP^1 model. In the commutative CP^1 model, they are given by [28, 29]

$$w = \lambda + \frac{\mu}{z-\nu} \qquad \text{(one-soliton solution)}, \tag{4.27}$$

$$w = \alpha + \frac{2\beta z + \gamma}{z^2 + \delta z + \epsilon} \qquad \text{(two-soliton solution)}, \tag{4.28}$$

where $\alpha, \beta, \cdots \in C$ are parameters of the solutions, called *moduli parameters.* We may set the moduli parameters λ and α to zero, using the global $SU(2)$ symmetry.

In the noncommutative case, $W = {}^t(z,\mu)$ and $W' = Wz^{-1} = {}^t(1,\mu z^{-1})$ are not gauge equivalent. This is because z^{-1} is not invertible. Here z^{-1} is defined by

$$z^{-1} \equiv (\bar{z}z)^{-1}\bar{z} \equiv \bar{z}(\bar{z}z+\theta)^{-1}. \tag{4.29}$$

and hence,

$$zz^{-1} = 1, \quad z^{-1}z = 1 - |0\rangle\langle 0|. \tag{4.30}$$

W satisfies the BPS equation but W' does not. One- and two-soliton solutions in noncommutaive space are given by

$$W = \begin{pmatrix} z-\nu \\ \mu \end{pmatrix} \qquad \text{(one-soliton)}, \tag{4.31}$$

$$W = \begin{pmatrix} z^2+\delta z+\epsilon \\ 2\beta z+\gamma \end{pmatrix} \qquad \text{(two-soliton)}. \tag{4.32}$$

5. Scattering of solitons in noncommutative CP^1 model

In the previous section, we have constructed static solutions of the 3D CP^1 model. We can go beyond static solutions and introduce time dependence to soliton solutions. It will allow solitons to move and scatter. The same type of problem already appeared in the case of magnetic monopoles in 4D Yang-Mills theories. Motion of mono-poles at low energies can be dealt with by Manton's prescription [30], introducing time dependence to moduli parameters, and thus reducing the problem to geodesics in the moduli space of mono-pole solutions.

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An interesting issue regarding moduli space, especially from mathematical physics point of view, is the singularities of the instanton/soliton solutions. Short-distance singularities (in the moduli space) appear in commutative 4D Yang-Mills theories. They are shown to be resolved in the noncommutative models [19, 20].

We have investigated the moduli space of the BPS solitons in the noncommutative CP^1 model from these poins of view [9]. In commutative space, the moduli space was investigated for one- and two-soliton solutions in [28]. It was noted that there is a singularity corresponding to the small scale limit of the moduli parameters.

Scattering of solitons in $d = 3$ was studied previously in other types of 3D noncommutative scalar field theories [31, 32, 33].

5.1 Low-energy dynamics and moduli space

At low energies (near the Bogomolnyi bound), it is a good approximation that only the moduli parameters depend on time t. Let us denote the moduli parameters generically by $\alpha(t), \beta(t), \cdots$. The time evolution of the moduli parameters is determined by the action

$$S = \int dtL[\alpha(t), \beta(t), \cdots], \tag{5.1}$$

It amounts to dealing with the kinetic energy term T.

$$\begin{aligned} T &= \mathrm{Tr}\left[\frac{1}{\sqrt{W^\dagger W}}\partial_t W^\dagger(1-P)\partial^t W \frac{1}{\sqrt{W^\dagger W}}\right] \\ &= \frac{1}{2}\mathbf{Tr}(\partial_t P)^2, \end{aligned} \tag{5.2}$$

where **Tr** consists of the trace over the Fock space and that over the 2×2 matrix indices. Other terms in L give the topological charge and thus become a constant term.

To give the metric on the moduli space, we rewrite the kinetic energy T as follows,

$$T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}g_{ab}\frac{d\zeta^a}{dt}\frac{d\zeta^b}{dt}. \tag{5.3}$$

Here ds is a line element of the moduli space $\mathcal{M}$. g_{ab} is the metric on $\mathcal{M}$. The moduli parameters ζ^a are the coordinates on $\mathcal{M}$. Eq. (5.3) means that dynamics of solitons is given by the geodesics in $\mathcal{M}$.

In the commutative CP^n model, the moduli space is known to be a Kähler manifold [34]. We have shown that this is also the case in noncommutive space [9]. To see this, we rewrite the projection operator P as [32]

$$\begin{aligned} P &= \sum_{n,m} |\psi_n\rangle h^{nm}\langle\psi_m|, \\ |\psi_n\rangle &= W|n\rangle, \\ h_{nm} = \langle\psi_n|\psi_m\rangle, &\quad h^{nm} = (h_{nm})^{-1}. \end{aligned} \tag{5.4}$$

$|\psi_n\rangle$ is a holomorphic function of the moduli parameters. One can show that when $|\psi_n\rangle$ is a holomorphic function of the moduli parameters, the moduli space $\mathcal{M}$ is a Kähler manifold with the Kähler potential [32]

$$K = \mathrm{Tr}\ln(h_{nm}) = \mathrm{Tr}\ln(W^\dagger W). \tag{5.5}$$

Hence we write

$$T = \frac{1}{2} g_{\bar{a}b} \frac{d\zeta^{\bar{a}}}{dt}\frac{d\zeta^b}{dt}, \quad g_{\bar{a}b} = \frac{\partial}{\partial\zeta^{\bar{a}}}\frac{\partial}{\partial\zeta^b}K. \tag{5.6}$$

5.2 One-soliton metric

Recall the one-soliton solution,

$$W = \begin{pmatrix} z-\nu \\ \mu \end{pmatrix}. \tag{5.7}$$

ν represents the position of the soliton. $|\mu|$ gives the size of the soliton. Subsitute (5.7) to the kinetic energy (5.2). Then,

$$\begin{aligned} T = \mathrm{Tr}\Bigg[& \frac{1}{\sqrt{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2}}\partial_t \begin{pmatrix} \bar{z}-\bar{\nu} & \bar{\mu} \end{pmatrix} \\ & \times \left\{ 1 - \begin{pmatrix} z-\nu \\ \mu \end{pmatrix} \frac{1}{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2} \begin{pmatrix} \bar{z}-\bar{\nu} & \bar{\mu} \end{pmatrix} \right\} \\ & \times \partial_t \begin{pmatrix} z-\nu \\ \mu \end{pmatrix} \frac{1}{\sqrt{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2}} \Bigg]. \end{aligned} \tag{5.8}$$

We look at the $\dot{\bar{\mu}}\dot{\mu}$ term in T.

$$2\pi\theta\dot{\bar{\mu}}\dot{\mu}\sum_{n\geq 0}\frac{1}{\theta n+|\mu|^2}\left[\frac{\theta n}{\theta n+|\mu|^2}+\frac{|\nu|^2}{\theta(n+1)+|\mu|^2}\right]. \tag{5.9}$$

The sum of the first terms in (5.9) diverges. The low-energy aproximation would fail unless $\dot{\mu}=0$. In the commutaitve case, this is a typical property of the CP^n model. The moduli space is restricted to a lower dimensional submanifold. Non-commutativity does not change this situation. We set μ to a constant.

Then we obtain,

$$T = 2\pi\frac{d\bar{\nu}}{dt}\frac{d\nu}{dt}, \quad \text{or} \quad ds^2 = 4\pi d\bar{\nu}d\nu. \tag{5.10}$$

This equation means that the geodesic is a straight line in the ν plane. Soliton moves straight without changing its size $|\mu|$.

5.3 Two-soliton metric

Recall the BPS two-soliton solution (4.32),

$$W = \begin{pmatrix} z^2+\epsilon \\ 2\beta z+\gamma \end{pmatrix}. \tag{5.11}$$

Here we concider the center-of-mass frame, setting δ to 0 in (4.32). Computing the kinetic energy in the low-energy limit, we find that the contribution of the $\dot{\bar{\beta}}\dot{\beta}$ term diverges in the same way as $\dot{\bar{\mu}}\dot{\mu}$ term in the one-soliton case. In the low-energy approximation, we should set β to a constant. Furthermore, for the sake of simplicity, we consider the case of $\beta = 0$. In this case, $\pm i\epsilon^{1/2}$ represent the locations of the solitons. $|\gamma/\epsilon^{1/2}|$ is the size of the solitons (same size for both solitons).

In the commutative model, the moduli space metric has been calculated by Ward [28]. It is written as

$$ds^2 = \xi R^{-1} dR^2 + \mu dR d\psi + \nu R d\psi^2 + R(\tau d\phi^2 + \sigma d\theta d\phi + \omega d\theta^2), \tag{5.12}$$

where $\gamma = Re^{i\phi}\sin\psi$, $\epsilon = Re^{i\theta}\cos\psi$. $\xi, \mu, \nu, \tau, \sigma$ and ω are functions of ψ only. The explicit forms of these functions are

$$\begin{aligned} &\xi = \frac{1}{2}E\ , \quad \mu = \tan\psi(K-E)\ , \quad \nu = K - \frac{1}{2}E\ , \\ &\tau = \nu\sin^2\psi\ , \quad \sigma = -\mu\sin\psi\cos\psi\ , \omega = \xi\cos^2\psi\ , \end{aligned} \tag{5.13}$$

where $K = K(\cos\psi), E = E(\cos\psi)$ are the complete elliptic functions of the first and second kind, respectively. One can show that at $(\epsilon = 0, \gamma \neq 0)$ or $(\epsilon \neq 0, \gamma = 0)$, there are no singularities by suitable coordinate redefinitions. Since the metric (5.12) is homogeneous in R, there is a singularity at $(\epsilon, \gamma) = (0, 0)$, otherwise the whole space must be flat. We will now see the disappearance of this singularity in the noncommutative model.

We have obtained the moduli space metric in the noncommutative model,

$$g_{\bar{\gamma}\gamma} = \mathrm{Tr}\left[\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon)}\left(1 - \frac{\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon)}\right)\right], \tag{5.14}$$

$$g_{\bar{\epsilon}\gamma} = -\mathrm{Tr}\left[\bar{\gamma}(z^2+\epsilon)\frac{1}{[\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon)]^2}\right], \tag{5.15}$$

$$g_{\bar{\gamma}\epsilon} = -\mathrm{Tr}\left[\gamma\frac{1}{[\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon)]^2}(\bar{z}^2+\bar{\epsilon})\right], \tag{5.16}$$

$$g_{\bar{\epsilon}\epsilon} = \mathrm{Tr}\left[\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon)}\frac{\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2+\bar{\epsilon})(z^2+\epsilon) + 4\theta\bar{z}z + 2\theta^2}\right]. \tag{5.17}$$

It is difficult to go further to compute the Tr for arbitrary values of θ. We can compute the Tr and obtain the moduli space metric explicitly in the two limiting cases: large values of θ and small values of θ.

(1) $|\gamma|, |\epsilon| \ll \theta$ case.

We obtain

$$ds^2 = \frac{2\pi}{\theta}\left(d\bar{\gamma}d\gamma + \frac{2}{3}d\bar{\epsilon}d\epsilon\right) + O(\theta^{-2}). \tag{5.18}$$

Results are summarized as follows.

(i) The moduli metric turns out to be flat.

(ii) Suppose that initialy, $\epsilon > 0$, $\dot{\epsilon} < 0$. The soliton locations $\pm i\epsilon^{1/2}$ move from imaginary axis to real axis, as shown schematically in fig.5.3. Note that this does not mean right-angle scattering since the expression (5.18) holds only when the soliton locations are close, i.e., $|\epsilon| \ll \theta$.

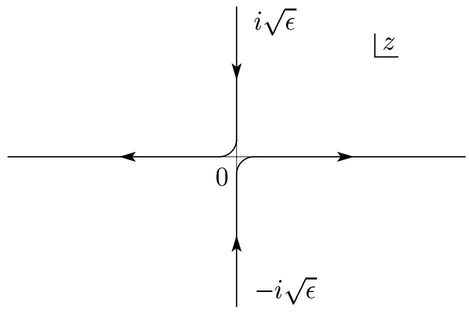

Figure 2: Scattering of solitons: Two solitons going in from the imaginary axis $\pm i\sqrt{\epsilon}\,(\epsilon > 0)$ come out to the real axis $\pm\sqrt{-\epsilon}\,(\epsilon < 0)$.

(iii) The singularity which exists in the commutative model at $(\epsilon, \gamma) = (0, 0)$ is resolved in the noncommutaive model.

(2) $|\gamma|, |\epsilon| \gg \theta$ case.

In this case, the metric can be calculated using the $\star$-product formalism rather than the operator formalism. We find

$$(g_{ab})_{\mathrm{NC}} = (g_{ab})_{\mathrm{C}} + O(\theta), \tag{5.19}$$

where a, b represent the moduli parameters. NC and C denote noncommutative and commutative cases respectively. From (5.19) one sees that the metric has the smooth commutative limit $\theta \to 0$ with moduli parameters fixed.

6. Non-BPS solitons in noncommutative CP^1 model

We have so far considered BPS solutions. In general there can be non-BPS solutions, i.e., solutions of the equation of motion which do not satisfy the BPS equations. It is known that there exists no *static* non-BPS solutions in the commutative 3D CP^1 moel [24]. We will show in this section that there exist static non-BPS solutions in the noncommutative 3D CP^1 model. We present solutions corresponding to soliton anti-soliton configurations and solutions of other types derived in [10].

6.1 Preliminary

We rewrite some of the formulae given previously. In terms of the projection operator P introduced in sec.4.2, the Lagrangian is written as

$$L = \frac{1}{2}\mathbf{Tr}(\partial_t P \partial_t P - 2\hat{\partial}_{\bar{z}} P \hat{\partial}_z P), \tag{6.1}$$

where **Tr** is defined in sec.5.1. The equation of motion is

$$[\partial_t^2 P - 2\hat{\partial}_{\bar{z}}\hat{\partial}_z P, P] = 0. \tag{6.2}$$

We deal with static configurations. Using the differentiation formula (4.5), eq. (6.2) is written as

$$[[\hat{z}, [\hat{\bar{z}}, P]], P] = 0, \tag{6.3}$$

The BPS equations (4.16), (4.17) are now written as

$$(1-P)\hat{z}P = 0 \quad \text{(self-dual solution)}, \tag{6.4}$$

$$(1-P)\hat{\bar{z}}P = 0 \quad \text{(anti-self-dual solution)}. \tag{6.5}$$

Note that solutions of the BPS equations automatically satisfy the equation of motion (6.3).

6.2 Soliton-antisoliton solution and its stability

As a candidate for non-BPS solutions, we consider the 2×2 projection operator of the form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{6.6}$$

We take P_1 and P_2 which are self-dual and anti-self-dual solutions, respectively.

$$(1-P_1)\hat{z}P_1 = 0, \tag{6.7}$$

$$(1-P_2)\hat{\bar{z}}P_2 = 0. \tag{6.8}$$

P satisfies the equation of motion (6.3), since P_1 and P_2 do. P does not satisfy the self-dual equation (6.4) (anti-self-dual equation (6.5)), since P_2 does not (P_1 does not). Hence, P is a desired non-BPS solution.

Non-trivial solutions of (6.7) are known as GMS solitons and are given in [32, 33, 35]. Since $1-P_2$ satisfies the self-dual equation, solutions of (6.8) are also given by GMS solitons. Therefore, P takes the following form

$$P = \begin{pmatrix} \sum_{i,j=1}^{r} |z^i\rangle h_{ij}^{-1} \langle z^j| & 0 \\ 0 & 1 - \sum_{k,l=1}^{s} |\tilde{z}^k\rangle \tilde{h}_{kl}^{-1} \langle \tilde{z}^l| \end{pmatrix}, \tag{6.9}$$

where

$$|z^i\rangle = e^{\theta^{-1}(z^i\hat{\bar{z}} - \bar{z}^i\hat{z})}|0\rangle, \tag{6.10}$$

$$h^{ij} = \langle z^i|z^j\rangle, \quad h^{-1}_{ij}h^{jk} = \delta_i{}^k. \tag{6.11}$$

A natural interpretation of the solution (6.9) is that it represents a soliton-antisoliton configuration with $Q = r - s$ and $E = 2\pi(r+s)$, where r and s are the numbers of solitons and anti-solitons, respectively. Then, z^i $(i = 1, \dots, r)$ and $\tilde{z}^k$ $(k = 1, \dots, s)$ are the positions of solitons and anti-solitons, respectively.

We analyze the stability of the solution of one soliton-antisoliton pair

$$P = \begin{pmatrix} |z\rangle\langle z| & 0 \\ 0 & 1 - |0\rangle\langle 0| \end{pmatrix}. \tag{6.12}$$

We can find a path which connects this solution to the vacuum solution

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.13}$$

One way of parametrizing the path is given by

$$P_\phi = \begin{pmatrix} \sin^2\phi|z\rangle\langle z| & \sin\phi\cos\phi|z\rangle\langle 0| \\ \sin\phi\cos\phi|0\rangle\langle z| & 1 - \sin^2\phi|0\rangle\langle 0| \end{pmatrix}, \quad \phi \in \left[0, \frac{\pi}{2}\right]. \tag{6.14}$$

The energy of this configuration is

$$E = 4\pi\sin^2\phi\left(1 + \frac{\bar{z}z}{\theta}\cos^2\phi\right). \tag{6.15}$$

It is easy to see that the stability of the solution (6.12) depends on the separation $|z|$ between the soliton and the anti-soliton. When $\bar{z}z < \theta$ the energy (6.15) has a local maximum at $\phi = \frac{\pi}{2}$ and decreases monotonically to zero at $\phi = 0$. In this case the solution (6.12) is unstable and the soliton-antisoliton pair annihilates. When $\bar{z}z > \theta$ the energy (6.15) has a local minimum at $\phi = \frac{\pi}{2}$ and therefore the solution (6.12) is metastable in this parameter space. We do not know whether the solution is unstable under fluctuations in other directions.

6.3 Time-dependent solution

Time-dependent solutions can be obtained by a boost accompanied by rescaling of the noncommutative parameter because the Lorentz symmetry is explicitly broken by the noncommutativity [36]. For the solution of the diagonal form (6.6), P_1 and P_2 can be boosted with arbitrary velocities v_1 and v_2. Boosted solutions take the same form as (6.9) but the coordinates $\hat{z}$ and $\hat{\bar{z}}$ are replaced by the boosted coordinates $\hat{z}_a$ and $\hat{\bar{z}}_a$ $(a = 1, 2)$ which obey the commutation relation

$$[\hat{z}_a, \hat{\bar{z}}_a] = \theta_a, \quad \theta_a = \frac{\theta}{\sqrt{1 - v_a^2}}, \quad a = 1, 2. \tag{6.16}$$

Note that time does not commute with spatial coordinates due to the boost but static solutions are the same as those with time being commutative. The solutions constructed in this way do not represent time-dependent multi-(anti-)soliton configurations. For, all (anti-) soliton peaks of our solutions move with a common velocity, whereas in time-dependent solutions (anti-)soliton peaks should exhibit relative motion.

6.4 Non-BPS solutions of other types

We can construct other non-BPS solutions of the form (6.6). For example, we construct the non-BPS solution

$$P = \begin{pmatrix} |n\rangle\langle n| & 0 \\ 0 & 1 \end{pmatrix}, \quad n > 0. \tag{6.17}$$

This configuration has the topological charge $Q = 1$ and the energy $E = 2\pi(2n+1)$. We do not know whether the solution (6.17) can be interpreted as a soliton-antisoliton configuration.

As mentioned in section 4.2, $W' = {}^t(\mu\hat{z}^{-1}, 1)$ is not a BPS solution. Moreover, W' is not a solution of the equation of motion. We can construct a solution by adding the correction to the projection operator $P' = W'(W'^\dagger W')^{-1}W'^\dagger$. Consider the projection operator

$$P = P' + \frac{1}{\mu^2+\theta}\begin{pmatrix} \theta|1\rangle\langle 1| & -\mu\sqrt{\theta}|1\rangle\langle 0| \\ -\mu\sqrt{\theta}|0\rangle\langle 1| & \mu^2|0\rangle\langle 0| \end{pmatrix}. \tag{6.18}$$

We have shown that P is a non-BPS solution which has the topological charge $Q = 1$ and the energy $E = 6\pi$. The parameter μ is related to the size of the solution. In the limit of $\mu \to 0$, (6.18) reduces to

$$P = \begin{pmatrix} |1\rangle\langle 1| & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.19}$$

This corresponds to (6.17) with $n = 1$. On the other hand, in the limit of $\mu \to \infty$, (6.18) reduces to

$$P = \begin{pmatrix} 1 - |0\rangle\langle 0| & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix}. \tag{6.20}$$

This corresponds to the solution representing a soliton-antisoliton pair sitting at the origin. We can interpret the non-BPS solution (6.18) as the configuration which contains a soliton of the size μ and a small soliton-antisoliton pair. In the large μ limit the soliton spreads over the space and disappears, and hence only the soliton-antisoliton pair exists.

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