PROCEEDINGS

Workshop on Integrable Theories, Solitons and Duality

Bogomolny Yang-Mills-Higgs solutions in 2+1 anti-de-Sitter spac

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Abstract: In this paper solutions (first obtained in [1]) of the Bogomolny Yang-Mills-Higgs equations in (2+1) anti-de Sitter space which are integrable are presented using analytical methods. In particular, families of soliton solutions have been constructed explicitly and their dynamics has been investigated in some detail.

1. Introduction

In this paper, we consider an integrable system [2] which is related to hyperbolic monopoles. Recall that the the monopole equations on hyperbolic space $\mathbb{H}^3$ are integrable [3] and that they turn out to be easier to study than the Euclidean (see, for example, [4]). The model we are going to investigate follows from replacing the positive defined space $\mathbb{H}^3$ of the hyperbolic monopole equations by a Lorentzian version, ie the anti-de Sitter space. In recent years, the n-dimensional anti-de Sitter spacetime has been of continuing interest since it is a possible vacuum of M-theory and a source of simple examples studying methods and spacetime concepts both on classical and quantum level. It also arises as the natural ground state of gauged supergravity theories when quantized [5].

The Bogomolny version of Yang-Mills-Higgs equations for Yang-Mills-Higgs fields on a three-dimensional Riemannian manifold ($\mathcal{M}$) with gauge group $SU(2)$ have the form

$$D_i\Phi = \frac{1}{2\sqrt{|g|}} g_{ij}\, \epsilon^{jkl} F_{kl}. \tag{1.1}$$

Here A_k, for $k = 0, 1, 2$, is the $su(2)$-valued gauge potential, with field strength $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and $\Phi = \Phi(x^\mu)$ is the $su(2)$-valued Higgs field; while $x^\mu = (x^0, x^1, x^2)$ represent the local coordinates on M. The action of the covariant derivative $D_i = \partial_i + A_i$ on Φ is: $D_i\Phi = \partial_i\Phi + [A_i, \Phi]$. Equation (1.1) is integrable in the sense that a Lax pair

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exists for constant curvature. In particular, the solutions of (1.1) correspond to Euclidean or hyperbolic BPS monopoles when $(\mathcal{M}, g)$ is Euclidean $\mathbb{R}^3$ or hyperbolic $\mathbb{H}^3$ space.

By definition the (2+1)-dimensional anti-de Sitter space is the universal covering space of the hyperboloid $\mathcal{H}$ satisfied by the equation

$$U^2 + V^2 - X^2 - Y^2 = 1 \tag{1.2}$$

with metric given by

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{1.3}$$

By parametrizing the hyperboloid $\mathcal{H}$ by

$$\begin{aligned} U &= \sec\rho\cos\theta \\ V &= \sec\rho\sin\theta \\ X &= \tan\rho\cos\phi \\ Y &= \tan\rho\sin\phi \end{aligned} \tag{1.4}$$

for $\rho \in [0, \pi/2)$, the corresponding metric takes the form

$$ds^2 = \sec^2\rho\left(-d\theta^2 + d\rho^2 + \sin^2\rho\, d\phi^2\right). \tag{1.5}$$

The spacetime contains closed timelike curves, due to the periodicity of θ (for more details, see Ref. [6]). In fact, anti-de Sitter space (as a manifold) is the product of an open spatial disc with θ and constant curvature equal to minus six in units of the cosmological constant (or vacuum energy) as the normalization of (1.1) requires. The variables (ρ, ϕ) correspond to polar coordinates and $\theta \in R$ being the time. Null spacelike infinity $\mathcal{I}$ consists of the timelike cylinder $\rho = \pi/2$ and this surface is never reached by timelike geodesics.

If the Poincaré coordinates (r, x, t) for $r > 0$ are defined as

$$\begin{aligned} r &= \frac{1}{U+X} \\ x &= \frac{Y}{U+X} \\ t &= \frac{-V}{U+X} \end{aligned} \tag{1.6}$$

the metric simplifies to the following form

$$ds^2 = r^{-2}(-dt^2 + dr^2 + dx^2). \tag{1.7}$$

Note that, the Poincaré coordinates cover a small part of anti-de Sitter space, ie that corresponding to half of the hyperboloid $\mathcal{H}$ for $U + X > 0$. The surface $r = 0$ is part of infinity $\mathcal{I}$.

Hitchin [7] shows that the minitwistor space corresponding to Poincaré space (1.7) is $CP^1 \times CP^1$ and can be visualized as a quadric Q in CP^3; while the points of spacetime correspond to certain plane sections (conics) of Q with space CP^3. The relevant conics which have to be real and nondegenerate, are given by the expression [2]

$$\omega = v - r^2 (\mu - u)^{-1} \tag{1.8}$$

where (ω, μ) are standard coordinates on the two CP^1 factors of Q, while $u = x + t$ and $v = x - t$. Note that the Poincaré coordinates (r, x, t) cover all of the space of these conics (which is the top half of RP^3) expect for a set of measure zero. In order to see the correspondence between spacetime and twistor space Q one needs to substitute (1.6) into (1.8).

Consider the set of linear equations

$$\begin{aligned}\left[rD_r - 2(\lambda - u)D_u - \Phi\right]\Psi &= 0\\ \left[2D_v + \frac{\lambda - u}{r}D_r - \frac{\lambda - u}{r^2}\Phi\right]\Psi &= 0.\end{aligned} \tag{1.9}$$

Here $\lambda \in \mathbb{C}$ and (r, u, v) are the Poincaré coordinates. The gauge fields (Φ, A_r, A_u, A_v) are 2×2 trace-free matrices depending only on (r, u, v) and $\Psi(\lambda, r, u, t)$ is a unimodular 2×2 matrix function satisfying the reality condition $\Psi(\lambda)\Psi(\bar{\lambda})^\dagger = I$ (where $\dagger$ denotes the complex conjugate transpose). The system (1.9) is overdetermined and in order for a solution Ψ to exist the following integrability conditions need to be satisfied

$$\begin{aligned}D_u\Phi &= rF_{ur}\\ D_v\Phi &= -rF_{vr}\\ D_r\Phi &= -2rF_{uv}.\end{aligned} \tag{1.10}$$

The above equations are consistent with the ones obtained from (1.1) using the Poincaré coordinates.

The gauge and Higgs fields in terms of the function Ψ can be obtained from the Lax pair (1.9). Note that, as $\lambda \to \infty$ the function Ψ goes to the identity matrix which implies that

$$A_u = 0, \qquad A_r = \frac{1}{r}\,\Phi. \tag{1.11}$$

On the other hand, for $\lambda = 0$ and using (1.11) the rest of the gauge fields are defined as

$$\begin{aligned}\Phi &= -\frac{r}{2}\,J_rJ^{-1} - u\,J_uJ^{-1}\\ A_v &= \frac{u}{2r}\,J_rJ^{-1} - J_vJ^{-1}\end{aligned} \tag{1.12}$$

where $J(r, u, v) \doteq \Psi(\lambda = 0, r, u, v)$. Note that, in this case, the first equation of the system (1.10) is automatically satisfied (due to the specific gauge choice).

Recently, Ward [2] has shown that holomorphic vector bundles V over Q determine multi-soliton solutions of (1.10) in anti-de Sitter space via the usual Penrose transform. This way a five-parameter family of soliton solutions can be obtained, in a similar way as for flat spacetime [8]. Later, more solutions of equations (1.10) were obtained by Zhou [9, 10] using Darboux transformations with constant and variable spectral parameters. In what follows, we use the Riemann problem with zeros to construct families of soliton solutions and observe the occurrence of different types of scattering behaviour. More precisely, we present families of multi-soliton solutions with trivial and nontrivial scattering.

2. Construction of Solitons

Using the standard method of Riemann problem with zeros in order to construct the multi-soliton solution, we assume [8] that the function Ψ has the simple form in λ, ie

$$\Psi = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k} \tag{2.1}$$

where M_k are 2×2 matrices independent of λ and n is the soliton number. The components of the matrix M_k are given in terms of a rational function $f_k(\omega_k) = a_k\,\omega_k + c_k$ of the complex variable: $\omega_k = v - r^2\,(\mu_k - u)^{-1}$. Here a_k, c_k and μ_k are complex constants which determine the size, position and velocity of the k-th solitons. *Remark*: The rational dependence of the solutions Ψ follows (directly) when the inverse spectral theory is considered. In [11] (for the flat spacetime), it was shown by solving the Cauchy problem that the spectral data is a function of a parameter similar to (1.8).

The matrix M_k has the form

$$M_k = \sum_{l=1}^{n} (\Gamma^{-1})^{kl} \bar{m}_a^l m_b^k \tag{2.2}$$

with Γ^{-1} the inverse of

$$\Gamma^{kl} = \sum_{a=1}^{2} (\bar{\mu}_k - \mu_l)^{-1} \bar{m}_a^k m_a^l \tag{2.3}$$

and m_a^k holomorphic functions of ω_k, of the form $m_a^k = (m_1^k, m_2^k) = (1, f_k)$. The Yang-Mills-Higgs fields (Φ, A_r, A_v, A_u) can then be read off from (1.11-1.12) and they automatically satisfy (1.10). The corresponding solitons are spatially localized since $\Phi \to 0$ at spatial infinity (ie at $r = 0$). In this case, the solitons scatter in a trivial way, that is they pass each other without any phase shift or change of velocity/shape.

3. Scattering Solutions

The Riemann problem with zeros approach assumes that the parameters μ_k are distinct and also $\bar{\mu}_k \neq \mu_l$ for all (k, l). However, examples of generalizations of these constructions can be obtained either involving higher order poles in μ_k or when $\bar{\mu}_k = \mu_l$. When this procedure has been applied in flat spacetime the corresponding solitons scatter in a nontrivial way. In particular, as it has been shown in [12, 13], in head-on collisions of N indistinguishable solitons the scattering angle of the emerging solitons relative to the incoming ones is π/N. As a result, it would be of great interest to see the scattering behaviour of the corresponding solitons in the anti-de Sitter spacetime.

○ Firstly, let us look at an example in which the function Ψ has a double pole in λ and no others. In this case, Ψ has the form

$$\Psi = I + \sum_{k=1}^{2} \frac{R_k}{(\lambda - \mu)^k} \tag{3.1}$$

where R_k are 2×2 matrices independent of λ. Then, as in flat spacetime [12], Ψ corresponds to a solution of (1.9) if and only if it factorizes as

$$\Psi(\lambda)=\left(1-\frac{\bar{\mu}-\mu}{(\lambda-\mu)}\frac{q^\dagger\otimes q}{|q|^2}\right)\left(1-\frac{\bar{\mu}-\mu}{(\lambda-\mu)}\frac{p^\dagger\otimes p}{|p|^2}\right) \tag{3.2}$$

for some two vectors q and p. One way to obtain the form of these vectors is by taking the formula (2.1) for $n=2$ and setting $\mu_1=\mu+\epsilon$, $\mu_2=\mu-\epsilon$, $f_1(\omega_1)=f(\omega_1)+\epsilon h(\omega_1)$, $f_2(\omega_2)=f(\omega_2)-\epsilon h(\omega_2)$, with f and h being rational functions of one variable. In the limit $\epsilon\to0$ the two vectors q and p can be obtained and are of the form:

$$\begin{aligned} q &= (1+|f|^2)(1,f)+(\bar{\mu}-\mu)\left(\frac{r^2 f'}{(\mu-u)^2}+h\right)(\bar{f},-1)\\ p &= (1,f). \end{aligned} \tag{3.3}$$

In this case, the constraint $f_2(\omega_2)-f_1(\omega_1)\to0$ as $\epsilon\to0$ has to be imposed in order for the resulting solution Ψ to be smooth for all (r,u,v), which is true due to (1.8). Note that the solution depends on the parameter μ and on the two arbitrary functions f and h.

Another way to obtain the aforementioned solutions is by using the Uhlenbeck construction [14]; ie by assuming that the function Ψ is a product of projectors which satisfy first-order partial differential equations and can easily be solved [15].

In order to illustrate the above family of solutions, two simple cases are going to be examined, by giving specific values to the parameters μ, $f(\omega)$ and $h(\omega)$.

(i) Let us study the simple case, where $\mu=i$, $f(\omega)=\omega$ and $h(\omega)=0$. Then, the gauge invariant Higgs density $-\mathrm{tr}\Phi^2$ simplifies to

$$-\mathrm{tr}\Phi^2=32r^2\frac{[(r^2+x^2-t^2+1)^2+4t^2][(r^2+x^2-t^2-1)^2+4x^2]}{\{[(r^2+x^2-t^2)^2+1+2t^2+2x^2]^2+4r^4\}^2}, \tag{3.4}$$

which is time reversible. The time-dependent solution is a traveling soliton configuration which for negative t, goes towards spatial infinity $(r=0)$; approaches it at $t=0$ and then bounces back at positive t. During this period the soliton configuration deforms.

(ii) Next, we investigate the solution which corresponds to a nontrivial scattering, at least in the flat spacetime; for $\mu=i$, $f(\omega)=\omega$ and $h(\omega)=\omega^4$. The picture consists of two solitons with nontrivial scattering since, for large (negative) t, the $-\mathrm{tr}\Phi^2$ is peaked at two points which changes to a lump at $t=0$ and then two solitons emerge, for large (positive) t, with the small one been shifted to the left.

This method can be extended to derive solutions which correspond to the case where the function Ψ has a higher order pole in λ (and no others). Then, Ψ can be written as a product of three (or more) factors with three (or more) arbitrary vectors (for more details, see [13]).

○ Secondly, let us construct a large family of solutions which correspond to the case where $\bar{\mu}_k=\mu_l$. One way of proceeding is to take the solution (2.1) with $n=2$, put $\mu_1=\mu+\epsilon$, $\mu_2=\bar{\mu}-\epsilon$ and take the limit $\epsilon\to0$. In order for the resulting Ψ to be smooth it is necessary to take $f_1(\omega_1)=f(\omega_1)$, $f_2(\omega_2)=-1/f(\omega_2)-\epsilon h(\omega_2)$, where f and h are rational functions of one variable. On taking the limit we obtain a solution Ψ of the form

$$\Psi = I + \frac{n^1 \otimes m^1}{\lambda - \mu} + \frac{n^2 \otimes m^2}{\lambda - \bar{\mu}} \tag{3.5}$$

where n^k, m^k for $k = 1, 2$ are complex valued two vector functions of the form

$$m^1 = (1, f), \qquad m^2 = (-\bar{f}, 1)$$

$$\begin{pmatrix} n^1 \\ n^2 \end{pmatrix} = \frac{2(\mu - \bar{\mu})}{4(1+|f|^2)^2 - (\mu-\bar{\mu})^2|w|^2} \begin{pmatrix} 2(1+|f|^2) & -(\mu-\bar{\mu})\bar{w} \\ (\mu-\bar{\mu})w & -2(1+|f|^2) \end{pmatrix} \begin{pmatrix} m^{1\dagger} \\ m^{2\dagger} \end{pmatrix} \tag{3.6}$$

with

$$w \equiv \frac{2r^2}{(\mu - u)^2} f' + \bar{h} f^2. \tag{3.7}$$

So we generate a solution which depends on the parameter μ and the two arbitrary rational functions $f = f(\omega)$ and $h = h(\bar{\omega})$.

For the choice: $\mu = i$, $f = \omega$, $h = \bar{\omega}$ the configuration consists of two solitons with nontrivial scattering behaviour. Again, the quantity $-\mathrm{tr}\Phi^2$ is peaked at two points, for (negative) t, which are still distinct at $t = 0$ and then two shifted (compared to the initial ones at $t = -3$) solitons emerge, for (positive) t. Throughout the time-evolution their sizes change.

Note that, the scattering solutions belong to a large family since f and h can be taken to be any rational functions of ω. For further details and snapshots of the soliton scattering look at [1].

4. Conclusions

Currently a great deal of attention has been focused on anti-de Sitter spacetimes since they may occur in black hole and p-branes. For the case of Yang-Mills theory with $\mathcal{N} = 4$ supersymmetries and a large number of colours it has been conjectured that gauge strings are the same as the fundamental strings but moving in a particular curved space: the product of five-dimensional anti-de Sitter space and a five sphere [16]. Then, using Poincaré coordinates the anti-de Sitter solutions play the role of classical sources for the boundary field correlators, as shown in [17]; while extensions of the corresponding statements can be applied to gravity theories, like the black holes which arise in anti-de Sitter backgrounds.

In this paper, we illustrate the construction of time-dependent solutions related to hyperbolic monopoles. In particular, families of solutions of the Bogomolny Yang-Mills-Higgs equations in the (2+1)-dimensional anti-de Sitter space have been constructed and their dynamics has been in studied in some detail. As a result, it would be interesting to understand the role of higher poles in algebraic-geometry approach like twistor theory (for example, the function Ψ (3.2) correspond to $n = 2$ bundles), and also to investigate the construction of the corresponding solutions and their dynamics in de Sitter space. Finally, it would be interesting to extend our construction in higher dimensional gauged theories and investigate the scattering behaviour of the corresponding classical solutions and, also, to consider and study its noncommutative version (see, for example, Ref. [18]).

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