Soliton Equations by the Noncommutative Zero-Curvature Formulation

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ABSTRACT: Defining the noncommutative zero-curvature equation, we show that many soliton equations can be written in this form. We demonstrate this by showing that various soliton equations are derived from this equation. The derived soliton equations differ according to different choices of manifolds in the reduction of the noncommutative zero-curvature equation.

The Moyal quantization is known to give an alternative to the quantization. Lately people are interested in noncommutative space-time, which is also formulated in the same way as the Moyal quantization[1]. The Moyal quantization expresses quantum theory not by operators but by functions of the phase space. The purpose of this note is to show that the noncommutative zero-curvature equation, which is defined by using the $\star$ product, can be an alternative to the zero-curvature equation of the matrix-valued potentials.

The soliton equations can be formulated in various ways, and one of which is the AKNS formulation[3, 4]. This is regarded as the geometrical zero-curvature equation. This is given by

$$\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu] = 0, \tag{1}$$

where $A_\mu = A_\mu(x_0, x_1; \zeta), (\mu = 0, 1)$ are the Lie-algebra valued matrices. These potentials include a parameter ζ, and the specific expansion in terms of the parameter yields a corresponding soliton equation.

We define the $\star$ product by

$$f \star g = \exp\left[\kappa\left(\frac{\partial}{\partial x}\frac{\partial}{\partial \tilde{p}} - \frac{\partial}{\partial p}\frac{\partial}{\partial \tilde{x}}\right)\right] f(\mathbf{x})g(\tilde{\mathbf{x}})|_{\mathbf{x}=\tilde{\mathbf{x}}}, \tag{2}$$

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where $\mathbf{x} = (x, p)$. Here, we denote a parameter by κ instead of $i\hbar/2$ in the case of the Moyal quantization. The implication of the parameter will be given below. The Moyal bracket is defined by

$$\{f, g\} = \frac{f \star g - g \star f}{2\kappa}, \tag{3}$$

which becomes the Poisson bracket of $f(x, p)$ and $g(x, p)$ as κ goes to zero. The variables x and p do not need to be the phase space variables in R^2. We regard them as the variables of a 2-dimensional manifold. When we specify the manifold to be torus, $x + 2\pi \equiv x$ and $q + 2\pi \equiv q$, the functions on the torus are given by $f(x, p) = e^{i(mx+np)}$ and $g(x, p) = e^{i(m'x+n'p)}$ with integers m, n, m' and n'. In this case, the above bracket is known to yield the trigonometric algebra, which can be identified as the $su(\infty)$ algebra[2]. This motivates us to rewrite the matrix form equation (1) to the $\star$ product form equation. We replace the commutation relation in Eq.(1) by the Moyal bracket to obtain

$$\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + \{A_\mu, A_\nu\} = 0, \tag{4}$$

where $A_\mu = A_\mu(x_0, x_1; x, p)$ are not matrices but functions of 4 variables.

We study the soliton equations derived from this "noncommutative zero-curvature equation". In order to obtain the 2-dimensional soliton equations, we reduce the 4 dimensions by one as a first step by the following assumptions:

In the time-dependent case, $A_\mu = A_\mu(t; x, p)$, Eq.(4) reads,

$$\frac{\partial A_1}{\partial t} + \{A_0, A_1\} = 0, \tag{5}$$

where $t = x_0$. In the space-dependent case, $A_\mu = A_\mu(s; x, p)$, Eq.(4) reads,

$$\frac{\partial A_0}{\partial s} - \{A_0, A_1\} = 0, \tag{6}$$

where $s = x_1$.

Next we reduce one more dimension to get the 2-dimensional soliton equations. In the case that a manifold is compact, we should adopt the periodic variables on the manifold like in the case of the torus mentioned above. Contrastingly, if a manifold is euclidean, the functions on the manifold is expanded in terms of power series of the variables with infinite ranges.

We first discuss the power series expansion of $A_\mu(t; x, p)$ of the type

$$A_0(t; x, p) = \sum_n p^n a_k(t; x), \tag{7}$$

$$A_1(t; x, p) = \sum_k p^n b_k(t; x), \tag{8}$$

then the KdV equation is shown to fall into this type.

A simple example of the expansions of this type is given by

$$A_0(t; x, p) = p^3 + p^2 g_2(t; x) + p g_1(t; x) + g_0(t; x), \tag{9}$$

$$A_1(t; x, p) = p^2 + u(t; x). \tag{10}$$

Substituting these expansions into Eq.(5), we get the KdV equation:

$$u_t = \frac{3}{2}u'u + \kappa^2 u'''. \tag{11}$$

Here, we have used the relations that g_0, g_1, and g_2 are expressed in terms of u and its derivatives.

In more general expansions

$$A_1(t;x,p) = p^2 + u(t;x), \tag{12}$$

$$A_0(t;x,p) = \sum_{k=0}^{2N+1} p^k g_k(t;x), \tag{13}$$

the KdV hierarchy equations can be obtained by the following way. Substitution of the above expansions into Eq.(5) yields

$$u_t = \sum_{m=1}^{N} \kappa^{2m} u^{(2m+1)} g_{2m+1}, \tag{14}$$

$$\frac{\partial g_{k-1}}{\partial x} = \frac{1}{2}\sum_{m=0}^{[\frac{2N-k}{2}]} \kappa^{(2m)} \binom{k+2m+1}{2m+1} u^{(2m+1)} g_{k+2m+1}, (k = 1, 2, \cdots, 2N) \tag{15}$$

$$\frac{\partial g_{2N}}{\partial x} = 0. \tag{16}$$

These equations determine g_i successively. For even suffices i, we obtain

$$g_{2k} = 0, (k = 1, 2, \cdots, N) \tag{17}$$

and for odd suffices i,

$$\frac{\partial g_{2k-1}}{\partial x} = \frac{1}{2}\sum_{m=0}^{N-k} \kappa^{(2m)} \binom{2k+2m+1}{2m+1} u^{(2m+1)} g_{2(k+m)+1}. \tag{18}$$

These determine g_i as

$$g_{2N+1} := 1 \to g_{2N-1} \to g_{2N-3} \to \cdots g_3 \to g_1. \tag{19}$$

Substituting these g_i into Eq.(14), we obtain a soliton equation for each N. The several examples are in order:

$$u_t = u^{(1)} = \mathcal{K}_1, \tag{20}$$

$$u_t = \frac{3}{2}uu^{(1)} + \kappa^2 u^{(3)} = \mathcal{K}_3, \tag{21}$$

$$u_t = \frac{15}{8}u^2u^{(1)} + 5\kappa^2 u^{(1)}u^{(2)} + \frac{5}{2}\kappa^2 uu^{(3)} + \kappa^4 u^{(5)} = \mathcal{K}_5, \tag{22}$$

$$\begin{aligned} u_t = {} & \frac{35}{16}u^3u^{(1)} + \frac{35}{8}\kappa^2(u^{(1)})^3 + \frac{35}{2}\kappa^2 uu^{(1)}u^{(2)} + \frac{21}{2}\kappa^4 u^{(1)}u^{(4)} \\ & + \frac{35}{8}\kappa^2u^2u^{(3)} + \frac{35}{2}\kappa^4 u^{(2)}u^{(3)} + \frac{7}{2}\kappa^4 uu^{(5)} + \kappa^6 u^{(7)} = \mathcal{K}_7. \end{aligned} \tag{23}$$

These constitute the KdV hierarchy equations, which can be summarized as a equation,

$$u_t = \mathcal{K}_{2n+1} = \mathcal{O}^n \mathcal{K}_1. \tag{24}$$

Here, the integro-differential operator $\mathcal{O}$ is given by

$$\mathcal{O} = \kappa^2 D^2 + u + u^{(1)} I. \tag{25}$$

The following type of expansions for $A_\mu(s;x,p)$ obtained by changing time variable by space variable from the previous case;

$$A_1(s;x,p) = p^2 + u(s;x), \tag{26}$$
$$A_0(s;x,p) = p^3 + p^2 g_2(s;x) + p g_1(s;x) + g_0(s;x), \tag{27}$$

lead to the Boussinesque equation, by substituting these expansions into Eq.(6),

$$u_{ss} + (uu')' + \kappa^2 u'''' = 0. \tag{28}$$

We next study the compact manifold case. We show how the Toda Lattice hierarchy equations are derived in the present formulation by adopting a compact manifold. We note that the power series expansion in terms of the variables are suitable for R^n, and each range of the variables is from -∞ to ∞. Contrastigly, the finite rage of variables should be legitimate for the compact manifolds. The power series expansion should be subject to modification if a compact manifold is adopted instead of the euclidean space. In the same way that we obtain the trigonometric algebra from the Moyal algebra, the expansion in terms of e^{ip} is available for such compact manifolds. We expand the potentials as

$$A_0(t;x,p) = \sum_m e^{imp} a_m(t;x), \tag{29}$$
$$A_1(t;x,p) = \sum_m e^{imp} b_m(t;x). \tag{30}$$

One of the examples is given by

$$A_0 = a_0(t;x) + a_{-1}(t;x)e^{-ip} + a_1(t;x)e^{ip}, \tag{31}$$
$$A_1 = b_0(t;x) + b_{-1}(t;x)e^{-ip} + b_1(t;x)e^{ip}. \tag{32}$$

A useful formula for these type expansions is given by

$$\{e^{imp} f(x), e^{inp} g(p)\} = \frac{i}{2k} e^{\{i(m+n)p\}} \Big\{ f(x+nk)g(x-mk) - g(x+mk)f(x-nk) \Big\}, \tag{33}$$

which is obtained by setting $k = -ik$ in Eq.(2). Substituting the above expansions (31) and (32) into Eq.(5), we obtain

$$0 = a_{-1}(x-k)b_{-1}(x+k) - b_{-1}(x-k)a_{-1}(x+k), \tag{34}$$
$$\frac{\mathrm{d}b_{-1}(x)}{\mathrm{d}t} = -\frac{1}{2k}[a_{-1}(x)(b_0(x+k) - b_0(x-k)) - b_{-1}(x)(a_0(x+k) - a_0(x-k))], \tag{35}$$

$$
\begin{aligned}
\frac{\mathrm{d}b_0(x)}{\mathrm{d}t} &= -\frac{1}{2k}[a_{-1}(x+k)b_1(x+k) - b_1(x-k)a_{-1}(x-k)) \\
&\quad -(a_1(x+k)b_{-1}(x+k) - b_{-1}(x-k)a_1(x-k))], \qquad (36) \\
\frac{\mathrm{d}b_1(x)}{\mathrm{d}t} &= -\frac{1}{2k}[a_1(x)(b_0(x+k) - b_0(x-k)) - b_1(x)(a_0(x+k) - a_0(x-k))], \qquad (37) \\
0 &= a_1(x-k)b_1(x+k) - b_1(x-k)a_1(x+k). \qquad (38)
\end{aligned}
$$

Assuming

$$
b_{\pm 1} = \gamma_{\pm 1} a_{\pm 1}, \qquad (39)
$$

we obtain

$$
\begin{aligned}
\mathrm{lna(x)}_{\pm 1} &= \mp\frac{i}{2k}[a_0(x+k) - a_0(x-k)) - \frac{1}{\gamma_{\pm 1}}(b_0(x+k) - b_0(x-k))], \qquad (40) \\
\frac{\mathrm{d}b_0(x)}{\mathrm{d}t} &= -\frac{i}{2k}(\gamma_1 - \gamma_{-1})\{a_{-1}(x+k)a_1(x+k) - a_1(x-k)a_{-1}(x-k)\}. \qquad (41)
\end{aligned}
$$

We then introduce $\rho(t;x)$ by

$$
\rho(t;x) = -\mathrm{ln}a_{-1}(t;x)a_1(t;x), \qquad (42)
$$

to obtain the Toda Lattice equation:

$$
\frac{\mathrm{d}^2\rho(t;x)}{\mathrm{d}t^2} = \frac{(\gamma_1 - \gamma_{-1})^2}{\gamma_1\gamma_{-1}}\frac{1}{(2k)^2}(e^{-\rho(x-2k)} - 2e^{-\rho(x)} + e^{-\rho(x+2k)}). \qquad (43)
$$

It is now possible to give the physical implication to $\kappa = -ik$ in the definition of the $\star$ product (2). In the Moyal quantization, the parameter appearing there is the Planch constant $\hbar$ that is a unit of spacing between energy levels. In the present case, the parameter k is the spacing between the adjacent particles on the lattice[5].

In the $k \to 0$ limit, we obtain the continuous Toda equation[6]

$$
\frac{\partial^2\rho}{\partial t^2} = \frac{\partial^2 \mathrm{e}^{-\rho}}{\partial x^2}. \qquad (44)
$$

We thus showed that the noncommutative zero-curvature equation has a rich structure. A various soliton equations were derived by the reduction of dimensions. In the reduction, we demonstrated that a choice of a manifold is deeply connected to a choice of the soliton equation.

References

[1] J. E. Moyal, Proc. Cambridge Phil. Soc. **45** (1949) 90.

[2] D. Fairlie and C. Zachos, Phys. Lett.**B244** (1989) 101.

[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, Phys. Rev. Lett.**31** (1973) 125; Stud. In Appl. Math. **53** (1974) 160.

[4] J. M. Alberty, T. Koikawa, R. Sasaki, Physica**5D**(1982)43.

[5] T. Koikawa, Prog. Theor. Phys.**105**(2001) 1045.

[6] T.Koikawa, Prog.Theor.Phys. **66** (1981) 1970.