What is the continuum limit of the integrable $Sp(2N)$ lattice models?.

M.J. Martins*

Departamento de Física, Universidade Federal de São Carlos
Caixa Postal 676, 13565-905, São Carlos, Brasil
E-mail: martins@df.ufscar.br

ABSTRACT: We summarize the contents of the talk presented at the Workshop on Integrable Theories, Solitons and Duality. A central theme in this lecture was the critical properties of the integrable $Sp(2N)$ lattice model and its applications to the physics of the spin-orbital chains and intersecting loop models

The theory of quantum one-dimensional integrable models has turned out to be a fruitful venture since the solution of the Heisenberg model by the Bethe ansatz [1]. One of the central quantities of quantum integrability is the S-matrix which describes the factorized scattering of particles of (1+1) quantum field theories and also the statistical weights of integrable two-dimensional lattice models. Of particular interest are the scattering amplitudes preserving bilinear antisymmetric metrics typical of systems with N component Dirac fermions invariant by the $Sp(2N)$ symmetry. More specifically, the scattering weight $S_{ab}^{cd}(\lambda)$ of this theory [2] is given by

$$S_{ab}^{cd}(\lambda) = \delta_{a,d}\delta_{c,b} + \lambda\delta_{a,c}\delta_{b,d} + \frac{\lambda}{\lambda+N+1}\epsilon_a\epsilon_c\delta_{a,\bar{b}}\delta_{c,\bar{d}} \tag{1}$$

where $\bar{a} = 2N+1-a$, $\epsilon_a = 1$ for $1 \leq a \leq N$ and $\epsilon_a = -1$ for $N+1 \leq a \leq 2N$. The variable λ denotes the two-particle scattering rapidity.

Although the exact solution of the above system has long been known [3] only recently the physical content of this model such as the nature of the low-energy behavior of the ground state and excitations have been unveiled. In general, integrable systems derivable from fundamental S-matrices such as (1) are expected to be massless and their critical behaviour is believed to be described in terms of the properties of a Wess-Zumino-Witten field theory on the respective group [4]. Here, however, we have very different situation and the criticality of model (1) is governed by the product of N c=1 conformal field theories which is certainly distinct from the one governed by the $Sp(2N)$ Wess-Zumino-Witten

*Speaker.

theory. The excitations consist of N elementary spinons and $N-1$ composite excitations made by special convolutions between the spinons. The first $N-1$ excitations behave much like that of generalized $SU(N)$ spinons while the Nth mode turns out to be a standard $s = 1/2$ spinon. It should be emphasized that the $c = 1$ theories are not independent from each other because one expects the manifestation of the $Sp(2N)$ symmetry in the continuum. The results for the lowest excitations lead us to conjecture that the additional constraints on the operator content should follow the rules of the canonical reduction procedure of $Sp(2N)$. We also remark that though the excitations possess linear dispersion relation their sound velocities are not all the same and consequently the underlying continuum theory is not Lorentz invariant. This briefly summarize the main results and the technical details omitted in the text can be found in ref.[5].

The results discussed above have at least two immediate applications. The first one is related to the process of surface diffusion which may be described by repeated scattering of mobile particles by impurities. This problem may be modeled assuming that the lattice sites are occupied by randomly placed scatterers and that the particle move along the lattice bonds [6]. In the two-dimensional square lattice the simplest non-trivial scattering rules one can think of is that the particle, arriving on a node, can be scattered to the left, to the right or pass freely in the case of absence of a scatterer. For instance, the scatterers can be viewed as double-sided mirrors allowing right-angle reflections when placed along the diagonals of the square lattice. Note that the presence of empty sites implies intersections between trajectories configurations making this loop model very interesting. Considering periodic boundary conditions each particle follows a closed path and if for every closed loop we assign a fugacity q, then the partition function Z of the system can be written as

$$Z = \sum_{scatter\ configurations} w_a^{n_a} w_b^{n_b} w_c^{n_c} q^{\#paths} \tag{2}$$

where n_a,n_b and n_c are the number of weights w_a, w_b and w_c appearing in a given configuration of left mirrors, right mirrors and empty sites, respectively.

The important point here is that the amplitudes (1) realize the scattering rules mentioned above providing us a theoretical framework to study the corresponding diffusive behavior. In fact the weights w_i are in one-to-one correspondence with the operators that generate the scattering amplitudes $S_{ab}^{cd}(\lambda)$. More precisely, the partition function of the statistical mechanics vertex model associate to the amplitudes (1) has exactly the form (2) where

$$w_a = 1, \quad w_b = \frac{\lambda}{\lambda + N + 1}, \quad w_c = \lambda, \quad q = -2N \tag{3}$$

As a consequence of this mapping one can compute exactly the exponent that governs the probability that two distant sites are visited by the same trajectory. This is the so called fractal dimension of the path $d_f = 2 - 2h$ where h is the conformal dimension associated to a spin-wave (spinon) excitation in the $Sp(2N)$ model. This identification together with the conformal content of the $Sp(2N)$ model described above lead us to predict that

$$d_f = (8 - N)/4 \tag{4}$$

This result can easily be tested for $N = 1$. In this case we can rely on the global isomorphism $Sp(2) \sim SU(2)$ in order to see that the amplitudes (1) coincide with that of the isotropic six-vertex model. This system can be related to percolation clusters of a bond percolation problem and the exponent governing the scaling of single paths is expected to be $d_f = 7/4$ [7] which agrees with our result (4).

Next, our results turn out also to be of utility to one-dimensional systems with coupled spin and orbital degrees of freedom such as the spin-orbital models [8, 9]. We recall that the effective spin-isospin Hamiltonian describing these systems may be written in the form

$$H_{SO}(J_0, J_1, J_2) = \sum_{i=1}^{L} \sum_{\alpha=0}^{2} J_\alpha P_{i,i+1}^{(\alpha)} \tag{5}$$

where J_α are superexchange constants and $P_{i,i+1}^{(\alpha)}$ denote the respective projections on the singlet, triplet and doublet spin-isospin states in a four dimensional Hilbert space. It is not difficult to write such projectors in terms of two commuting sets of Pauli matrices leading us to identify the integrable $Sp(4)$ model with the point $J_0/J_1 = J_0/J_2 = 1/3$.

The interesting feature is that while the spin degrees of freedom are indeed $SU(2)$ symmetric, the orbital variables possess only $U(1)$ invariance and the system altogether is both asymmetric and anisotropic in the Heisenberg couplings. It turns out that models of this sort have been proposed to explain peculiar properties of certain compounds such as NaV_2O_5 [10]. The excitations we predicted for the $Sp(4)$ model consist of two spinons and one additional spinless composite mode representing an extra "charge" excitation. We also note that one of the spinons travels with different velocity than the composite $s = 0$ excitation which makes it possible the phenomena of spin-charge separation. Remarkably, this scenario is in qualitative aggrement with experiments performed in this material [11]. These results suggest the $Sp(4)$ model can be used as a test case to check the reliability of non-perturbative methods for more general values of the Heisenberg coupling constants. This study also indicates that the nature of the excitations in spin-orbital systems can be rather involving. In fact, the isotropic point $J_0 = J_1 = J_2$ is known to have three basic excitations [12] and it is the lattice realization of a $SU(4)$ Wess-Zumino-Witten field theory which has $c = 3$. On the other hand the anisotropic point $J_0/J_1 = J_0/J_2 = 1/3$ has only two independent excitations and one composite mode that do not contribute to the low-energy limit and the total central charge is now $c = 2$. A natural question one would like to make is: what is the nature of the excitations of the spin-orbital model (5) in the crossover regime $1/3 \leq J_0/J_1 = J_0/J_2 \leq 1$?. The other one concerns with the mechanism that made one of the excitations to become a composite state. Preliminary numerical results suggest that in the above region all the excitations are gapless leading us to conjecture that the composite mode should be a goldstone excitation responsible by the crossover $c = 3$ to $c = 2$.

Finally, this work prompts us to ask a question that may open up new interesting avenues in integrable models. Namely, what is the integrable lattice $Sp(2N)$ model whose continuum limit corresponds to the $Sp(2N)$ Wess-Zumino-Witten theory?. The answer to this question has eluded us so far.

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