Continuum limit of the N=(1|1) supersymmetric Toda lattice hierarchy

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Abstract: Generalizing the graded commutator in superalgebras, we propose a new bracket operation on the space of graded operators with an involution. We show that the Lax representation of the two-dimensional N=(1|1) supersymmetric Toda lattice hierarchy can be realized via the generalized bracket operation; this is important in constructing the semiclassical (continuum) limit of this hierarchy. We construct the continuum limit of the N=(1|1) Toda lattice hierarchy, the dispersionless N=(1|1) Toda hierarchy. In this limit, we obtain the Lax representation, with the generalized graded bracket becoming the corresponding Poisson bracket on the graded phase superspace.

1. Introduction

An integrable N=(1|1) supersymmetric generalization of the two-dimensional bosonic Toda lattice hierarchy (2DTL hierarchy) [1] was proposed in [2], [3]. It is given by an infinite system of evolution equations (flows) for an infinite set of bosonic and fermionic lattice fields evolving in two bosonic and two fermionic infinite "towers" of times; as a subsystem, it involves an N=(1|1) supersymmetric integrable generalization of the 2DTL equation, which is called the N=(1|1) 2DTL in what follows.

Two new infinite series of fermionic flows of the N=(1|1) 2DTL hierarchy were constructed in [4]–[6]. This hierarchy was shown to actually have a higher symmetry, namely, the $N = (2|2)$ supersymmetry. Together with the previously known bosonic flows of the N=(1|1) 2DTL hierarchy, these flows are symmetries of the N=(1|1) 2DTL equation. The continuum limit of that equation with respect to the lattice constant [7] is a three-dimensional nonlinear equation, called the continuum or dispersionless N=(1|1) 2DTL equation.

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Although the N=(1|1) 2DTL hierarchy and the dispersionless N=(1|1) 2DTL equation have been known for a relatively long time, the problem of constructing the continuum (semiclassical) limit with respect to the lattice constant (which plays the role of the Planck constant here) for all the N=(1|1) 2DTL hierarchy flows was solved only quite recently in [8]. The present talk, based on the paper [8], addresses this problem. Apart from the purely academic significance of this problem, its solution is also interesting in relation to a number of important physical and mathematical applications. In particular, these include the semiclassical limit of the bosonic preimage of the N=(1|1) 2DTL hierarchy, the dispersionless 2DTL hierarchy [9], which unifies the 2DTL hierarchy flows arising in the leading semiclassical approximation constructed in [10] (see also [11]). In view of a deep relation between the 2DTL and N=(1|1) 2DTL hierarchies, it is natural to hypothesize that the dispersionless N=(1|1) supersymmetric 2DTL hierarchy must also admit similar applications in supersymmetric generalizations of the corresponding theories. This motivates our construction of the dispersionless N=(1|1) 2DTL hierarchy here.

The structure of this paper is as follows. In Sec. 2, we introduce the generalized graded bracket operation on the space of graded operators with an involution that generalizes the graded commutator for superalgebras. We then obtain the Lax representation of the N=(1|1) 2DTL hierarchy and all the basic defining relations in terms of the generalized graded bracket. We give an explicit expression for the N=(1|1) 2DTL hierarchy flows which are consequently used in Sec. 3 to obtain the respective dispersionless analogues.

In Sec. 3, we also find the semiclassical limit of the N=(1|1) 2DTL hierarchy and postulate the corresponding asymptotic behavior of the fermionic and bosonic fields parameterizing the Lax operators. Using these data, we then evaluate the asymptotic behavior of all the composite operators entering the Lax representation and the corresponding field evolution equations. We next obtain regular leading terms in the semiclassical expansion of these evolution equations, which are by definition the flows of the dispersionless N=(1|1) 2DTL hierarchy. This a posteriori demonstrates self-consistency of the postulates underlying all our calculations. The next step is to model the Poisson superbracket on the phase superspace obtained by extending the phase space of the dispersionless 2DTL hierarchy by one Grassmann coordinate. Replacing the Lax operators with their symbols and replacing the generalized graded bracket with the above Poisson superbracket in the Lax representation of the N=(1|1) 2DTL hierarchy and in all its defining relations, we show by a direct calculation that the operator representation thus obtained correctly reproduces the flows of the dispersionless N=(1|1) 2DTL hierarchy that we constructed previously and is therefore the sought Lax representation of the above hierarchy.

2. N=(1|1) 2DTL hierarchy

In this section, we introduce a new graded bracket operation and use it to propose a new form of the Lax representation for the N=(1|1) 2DTL hierarchy.

2.1 Generalized graded brackets

We consider the space of operators $\mathbb{O}_k$ with the grading $d_{\mathbb{O}_k}$ ($d_{\mathbb{O}_k} \in \mathbb{Z}$),

$$d_{\mathbb{O}_1\mathbb{O}_2} = d_{\mathbb{O}_1} + d_{\mathbb{O}_2}, \tag{2.1}$$

and the involution $*$,

$$\mathbb{O}_k^{*(2)} = \mathbb{O}_k, \tag{2.2}$$

where ${\mathbb{O}_k}^{*(m)}$ denotes the m-fold action of the involution $*$ on the operator $\mathbb{O}_k$. On this space, we can define the generalized graded bracket operation $[\cdot, \cdot\}$,

$$\Big[\mathbb{O}_1, \mathbb{O}_2\Big\} := \mathbb{O}_1\mathbb{O}_2 - (-1)^{d_{\mathbb{O}_1} d_{\mathbb{O}_2}}\ {\mathbb{O}_2}^{*(d_{\mathbb{O}_1})}\ {\mathbb{O}_1}^{*(d_{\mathbb{O}_2})} \tag{2.3}$$

with the easily verified properties of

a. *symmetry*

$$\Big[\mathbb{O}_1, \mathbb{O}_2\Big\} = (-1)^{d_{\mathbb{O}_1} d_{\mathbb{O}_2}+1}\Big[{\mathbb{O}_2}^{*(d_{\mathbb{O}_1})}, {\mathbb{O}_1}^{*(d_{\mathbb{O}_2})}\Big\}, \tag{2.4}$$

b. *derivation*

$$\Big[\mathbb{O}_1, \mathbb{O}_2\mathbb{O}_3\Big\} = \Big[\mathbb{O}_1, \mathbb{O}_2\Big\}\mathbb{O}_3 + (-1)^{d_{\mathbb{O}_1} d_{\mathbb{O}_2}}\mathbb{O}_2^{*(d_{\mathbb{O}_1})}\Big[{\mathbb{O}_1}^{*(d_{\mathbb{O}_2})}, \mathbb{O}_3\Big\}, \tag{2.5}$$

c. *Jacobi identity*

$$\begin{aligned}
&(-1)^{d_{\mathbb{O}_1} d_{\mathbb{O}_3}}\Big[\Big[{\mathbb{O}_1}^{*(d_{\mathbb{O}_3})}, \mathbb{O}_2\Big\}, {\mathbb{O}_3}^{*(d_{\mathbb{O}_1})}\Big\}\\
&+ (-1)^{d_{\mathbb{O}_2} d_{\mathbb{O}_1}}\Big[\Big[{\mathbb{O}_2}^{*(d_{\mathbb{O}_1})}, \mathbb{O}_3\Big\}, {\mathbb{O}_1}^{*(d_{\mathbb{O}_2})}\Big\}\\
&+ (-1)^{d_{\mathbb{O}_3} d_{\mathbb{O}_2}}\Big[\Big[{\mathbb{O}_3}^{*(d_{\mathbb{O}_2})}, \mathbb{O}_1\Big\}, {\mathbb{O}_2}^{*(d_{\mathbb{O}_3})}\Big\} = 0.
\end{aligned} \tag{2.6}$$

Relations (2.4–2.6) generalize the corresponding properties of the graded commutator in Lie superalgebras. We emphasize that in the particular case where the involution $*$ acts as the identity transformation, bracket (2.3) reproduces the graded Lie superalgebra commutator. In the general case of the involution action, this bracket is a nontrivial generalization of that commutator.

2.2 Lax representation and flows

We begin this section by detailing the space of operators, their grading, and involution which are relevant in context of the N=(1|1) 2DTL hierarchy. These operators can be represented in the general form

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} e^{(k-m)\partial}, \quad m \in \mathbb{Z}, \tag{2.7}$$

parameterized by the functions $f_{2k,j}^{(m)}$ ($f_{2k+1,j}^{(m)}$) that are Z_2-graded bosonic (fermionic) lattice fields ($j \in \mathbb{Z}$),

$$d'_{f_{k,j}^{(m)}} = |k| \mod 2; \tag{2.8}$$

the operator $e^{l\partial}$ ($l \in \mathbb{Z}$) acting on these fields as the discrete lattice shift

$$e^{l\partial} f_{k,j}^{(m)} \equiv f_{k,j+l}^{(m)} e^{l\partial} \tag{2.9}$$

has another Z_2-grading given by

$$d_{e^{l\partial}} = |l| \mod 2. \tag{2.10}$$

Operators (2.7) allow specifying only one diagonal Z_2-grading

$$d_{\mathbb{O}_m} = d'_{f_{k,j}^{(m)}} + d_{e^{(k-m)\partial}} = |m| \mod 2 \tag{2.11}$$

and the involution

$$\mathbb{O}_m^* = \sum_{k=-\infty}^{\infty} (-1)^k f_{k,j}^{(m)} e^{(k-m)\partial}. \tag{2.12}$$

In what follows, we also need the projections $(\mathbb{O}_m)_\pm$ of the operators $\mathbb{O}_m$ (2.7) defined as

$$(\mathbb{O}_m)_+ = \sum_{k=m}^{\infty} f_{k,j}^{(m)} e^{(k-m)\partial}, \quad (\mathbb{O}_m)_- = \sum_{k=-\infty}^{m-1} f_{k,j}^{(m)} e^{(k-m)\partial}. \tag{2.13}$$

The Lax operators $L^\pm$ of the N=(1|1) 2DTL hierarchy belong to the space of operators (2.7) [2, 6]

$$L^+ = \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad u_{0,j} = 1, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-1)\partial}, \quad v_{0,j} \neq 0 \tag{2.14}$$

and have the grading $d_{L^\pm} = 1$.

We now have all the ingredients necessary to express the Lax representation of the N=(1|1) 2DTL hierarchy in terms of bracket operation (2.3), thereby bringing it to a very simple form,

$$D_n^\pm L^\alpha = \mp\alpha(-1)^n \Big[(((L^\pm)_*^n)_{-\alpha})^*, L^\alpha\Big\}, \quad \alpha = +,-, \quad n \in \mathbb{N}, \tag{2.15}$$

where $D_{2n}^\pm$ ($D_{2n+1}^\pm$) are bosonic (fermionic) evolution derivatives.

For the composite operators $(L^\pm)_*^n$ entering this representation, we can also obtain very simple expressions in terms of the Lax operators and bracket operation (2.3),

$$(L^\alpha)_*^{2n} := \Big(\frac{1}{2} \Big[(L^\alpha)^*, (L^\alpha)\Big\} \Big)^n, \quad (L^\alpha)_*^{2m+1} := L^\alpha \, (L^\alpha)_*^{2n}. \tag{2.16}$$

Similarly to the Lax operators $L^\pm$, the operators $(L^\pm)_*^n$ belong to the space of operators (2.7) and can be represented as

$$(L^+)_*^m := \sum_{k=0}^{\infty} u_{k,j}^{(m)} e^{(m-k)\partial}, \quad u_{0,j}^{(m)} = 1, \quad (L^-)_*^m := \sum_{k=0}^{\infty} v_{k,j}^{(m)} e^{(k-m)\partial}, \tag{2.17}$$

where $u^{(m)}_{k,j}$ and $v^{(m)}_{k,j}$ (with $u^{(1)}_{k,j} \equiv u_{k,j}$, $v^{(1)}_{k,j} \equiv v_{k,j}$) are functionals of the original fields $\{u_{k,j},\ v_{k,j}\}$. It must be noted here that in Lax representation (2.15), the Z_2-grading of the operator $(L^{\pm})^n_*$, which has the form $d_{(L^{\pm})^{2n}_*} = 0$ and $d_{(L^{\pm})^{2n+1}_*} = 1$, agrees with another Z_2-grading $d^{'}_{D^{\pm}_{2n}} = 0$ and $d^{'}_{D^{\pm}_{2n+1}} = 1$ that corresponds to the statistics of the evolution derivatives $D^{\pm}_n$.

Using bracket properties (2.4–2.6) and relations (2.16) as definitions of $(L^{\pm})^n_*$, we can easily obtain the useful identities

$$\begin{aligned}
&\Big[(L^{\alpha})^{2n}_*, (L^{\alpha})^{2m}_*\Big\}= 0,\\
&\Big[((L^{\alpha})^{2n}_*)^*, (L^{\alpha})^{2m+1}_*\Big\}= 0, \quad \Big[(L^{\alpha})^{2n+1}_*, (L^{\alpha})^{2m}_*\Big\}= 0,\\
&\Big[((L^{\alpha})^{2n+1}_*)^*, (L^{\alpha})^{2m+1}_*\Big\}= 2(L^{\alpha})^{2(n+m+1)}_*.
\end{aligned} \tag{2.18}$$

Next, using Eqs. (2.4–2.6) and (2.15–2.16), we can derive equations of motion for the composite operators $(L^{\pm})^n_*$,

$$D^{\pm}_n (L^{\alpha})^m_* = \mp\alpha(-1)^{nm}\Big[(((L^{\pm})^n_*)_{-\alpha})^{*(m)}, (L^{\alpha})^m_*\Big\}, \tag{2.19}$$

and the evolution equations for the functionals $\{u^{(m)}_{k,j},\ v^{(m)}_{k,j}\}$ in (2.17) implied by Eqs. (2.19) found above,

$$\begin{aligned}
D^{+}_n u^{(2m)}_{k,j} &= \sum_{p=0}^{n}(u^{(n)}_{p,j} u^{(2m)}_{k-p+n,j-p+n}\\
&- (-1)^{(p+n)(k-p+n)} u^{(n)}_{p,j-k+p-n+2m} u^{(2m)}_{k-p+n,j}),\\
D^{+}_{2n} u^{(2m+1)}_{k,j} &= \sum_{p=0}^{2n}((-1)^p u^{(2n)}_{p,j} u^{(2m+1)}_{k-p+2n,j-p+2n}\\
&- (-1)^{p(k-p)} u^{(2n)}_{p,j-k+p-2n+2m+1} u^{(2m+1)}_{k-p+2n,j}),\\
D^{+}_{2n+1} u^{(2m+1)}_{k,j} &= \sum_{p=1}^{k}((-1)^{p+1} u^{(2n+1)}_{p+2n+1,j} u^{(2m+1)}_{k-p,j-p}\\
&+ (-1)^{p(k-p)} u^{(2n+1)}_{p+2n+1,j-k+p+2m+1} u^{(2m+1)}_{k-p,j}),
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
D^{-}_n u^{(m)}_{k,j} &= \sum_{p=0}^{n-1}((-1)^{(p+n)m} v^{(n)}_{p,j} u^{(m)}_{k+p-n,j+p-n}\\
&- (-1)^{(p+n)(k+p-n)} v^{(n)}_{p,j-k-p+n+m} u^{(m)}_{k+p-n,j}),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
D^{+}_n v^{(m)}_{k,j} &= \sum_{p=0}^{n}((-1)^{(p+n)m} u^{(n)}_{p,j} v^{(m)}_{k+p-n,j-p+n}\\
&- (-1)^{(p+n)(k+p-n)} u^{(n)}_{p,j+k+p-n-m} v^{(m)}_{k+p-n,j}),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
D_n^- v_{k,j}^{(2m)} &= \sum_{p=0}^{n-1} (v_{p,j}^{(n)} v_{k-p+n,j+p-n}^{(2m)} \\
&- (-1)^{(p+n)(k-p+n)} v_{p,j+k-p+n-2m}^{(n)} v_{k-p+n,j}^{(2m)}), \\
D_{2n}^- v_{k,j}^{(2m+1)} &= \sum_{p=0}^{2n-1} ((-1)^p v_{p,j}^{(2n)} v_{k-p+2n,j+p-2n}^{(2m+1)} \\
&- (-1)^{p(k-p)} v_{p,j+k-p+2n-2m-1}^{(2n)} v_{k-p+2n,j}^{(2m+1)}), \\
D_{2n+1}^- v_{k,j}^{(2m+1)} &= \sum_{p=0}^{k} ((-1)^{p+1} v_{p+2n+1,j}^{(2n+1)} v_{k-p,j+p}^{(2m+1)} \\
&+ (-1)^{p(k-p)} v_{p+2n+1,j+k-p-2m-1}^{(2n+1)} v_{k-p,j}^{(2m+1)})
\end{aligned} \tag{2.23}$$

(in the right-hand side of these equations, all the fields $\{u_{k,j}^{(m)},\ v_{k,j}^{(m)}\}$ with $k < 0$ must be set equal to zero).

Lax representation (2.15) generates a non-Abelian algebra of flows of the $N{=}(1|1)$ 2DTL hierarchy,

$$[D_n^+ \ ,\ D_l^-\} = [D_n^\pm \ ,\ D_{2l}^\pm] = 0, \quad \{D_{2n+1}^\pm \ ,\ D_{2l+1}^\pm\} = 2D_{2(n+l+1)}^\pm, \tag{2.24}$$

which can be realized as

$$D_{2n}^\pm = \partial_{2n}^\pm, \quad D_{2n+1}^\pm = \partial_{2n+1}^\pm + \sum_{l=1}^{\infty} t_{2l-1}^\pm \partial_{2(k+l)}^\pm, \quad \partial_n^\pm := \frac{\partial}{\partial t_n^\pm}, \tag{2.25}$$

where $t_{2n}^\pm$ ($t_{2n+1}^\pm$) are bosonic (fermionic) evolution times.

The supersymmetric $N{=}(1|1)$ 2DTL equation

$$D_1^+ D_1^- \ln v_{0,j} = v_{0,j+1} - v_{0,j-1} \tag{2.26}$$

belongs to system of equations (2.20–2.23). It can be obtained from Eq. (2.21) with $\{n = m = k = 1\}$,

$$D_1^- u_{1,j} = -v_{0,j} - v_{0,j+1}, \tag{2.27}$$

and from Eq. (2.22) with $\{n = m = 1,\ k = 0\}$,

$$D_1^+ v_{0,j} = v_{0,j}(u_{1,j} - u_{1,j-1}), \tag{2.28}$$

after eliminating the field $u_{1,j}$.

The dispersionless $N{=}(1|1)$ 2DTL hierarchy flows are based on the continuum limit of flows (2.20–2.23) constructed in Sec. 3.1.

3. The dispersionless $N{=}(1|1)$ 2DTL hierarchy

In this section, we construct the continuum (semiclassical) limit of the $N{=}(1|1)$ 2DTL hierarchy with respect to the lattice constant, which gives the dispersionless $N{=}(1|1)$ 2DTL hierarchy, and construct the corresponding Lax representation.

3.1 Semiclassical limit

Flows (2.20–2.23) of the N=(1|1) 2DTL hierarchy do not involve an explicit dependence on dimensional constants, and the lattice with the dimensionless coordinate j ($j \in \mathbb{Z}$) has the unit spacing constant. To study the continuum limit, we explicitly introduce the lattice spacing length. This parameter is denoted by $\hbar$ because it plays the role of the Planck constant in what follows. Instead of j, there then arises the combination

$$\hbar j \equiv s, \tag{3.1}$$

and all the lattice fields acquire a dependence on the parameter $\hbar$. The continuum (semi-classical) limit can then be defined as

$$\hbar \to 0, \quad s = \lim_{\hbar \to 0,\ j>>1}(\hbar j), \tag{3.2}$$

with s playing the role of the continuum "lattice" coordinate.

For the flows in Eqs. (2.20–2.23) to be nontrivial and regular in the limit in Eq. (3.2), we must additionally perform several scaling transformations of the dependent and independent variables in the system. We postulate the rules for transition to the new evolution times and fields of the hierarchy given by

$$t^{\pm}_{2n+1} \to \frac{1}{\sqrt{\hbar}} t^{\pm}_{2n+1}, \quad t^{\pm}_{2n} \to \frac{1}{\hbar} t^{\pm}_{2n} \ \Leftrightarrow\ D^{\pm}_{2n+1} \to \sqrt{\hbar} D^{\pm}_{2n+1}, \quad D^{\pm}_{2n} \to \hbar D^{\pm}_{2n}, \tag{3.3}$$

$$\begin{aligned} u_{2k,j} &\to u_{2k}(\hbar j), & v_{2k,j} &\to v_{2k}(\hbar j), \\ u_{2k+1,j} &\to \frac{1}{\sqrt{\hbar}}\, u_{2k+1}(\hbar j), & v_{2k+1,j} &\to \frac{1}{\sqrt{\hbar}}\, v_{2k+1}(\hbar j), \end{aligned} \tag{3.4}$$

and assume they are nonsingular at $\hbar = 0$.

We can now establish two important properties of semiclassical limit (3.2–3.4), which have a key importance in what follows and are verified by direct calculations.

The first property is that the new composite fields defined in accordance with the rules

$$\begin{aligned} u^{(m)}_{2k,j} &\to u^{(m)}_{2k}(\hbar j), & v^{(m)}_{2k,j} &\to v^{(m)}_{2k}(\hbar j), \\ u^{(2m+1)}_{2k+1,j} &\to \frac{1}{\sqrt{\hbar}}\, u^{(2m+1)}_{2k+1}(\hbar j), & v^{(2m+1)}_{2k+1,j} &\to \frac{1}{\sqrt{\hbar}}\, v^{(2m+1)}_{2k+1}(\hbar j), \\ u^{(2m)}_{2k+1,j} &\to \sqrt{\hbar}\, u^{(2m)}_{2k+1}(\hbar j), & v^{(2m)}_{2k+1,j} &\to \sqrt{\hbar}\, v^{(2m)}_{2k+1}(\hbar j), \end{aligned} \tag{3.5}$$

are regular in the semiclassical limit.

From rules (3.2–3.5) and the obvious identities

$$(L^{\alpha})^{2(m+1)}_{*} := (L^{\alpha})^{2}_{*}(L^{\alpha})^{2m}_{*}, \quad (L^{\alpha})^{(2m+1)}_{*} := L^{\alpha}(L^{\alpha})^{2m}_{*}, \tag{3.6}$$

which follow from Eqs. (2.16), we can, for example, obtain the important recursive relations for the leading terms of the functionals $u^{(m)}_{k} \equiv u^{(m)}_{k}(s)$,

$$\begin{aligned} u^{(2(l+1))}_{2k} &= \sum_{n=0}^{k} u^{(2)}_{2n} u^{(2l)}_{2(k-n)}, & u^{(2(l+1))}_{2k+1} &= \sum_{n=0}^{2k+1} u^{(2)}_{n} u^{(2l)}_{2k-n+1}, \\ u^{(2l+1)}_{2k} &= \sum_{n=0}^{2k} u_n u^{(2l)}_{2k-n}, & u^{(2l+1)}_{2k+1} &= \sum_{n=0}^{k} u_{2n+1} u^{(2l)}_{2(k-n)}, \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
u^{(2)}_{2k} &= \sum_{n=0}^{k} u_{2n}u_{2(k-n)} + 2\sum_{n=0}^{k-1}(k-n-1)u_{2(k-n)-1}\partial_s u_{2n+1}, \\
u^{(2)}_{2k+1} &= \sum_{n=0}^{k}\Big[(1-2n)u_{2n}\partial_s u_{2(k-n)+1} + 2(k-n)u_{2(k-n)+1}\partial_s u_{2n}\Big],
\end{aligned} \tag{3.8}
$$

where $\partial_s := \frac{\partial}{\partial s}$.

The second property is that in semiclassical limit (3.2–3.5) flows (2.20–2.23) of the N=(1|1) 2DTL hierarchy are nontrivial and regular. Explicit expressions for their leading terms are as follows.

For $\{m=2l,\ k=2r\}$ or $\{m=2l+1,\ k=2r+1\}$, we have

$$
\begin{aligned}
D^-_{2n+1}u^{(m)}_k &= \sum_{p=0}^{n-1}\Big[2(p-n)v^{(2n+1)}_{2p+1}\partial_s u^{(m)}_{k+2(p-n)} \\
&+ (k-m+2(p-n))(\partial_s v^{(2n+1)}_{2p+1})u^{(m)}_{k+2(p-n)}\Big] \\
&+ 2(-1)^k\sum_{p=0}^{n} v^{(2n+1)}_{2p}u^{(m)}_{k+2(p-n)-1}, \\
D^-_{2n}u^{(m)}_k &= \sum_{p=0}^{n-1}\Big[2(p-n)v^{(2n)}_{2p}\partial_s u^{(m)}_{k+2(p-n)} \\
&+ (k-m+2(p-n))(\partial_s v^{(2n)}_{2p})u^{(m)}_{k+2(p-n)} \\
&+ 2(-1)^k v^{(2n)}_{2p+1}u^{(m)}_{k+2(p-n)+1}\Big], \\
D^+_{2n}u^{(m)}_k &= \sum_{p=0}^{n}\Big[2(n-p)u^{(2n)}_{2p}\partial_s u^{(m)}_{k+2(n-p)} \\
&+ (k-m+2(n-p))(\partial_s u^{(2n)}_{2p})u^{(m)}_{k+2(n-p)}\Big] \\
&+ 2(-1)^k\sum_{p=0}^{n-1} u^{(2n)}_{2p+1}u^{(m)}_{k+2(n-p)-1}, \\
D^+_{2n+1}v^{(m)}_k &= \sum_{p=0}^{n}\Big[2(n-p)u^{(2n+1)}_{2p+1}\partial_s v^{(m)}_{k+2(p-n)} \\
&- (k-m+2(p-n))(\partial_s u^{(2n+1)}_{2p+1})v^{(m)}_{k+2(p-n)} \\
&+ 2(-1)^k u^{(2n+1)}_{2p}v^{(m)}_{k-1+2(p-n)}\Big], \\
D^+_{2n}v^{(m)}_k &= \sum_{p=0}^{n}\Big[2(n-p)u^{(2n)}_{2p}\partial_s v^{(m)}_{k+2(p-n)} \\
&- (k-m+2(p-n))(\partial_s u^{(2n)}_{2p})v^{(m)}_{k+2(p-n)}\Big] \\
&+ 2(-1)^k\sum_{p=0}^{n-1} u^{(2n)}_{2p+1}v^{(m)}_{k+1+2(p-n)},
\end{aligned}
$$

$$
\begin{aligned}
D^{-}_{2n} v^{(m)}_{k} &= \sum_{p=0}^{n-1} \Big[2(p-n) v^{(2n)}_{2p} \partial_s v^{(m)}_{k+2(n-p)} \\
&- (k-m+2(n-p))(\partial_s v^{(2n)}_{2p}) v^{(m)}_{k+2(n-p)} \\
&+ 2(-1)^k v^{(2n)}_{2p+1} v^{(m)}_{k+2(n-p)-1} \Big].
\end{aligned} \tag{3.9}
$$

For $\{m = 2l,\ k = 2r+1\}$ or $\{m = 2l+1,\ k = 2r\}$, we have

$$
\begin{aligned}
D^{-}_{n} u^{(m)}_{k} &= \sum_{p=0}^{n-1} (-1)^{m(p+n)} \Big[(p-n) v^{(n)}_{p} \partial_s u^{(m)}_{k+p-n} \\
&+ (k+p-n-m)(\partial_s v^{(n)}_{p}) u^{(m)}_{k+p-n} \Big], \\
D^{+}_{2n} u^{(m)}_{k} &= \sum_{p=0}^{2n} (-1)^{mp} \Big[(2n-p) u^{(2n)}_{p} \partial_s u^{(m)}_{k-p+2n} \\
&+ (k-p+2n-m)(\partial_s u^{(2n)}_{p}) u^{(m)}_{k-p+2n} \Big], \\
D^{+}_{n} v^{(m)}_{k} &= \sum_{p=0}^{n} (-1)^{m(p+n)} \Big[(n-p) u^{(n)}_{p} \partial_s v^{(m)}_{k+p-n} \\
&- (k+p-n-m)(\partial_s u^{(n)}_{p}) v^{(m)}_{k+p-n} \Big], \\
D^{-}_{2n} v^{(m)}_{k} &= \sum_{p=0}^{2n-1} (-1)^{mp} \Big[(p-2n) v^{(2n)}_{p} \partial_s v^{(m)}_{k-p+2n} \\
&- (k-p+2n-m)(\partial_s v^{(2n)}_{p}) v^{(m)}_{k-p+2n} \Big],
\end{aligned} \tag{3.10}
$$

and also

$$
\begin{aligned}
D^{+}_{2n+1} u^{(2m)}_{2k} &= 2 \sum_{p=0}^{n} \Big[(n-p) u^{(2n+1)}_{2p+1} \partial_s u^{(2m)}_{2(k-p+n)} \\
&+ (k-p+n-m)(\partial_s u^{(2n+1)}_{2p+1}) u^{(2m)}_{2(k-p+n)} \\
&+ u^{(2n+1)}_{2p} u^{(2m)}_{2(k-p+n)+1} \Big], \\
D^{+}_{2n+1} u^{(2m+1)}_{2k+1} &= 2 \sum_{p=1}^{k} \Big[p u^{(2n+1)}_{2(p+n)+1} \partial_s u^{(2m+1)}_{2(k-p)+1} \\
&+ (m+p-k)(\partial_s u^{(2n+1)}_{2(p+n)+1}) u^{(2m+1)}_{2(k-p)+1} \Big] \\
&+ 2 \sum_{p=0}^{k} u^{(2n+1)}_{2(p+n+1)} u^{(2m+1)}_{2(k-p)}, \\
D^{+}_{2n+1} u^{(2m+1)}_{2k} &= \sum_{p=1}^{2k} (-1)^p \Big[p u^{(2n+1)}_{p+2n+1} \partial_s u^{(2m+1)}_{2k-p} \\
&+ (2(m-k)+p+1)(\partial_s u^{(2n+1)}_{p+2n+1}) u^{(2m+1)}_{2k-p} \Big],
\end{aligned}
$$

$$
\begin{aligned}
D^{+}_{2n+1}u^{(2m)}_{2k+1} &= \sum_{p=0}^{2n+1}\Big[(2n-p+1)u^{(2n+1)}_{p}\partial_s u^{(2m)}_{2(k+n+1)-p} \\
&+ (2(k+n-m+1)-p)(\partial_s u^{(2n+1)}_{p})u^{(2m)}_{2(k+n+1)-p}\Big], \\
D^{-}_{2n+1}v^{(2m)}_{2k} &= 2\sum_{p=0}^{n-1}\Big[(p-n)v^{(2n+1)}_{2p+1}\partial_s v^{(2m)}_{2(k-p+n)} \\
&- (k-p+n-m)(\partial_s v^{(2n+1)}_{2p+1})v^{(2m)}_{2(k-p+n)}\Big] \\
&+ 2\sum_{p=0}^{n} v^{(2n+1)}_{2p}v^{(2m)}_{2(k-p+n)+1}, \\
D^{-}_{2n+1}v^{(2m+1)}_{2k+1} &= 2\sum_{p=0}^{k}\Big[-pv^{(2n+1)}_{2(p+n)+1}\partial_s v^{(2m+1)}_{2(k-p)+1} \\
&+ (k-p-m)\partial_s v^{(2n+1)}_{2(p+n)+1})v^{(2m+1)}_{2(k-p)+1} \\
&+ v^{(2n+1)}_{2(p+n+1)}v^{(2m+1)}_{2(k-p)}\Big], \\
D^{-}_{2n+1}v^{(2m+1)}_{2k} &= \sum_{p=0}^{k}(-1)^p\Big[-pv^{(2n+1)}_{p+2n+1}\partial_s v^{(2m+1)}_{2k-p} \\
&- (2(m-k)+p+1)(\partial_s v^{(2n+1)}_{p+2n+1})v^{(2m+1)}_{2k-p}\Big], \\
D^{-}_{2n+1}v^{(2m)}_{2k+1} &= \sum_{p=0}^{2n}\Big[-(2n-p+1)v^{(2n+1)}_{p}\partial_s v^{(2m)}_{2(k+n+1)-p} \\
&- (2(k+n-m+1)-p)(\partial_s v^{(2n+1)}_{p})v^{(2m)}_{2(k+n+1)-p}\Big], \qquad (3.11)
\end{aligned}
$$

where all the fields $\{u^{(m)}_k,\ v^{(m)}_k\}$ with $k<0$ must be set equal to zero in the right-hand sides. Flows (3.9–3.11) thus derived are said to constitute the dispersionless $N{=}(1|1)$ 2DTL hierarchy. The corresponding Lax representation is constructed in Sec. 3.3.

The dispersionless limit of $N{=}(1|1)$ 2DTL equation (2.26) can be easily obtained using Eqs. (3.2–3.5)

$$D^{+}_{1}D^{-}_{1}\ln v_0 = 2\partial_s v_0. \qquad (3.12)$$

3.2 Lax representation

In Sec. 3.1, we constructed flows (3.9–3.11) of the dispersionless $N{=}(1|1)$ 2DTL hierarchy. We now find the corresponding Lax representation. Because Lax representation (2.15) of the $N{=}(1|1)$ 2DTL hierarchy can be expressed in terms of generalized graded bracket (2.3), it is natural to expect that to derive its semiclassical limit, the corresponding dispersionless $N{=}(1|1)$ 2DTL hierarchy, it would suffice to replace all occurrences of this bracket with a certain Poisson superbracket and the operators with their symbols defined on the corresponding phase space.

Our immediate problem is to model the phase space and the Poisson superbracket starting from certain properties with which they must be endowed. Recalling that the

Lax operators of the $N{=}(1|1)$ 2DTL hierarchy, being defined on the space of operators graded by two different Z_2-gradings (2.8) and (2.10), have only one diagonal Z_2-grading (2.11), we can assume that the phase space inherits these properties. With the Z_2-grading $d_{e^\partial} = 1,\ d_{e^{2\partial}} = 0$ of the lattice shift operator $e^{l\partial}$ taken into account, (2.10), we assume that its counterpart on the phase space is given by two coordinates, the Grassmann coordinate π (with $\pi^2 = 0$) and the bosonic coordinate p, via

$$e^{\partial} \ \to \ \frac{1}{\sqrt{\hbar}}\,\pi, \quad e^{2\partial} \ \to \ p. \tag{3.13}$$

Although the coordinate π is Grassmann, it must have a property, not typical of Grassmann coordinates, to commute with not only bosonic but also fermionic fields of the hierarchy, as follows from the continuum limit of relation (2.9).

It is obvious that in addition to the coordinates π and p, the phase space must include a continuum "lattice" coordinate s, Eq. (3.1). The resulting phase superspace $\{\pi,\ p,\ s\}$ of the dispersionless $N{=}(1|1)$ 2DTL hierarchy contains the phase space $\{p,\ s\}$ of the dispersionless 2DTL hierarchy as a subspace.

Having established the coordinates on the phase space, we can construct the Poisson superbracket for them. Recalling that the Poisson superbracket must agree with the superalgebra of the Z_2-graded operators $\{\sqrt{\hbar}e^{\partial},\ e^{2\partial},\ \hbar j\}$, to which the phase superspace coordinates $\{\pi,\ p,\ s\}$ correspond, we can find these superbrackets relatively easily. We here give the Poisson superbrackets between two arbitrary functions $\mathbb{F}_{1,2} \equiv \mathbb{F}_{1,2}(\pi, p, s)$ on the phase space obtained as explained above,

$$\begin{aligned}\Big\{\mathbb{F}_1, \mathbb{F}_2\Big\} = 2p\,\Big(\frac{\partial\mathbb{F}_1}{\partial p}\,\frac{\partial\mathbb{F}_2}{\partial s} - \frac{\partial\mathbb{F}_1}{\partial s}\,\frac{\partial\mathbb{F}_2}{\partial p} + \frac{\partial\mathbb{F}_1}{\partial \pi}\,\frac{\partial\mathbb{F}_2}{\partial \pi}\Big)\\ + \pi\,\Big(\frac{\partial\mathbb{F}_1}{\partial \pi}\,\frac{\partial\mathbb{F}_2}{\partial s} - \frac{\partial\mathbb{F}_1}{\partial s}\,\frac{\partial\mathbb{F}_2}{\partial \pi}\Big).\end{aligned} \tag{3.14}$$

We note that after the transition to the new basis $\{\widetilde{s}, \widetilde{p}, \widetilde{\pi}\}$ in the phase space given by the formulas

$$\begin{aligned}&\widetilde{s} := \frac{s}{2}, \quad \widetilde{p} := \ln p, \quad \widetilde{\pi} := \frac{\pi}{\sqrt{2p}},\\ &s := 2\widetilde{s}, \quad p := e^{\widetilde{p}}, \quad \pi := \sqrt{2}\widetilde{\pi}e^{\frac{\widetilde{p}}{2}},\end{aligned} \tag{3.15}$$

Poisson superbrackets (3.14) become

$$\Big\{\mathbb{F}_1, \mathbb{F}_2\Big\} = \frac{\partial\mathbb{F}_1}{\partial \widetilde{p}}\,\frac{\partial\mathbb{F}_2}{\partial \widetilde{s}} - \frac{\partial\mathbb{F}_1}{\partial \widetilde{s}}\,\frac{\partial\mathbb{F}_2}{\partial \widetilde{p}} + \frac{\partial\mathbb{F}_1}{\partial \widetilde{\pi}}\,\frac{\partial\mathbb{F}_2}{\partial \widetilde{\pi}}, \tag{3.16}$$

which corresponds to the canonical orthosymplectic structure of the phase superspace.

We now proceed to the next stage of deriving the Lax representation of the dispersionless $N{=}(1|1)$ 2DTL hierarchy. Heuristic formulas for the symbols $\mathcal{L}^{\pm}$ and $(\mathcal{L}^{\pm})^n_*$ of the Lax operators $L^{\pm}$ (2.14) and of the composite operators $(L^{\pm})^n_*$ (2.16–2.17) are

$$\begin{aligned}&L^{\pm} \ \to \ \frac{1}{\sqrt{\hbar}}\,\mathcal{L}^{\pm},\\ &\mathcal{L}^{+} = \sum_{k=0}^{\infty}(u_{2k+1} + u_{2k}\pi)p^{-k}, \qquad \mathcal{L}^{-} = \sum_{k=0}^{\infty}(v_{2k-1} + v_{2k}\pi)p^{k-1}\end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
(L^{\pm})_*^{2m} \;\rightarrow\; (\mathcal{L}^{\pm})_*^{2m}, \quad (L^{\pm})_*^{2m+1} \;\rightarrow\; \frac{1}{\sqrt{\hbar}}\,(\mathcal{L}^{\pm})_*^{2m+1}, \\
(\mathcal{L}^{+})_*^{2m} &:= \sum_{k=0}^{\infty}(u_{2k}^{(2m)} + u_{2k-1}^{(2m)}\pi)p^{m-k}, \\
(\mathcal{L}^{-})_*^{2m} &:= \sum_{k=0}^{\infty}(v_{2k}^{(2m)} + v_{2k+1}^{(2m)}\pi)p^{k-m}, \\
(\mathcal{L}^{+})_*^{2m+1} &:= \sum_{k=0}^{\infty}(u_{2k+1}^{(2m+1)} + u_{2k}^{(2m+1)}\pi)p^{m-k}, \\
(\mathcal{L}^{-})_*^{2m+1} &:= \sum_{k=0}^{\infty}(v_{2k-1}^{(2m+1)} + v_{2k}^{(2m+1)}\pi)p^{k-m-1},
\end{aligned} \tag{3.18}
$$

respectively. By definition, all the fields $\{u_k^{(m)},\ v_k^{(m)}\}$ with $k<0$ must be set equal to zero. In obtaining these expressions we have used substitutions (3.5) and (3.13). We note that the above symbols are not commutative in general,

$$
(\mathcal{L}^{\alpha})_*^{k}(\mathcal{L}^{\beta})_*^{m} = (-1)^{km}((\mathcal{L}^{\beta})_*^{m})^{*(k)}((\mathcal{L}^{\alpha})_*^{k})^{*(m)}, \quad \alpha,\beta = +,-, \tag{3.19}
$$

which is related to the atypical property of the Grassmann coordinate π noted above (see the remark after Eq. (3.13)).

In Lax representations (2.15) and (2.19) for the N=(1|1) 2DTL hierarchy and in defining relations (2.16), we finally replace the Lax operators with their symbols (3.17–3.18) and the generalized graded bracket (2.3) with Poisson superbracket (3.14) in accordance with

$$
\Big[\ .\ ,\ .\ \Big\} \;\rightarrow\; \hbar\,\Big\{\ .\ ,\ .\ \Big\} \tag{3.20}
$$

and then perform substitution (3.3) and take limit (3.2). This gives the expressions

$$
D_n^{\pm}\mathcal{L}^{\alpha} \;=\; \mp\alpha(-1)^{n}\Big\{(((\mathcal{L}^{\pm})_*^{n})_{-\alpha})^{*},\mathcal{L}^{\alpha}\Big\}, \quad n\in\mathbb{N}, \quad \alpha=+,-, \tag{3.21}
$$

$$
D_n^{\pm}(\mathcal{L}^{\alpha})_*^{m} \;=\; \mp\alpha(-1)^{nm}\Big\{(((\mathcal{L}^{\pm})_*^{n})_{-\alpha})^{*(m)},(\mathcal{L}^{\alpha})_*^{m}\Big\}, \quad n,m\in\mathbb{N}, \tag{3.22}
$$

$$
(\mathcal{L}^{\alpha})_*^{2m} := \Big(\ \frac{1}{2}\,\Big\{(\mathcal{L}^{\alpha})^{*},\mathcal{L}^{\alpha}\Big\}\ \Big)^{m}, \quad (\mathcal{L}^{\alpha})_*^{2m+1} := \mathcal{L}^{\alpha}\ (\mathcal{L}^{\alpha})_*^{2m} \tag{3.23}
$$

for the dispersionless N=(1|1) 2DTL hierarchy. Direct calculations confirm that obtained relations (3.22–3.23) correctly reproduce dispersionless flows (3.7–3.11) and are therefore the sought Lax representation for the dispersionless N=(1|1) 2DTL hierarchy.

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