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Free Fermion Branches in some Quantum Spin Models

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Abstract: Based on numerical observations of the eigenspectra of the $SU(N)$ Perk-Schultz model at the special value of the anisotropy $q = \exp(i\pi(N-1)/N)$, we formulate a set of conjectures concerning the existence of free-fermion like eigenenergies. We prove analytically part of these conjectures

1. Introduction

The Bethe ansatz and its generalizations (see [1]) proved along the years to be a quite efficient method for the derivation of solutions of exact integrable quantum chains and transfer matrices in statistical mechanics. In the framework of this ansatz the wave functions are given in terms of combinations of plane waves whose quasi-momenta are given by a highly non-trivial set of equations, i. e., the so called Bethe ansatz equations (BAE). Despite the integrability of the model being a virtue that independs of its lattice size, these BAE are in general quite complicated and analytical solutions are known only in the thermodynamic limit, or in some exceptional cases, for general values of the lattice size. These exceptions happen for the XXZ chain at the special values of its anisotropy $\Delta = 0$ and $\Delta = -1/2$. In the first case we can derive all the solutions of the BAE since the model reduces to a free-fermion model. However for the special anisotropy $\Delta = -1/2$ the model is interacting and the analytical solution for the BAE is known only for the ground state [2, 3].

Motivated by these analytical solutions of the BAE, and seeking for new solutions, we study numerically a generalization of the XXZ ($SU(2)$), namely the anisotropic Sutherland model or the N-component Perk-Schultz model ($SU(N)$). In this case the BAE are even more complicated than those of the XXZ chain ($SU(2)$), since it is of nested type (NBAE).

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Our numerical observations show us regularities in the eigenspectra that could be explained analytically by the existence of free-fermion-like solutions, i. e., solutions where the roots of the NBAE depends only on the value of the lattice size, like in the free-fermion model. In this report we present in terms of conjectures our numerical observations, and we are going to prove some of them. This report is organized as follows. In the next section we introduce the model and formulate the associated NBAEs. In §3 we state nine conjectures that merge from our extensive numerical investigations. In §4 and §5 we explain analytically two of these conjectures and in §6 we present our conclusion.

2. The $SU(N)$ Perk-Schultz model

The $SU(N)$ Perk-Schultz model [7] is the anisotropic version of the $SU(N)$ Sutherland model [8] with the Hamiltonian, in a L-site chain, given by

$$H_q^p = \sum_{j=1}^{L-1} H_{j,j+1} + pH_{L,1} \qquad (p = 0, 1), \tag{1}$$

where

$$H_{i,j} = -\sum_{a=0}^{N-1} \sum_{b=a+1}^{N-1} (E_i^{ab} E_j^{ba} + E_i^{ba} E_j^{ab} - qE_i^{aa} E_j^{bb} - 1/q E_i^{bb} E_j^{aa}). \tag{2}$$

The $N \times N$ matrices E^{ab} have elements $(E^{ab})_{cd} = \delta_c^a \delta_d^b$ and $q = \exp(i\eta)$ plays the role of the anisotropy of the model. The cases of free and periodic boundary conditions are obtained by setting $p = 0$ and $p = 1$ in (1), respectively. This Hamiltonian describes the dynamics of a system containing N classes of particles $(0, 1, ..., N-1)$ with on-site hard-core exclusion. The number of particles belonging to each specie is conserved separately. Consequently the Hilbert space can be splitted into block disjoint sectors labeled by $(n_0, n_1, ..., n_{N-1})$, where $n_i = 0, 1, ..., L$ is the number of particles of specie i (i=0,1,...,N-1). The Hamiltonian (1) has a S_N symmetry due to its invariance under the permutation of distinct particles species, that implies that all the energies can be obtained from the sectors $(n_0, n_1, ..., n_{N-1})$, where $n_0 \leq n_1 \leq ... \leq n_{N-1}$ and $n_0 + n_1 + ... + n_{N-1} = L$.

At $q = 1$ the model is $SU(N)$ invariant and for $q \neq 1$ the model has a $U(1)^{\otimes N-1}$ symmetry as a consequence of the above mentioned conservation. Moreover in the special case of free boundaries ($p = 0$), the quantum chain (1) has a larger symmetry, being invariant under the additional quantum $SU(N)_q$ symmetry. This last invariance implies that all the eigenenergies belonging to the sector $(n'_0, n'_1, ..., n'_{N-1})$ with $n'_0 \leq n'_1 \leq ... \leq n'_{N-1}$ are degenerated with the energies belonging to the sectors $(n_0, n_1, ..., n_{N-1})$ with $n_0 \leq n_1 \leq ... \leq n_{N-1}$, if $n'_0 \leq n_0$ and $n'_0 + n'_1 \leq n_0 + n_1$ and so on up to $n'_0 + n'_1 + \ldots + n'_{N-2} \leq n_0 + n_1 + ... n_{N-2}$.

The NBAE that give the eigenenergies of the $SU_q(N)$ Perk-Schultz model in the sector whose number of particles is $(n_i, i = 0, \ldots, N-1)$ are given by (see e. g. [9],[10])

$$\prod_{j=1, j\neq i}^{p_k} F(u_i^{(k)}, u_j^{(k)}) = \prod_{j=1}^{p_{k-1}} f(u_i^{(k)}, u_j^{(k-1)}) \prod_{j=1}^{p_{k+1}} f(u_i^{(k)}, u_j^{(k+1)}), \tag{3}$$

where $k = 0, 1, ..., N-2$ and $i = 1, 2..., p_k$. The integer parameters p_k are given by

$$p_k = \sum_{i=0}^{k} n_i, \quad k = 0, 1, ..., N-2; \quad p_{-1} = 0, \quad p_{N-1} = L. \tag{4}$$

The functions $F(x, y)$ and $f(x, y)$ are defined by:

$$F(x, y) = \frac{\sin(x - y - \eta)}{\sin(x - y + \eta)}, \quad f(x, y) = \frac{\sin(x - y - \eta/2)}{\sin(x - y + \eta/2)}, \tag{5}$$

for periodic boundary conditions, and by

$$F(x, y) = \frac{\cos(2y) - \cos(2x - 2\eta)}{\cos(2y) - \cos(2x + 2\eta)}, \quad f(x, y) = \frac{\cos(2y) - \cos(2x - \eta)}{\cos(2y) - \cos(2x + \eta)}, \tag{6}$$

for the free boundary case. In using the NBAE (3) we deal with variables of different order $\{u_j^{\{k\}}\}$ $(k = 0, \ldots, N-2)$. The number of variables $u_i^{(k)}$ of order k is equal to p_k. The whole set of NBAE consists of subsets of order k which contain precisely p_k equations $(k = 0, 1, ..., N-2)$.

The eigenenergies of the Hamiltonian (1) in the sector $(n_0, n_1, ..., n_{N-1})$ are given in terms of the roots $\{u_j^{(N-2)}\}$:

$$E = -\sum_{j=1}^{p_{N-2}} \left(-q - \frac{1}{q} + \frac{\sin(u_j - \eta/2)}{\sin(u_j + \eta/2)} + \frac{\sin(u_j + \eta/2)}{\sin(u_j - \eta/2)} \right), \tag{7}$$

where to simplify the notation we write $u_j \equiv u_j^{(N-2)}$, $(j = 1, \ldots, p_{N-2})$.

All the solutions of the NBAE (3) which are going to be described in this report satisfy the additional "free-fermion" conditions (FFC):

$$f^L(u_i, 0) = 1, \quad i = 1, ..., p_{N-2}. \tag{8}$$

In this case, from (6) and (7) the corresponding eigenenergies of the Hamiltonian (1) for the case of free boundaries are given by

$$E = -2 \sum_{j=1}^{p_{N-2}} \left(-\cos\eta + \cos\frac{\pi k_j}{L} \right), \ 1 \le k_j \le L-1. \tag{9}$$

On the other hand, for periodic boundaries we have found solutions for the $SU(3)$ model with $\eta = 2\pi/3$, and for this case relations (5), (7) and (8) give us

$$E = -\sum_{j=1}^{p_1} \left(1 + 2\cos\frac{2\pi k_j}{L} \right), \ 1 \le k_j \le L. \tag{10}$$

With the additional set of equations merged from the FFC (8), the number of equations in (3) and (8) exceeds the number of variables by p_{N-2}. At the first glance we would expect no chance to satisfy the whole system given by (3) and (8). But, surprisingly, the NBAE possess some hidden symmetry.

3. Conjectures merged from numerical studies

In this section we state a series of conjectures that are consistent with the exact brute-force diagonalization of the Hamiltonian (1) with free ($p = 0$) and periodic ($p = 1$) boundary conditions. Some of these conjectures are going to be proved in the following sections. Let us consider separately the case of periodic and free boundaries.

3a - Periodic chain.

In the periodic case we only found important regularities in the eigenspectrum of the $SU(3)$ model at $q = \exp(i2\pi/3)$ and we state the following conjectures.

Conjecture 1. *The Hamiltonian (1) with L sites at $q = \exp(2i\pi/3)$ has eigenvectors (not all of them) with energy and momentum given by*

$$E_I = -\sum_{j\in I}(1 + 2\cos\frac{2\pi j}{L}), \tag{11}$$

$$P_I = \frac{2\pi}{L}\sum_{j\in I} j, \tag{12}$$

with I being a subset of $\mathcal{I}$ unequal elements of the set $\{1, 2, ..., L\}$. The number $\mathcal{I}$ has to be odd $\mathcal{I} = 2k+1$ and the sector of appearance of the above levels is $S_k \equiv (k, k+1, L-2k-1)$, $0 \le k \le (L-1)/2$.

The lowest eigenenergy among the above conjectured values (11) is obtained for the particular set $I_0^{(k)} = \{1, 2, ..., k\} \cup \{L-k, ..., L\}$, since in this case all contributions $-(1 + 2\cos\frac{2\pi j}{L})$ to (11) have the lowest possible values. The corresponding eigenstate has zero momentum and energy given by

$$E_0^{(k)} = -\sum_{j\in I_0^{(k)}}(1 + 2\cos\frac{2\pi j}{L}) = -2k - 1 - 2\frac{\sin(\pi(2k+1)/L)}{\sin(\pi/L)}. \tag{13}$$

Conjecture 2. *For arbitrary $L = 3n + l$ ($l = 1, 2, 3$), the eigenenergy $E_0^{(n)}$ is the lowest one in the sector S_n.*

Conjecture 3. *For arbitrary $L = 3n+l$ ($l = 1, 2$), the eigenenergy $E_0^{(n)}$ is the ground-state energy of the model.*

3b - Free boundaries.

In order to announce the conjectures let us define the special sectors

$$S_k = ([\frac{k}{N-1}], [\frac{k+1}{N-1}], ..., [\frac{k+N-2}{N-1}], L-k), \qquad k = 0, 1, ..., L, \tag{14}$$

where $[x]$ means the integer part of x. For example, for $N = 4$ and $L = 7$ the sectors are

$$\begin{aligned} &S_0 = (0,0,0,7), \quad S_1 = (0,0,1,6), \quad S_2 = (0,1,1,5), \\ &S_3 = (1,1,1,4), \quad S_4 = (1,1,2,3), \quad S_5 = (1,2,2,2), \\ &S_6 = (2,2,2,1), \quad S_7 = (2,2,3,0), \end{aligned} \tag{15}$$

and due to the quantum symmetry of the Hamiltonian we have the special ordering:

$$S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \equiv S_6 \supset S_7. \tag{16}$$

This means that, for example, all eigenvalues found in the sector S_3 can also be found in the sectors S_4, S_5 and S_6, and all eigenvalues appearing in the sector S_7 can also be found in the sectors S_6 and S_5. The sectors S_5 and S_6 are totally equivalent. In this example let us call the sectors $S_0, S_1, S_2, S_3, S_4, S_5$ as the *left* sectors and S_6, S_7 as the *right* ones.

We can generalize this definition to any $L = Nn+r$, where n and r are natural numbers and $r < N$, obtaining $L-n$ left sectors and $n+1$ right ones. Now we can formulate the conjectures merged form our bruteforce diagonalizations.

Conjecture 4. *For $L = Nn + r$ $(r = 1, 2, \dots, N-1)$ the Hamiltonian (1) with $p = 0$ and $q = \exp(i\pi/N)$ has eigenvalues given by*

$$E_I = -2 \sum_{j \in I} (\cos(\frac{\pi}{N}) + \cos(\frac{j\pi}{L})), \tag{17}$$

where I is an arbitrary subset of the set $\{1, 2, ...L-1\}$. If k is the number of elements of the subset I and also S_k is a left sector then we find the eigenvalues (17) in the sectors $S_k, S_{k+1}, ...S_{L-n}$. On the other hand if S_k is a right sector then we find the eigenvalues (17) in the sectors $S_{L-n-1}, S_{L-n}, \dots, S_{k+1}$.

For $L = Nn$ we have slightly more delicate picture. In this case we have the left sectors, the right sectors and a central one (n, n, n, n).

Conjecture 5. *For $L = Nn$ we can use conjecture 4 considering the central sector as a left or a right one.* This is possible due to the coincidence, apart of degeneracies, of the eigenenergies in the eigensectors $(n, n, n, \dots, n)$ and $(n-1, n, n, \dots, n+1)$.

Consider now the a special subsets $I = \{1, 2, 3, \dots, k\}$, $k = 0, 1, \dots, L-1$. Due to the conjectures 4 and 5 the Hamiltonian (1) has the corresponding eigenvalues

$$E^{(k)} = -2 \sum_{j=1}^{k} (\cos(\frac{\pi}{N}) + \cos(\frac{j\pi}{L})) = 1 - 2k\cos(\frac{\pi}{N}) - \frac{\sin \pi(2k+1)/2L}{\sin \pi/2L}. \tag{18}$$

We can now formulate the remarkable conjecture:

Conjecture 6. *The eigenvalues (18) are the lowest eigenenergies in the special sectors (14) of the Hamiltonian (1) with anisotropy $q = \exp(i(N-1)\pi/N)$. Namely, for the left sectors we have $E_{min}(S_k) = E^{(k)}$ while for the right sectors $E_{min}(S_k) = E^{(k-1)}$.*

We can add here an additional conjecture:

Conjecture 7. *The eigenvalues (18) for $k = L-n-1$ gives the ground state energy of the Hamiltonian:*

$$E_0 = 1 + 2(1-L+n)\cos(\frac{\pi}{N}) - \frac{\sin \pi(2n+1)/2L}{\sin \pi/2L}. \tag{19}$$

The last of the special sectors is

$$S_L = ([\frac{L}{N-1}], [\frac{L+1}{N-1}], ..., [\frac{L+N-2}{N-1}], 0), \qquad k = 0, 1, ..., L. \tag{20}$$

In this sector we have only $N-1$ classes of particles and the Hamiltonian (1) is reduced effectively to a $SU_q(N-1)$-invariant quantum spin model. The sector is of right type and due to conjecture 5 we can state the following conjecture:

Conjecture 8. *The eigenvalue (18) for $k = L-1$ gives the ground state energy of the $SU_q(N-1)$ Hamiltonian with anisotropy $q = \exp(i(N-1)\pi/N)$. This eigenvalue can be written as*

$$E_0 = -2(L-1)\cos(\frac{\pi}{N}). \tag{21}$$

For $N = 3$, for example, we get the XXZ model with anisotropy $\Delta = -1/2$ and $E_0 = 1-L$, a result that was first observed in [2] and produced quite interesting consequences [3, 4].

In addition to the above conjectures we have also verified the following conjecture that is valid for arbitrary values of the anisotropy.

Conjecture 9. The special sector $(n_0, \ldots, n_{N-1}) = (1, \ldots, 1, L-N+1)$ of the $SU_q(N)$ model have energies

$$E = \sum_{j=1}^{N}(q + \frac{1}{q} - 2\cos(\frac{2\pi}{L}k_j)), \quad 1 \le k_j \le L, \tag{22}$$

where q is arbitrary.

We are able to explain analytically the conjectures 1,4 and 9. For brevity in the following two sections we are going to present these explanations for the conjectures 1 and 4.

3. Free-fermion spectrum for the $SU(3)$ model with $q = \exp(2i\pi/3)$: NBAE solutions.

In this section we give an analytical explanation for the conjecture 1 of previous section. These solutions happen in the sector $(n_0, n_1, n_2) = (k, k+1, L-2k-1)$ $(0 \le k \le (L-1)/2)$ of the periodic $SU(3)$ model at $q = \exp(i\frac{2\pi}{3})$. In [5] exploring the functional equations related to the NBAE we were able to explain partially this conjecture. These solutions however do not belong to the numerically predicted sectors and do not satisfy the usual bounds $p_0 = n_0 < p_1 = n_0 + n_1 < L$. Moreover it is not clear if they correspond to non-zero norm eigenfunctions. In this section we review [6] direct explanation of these conjectures without the use of functional equations.

The NBAEs for the anisotropic $SU(3)$ Perk-Schultz model with anisotropy $q = \exp(2i\pi/3)$, with roots satisfying the FFC, can be written as follows:

$$\prod_{j=1,j\neq i}^{p_0} f(v_i, v_j) \prod_{j=1}^{p_1} f(v_i, u_j) = 1, \quad i = 1, 2, \ldots, p_0, \tag{23}$$

$$\prod_{j=1,j\neq i}^{p_1} f(u_i, u_j) \prod_{j=1}^{p_0} f(u_i, v_j) = 1, \quad i = 1, 2, \ldots, p_1, \tag{24}$$

$$f^L(u_i, 0) = 1, \quad i = 1, 2, ..., p_1, \tag{25}$$

where $p_0 = n_0$, $p_1 = n_0 + n_1$ and we have used the relation

$$F(x, y) = 1/f(x, y), \tag{26}$$

which is valid for $q = \exp(\frac{2i\pi}{3})$ or $\eta = \frac{2\pi}{3}$.

Before considering the general case let us restrict ourselves initially to the particular eigensector $(1, 2, L-3)$. We have in this case $p_0 = n_0 = 1$ and $p_1 = n_0 + n_1 = 3$. For this simple case the first subsystem (23) consists of a single equation:

$$f(v, u_1)f(v, u_2)f(v, u_3) = 1. \tag{27}$$

From the definition of the function f given in (5) we can show that the last equation (27) is equivalent to

$$\begin{aligned} &\cos(v + u_1 - u_2 - u_3) + \cos(v - u_1 + u_2 + u_3) + \\ &+ \cos(v - u_1 - u_2 + u_3) = 0. \end{aligned} \tag{28}$$

It is clear that this relation has the S_3 symmetry under the permutation of the variables u_1, u_2 and u_3. Surprisingly it has also the S_4 symmetry under the permutation of the variables u_1, u_2, u_3 and v!

The three equations of the second subsystem (24) can be obtained from (23) by the permutations $v \leftrightarrow u_i$, so they also can be reduced to (28). Fixing the variables u_i, $i = 1, 2, 3$, satisfying the FFC (25) and finding v from equation (28) we obtain the solution for the whole system (23-25) in the eigensector (1,2,L-3).

We show now that this method can be generalized to any sector $(k, k+1, L-2k-1)$, supporting the conjecture 1 of our previous paper [5]. Let us consider the set $\{x_i, i = 1, \ldots, p_0 + p_1\}$ of variables formed by the union of the two systems of variables:

$$\begin{aligned} x_i &= v_i \quad (i = 1, 2, \ldots, p_0), \\ x_{i+p_0} &= u_i \quad (i = 1, 2, \ldots, p_1). \end{aligned} \tag{29}$$

In terms of these variables the system of equations (23-25) becomes

$$r(x_i) = f(x_i, x_i), \quad i = 1, 2, \ldots, p_0 + p_1, \tag{30}$$

$$f^L(x_i, 0) = 1, \quad i = p_0 + 1, \ldots, p_0 + p_1, \tag{31}$$

where $r(x) = \prod_{j=1}^{p_0+p_1} f(x, x_j)$. The first of these subsystems (30) possess $S_{p_0+p_1}$ permutation symmetry. We intend to show now that many of the equations in this subsystem are dependent, and for $p_0 = k$ and $p_1 = 2k+1$ we can also satisfy independently the FFC (31). From (5) we may write

$$r(x) = \prod_{j=1}^{p_0+p_1} \frac{\sin(x - x_j - \pi/3)}{\sin(x - x_j + \pi/3)} = \prod_{j=1}^{p_0+p_1} \frac{b - qb_j}{qb - b_j}, \tag{32}$$

where for convenience we introduced the new variables

$$b = \exp(2ix), \quad b_j = \exp(2ix_j), \quad q = \exp(2i\pi/3). \tag{33}$$

The first subsystem (30) becomes now a system of algebraic equations:

$$\prod_{j=1}^{p_0+p_1}(b_i - q\, b_j) + \prod_{j=1}^{p_0+p_1}(q\, b_i - b_j) = 0 \quad i = 1, 2, \ldots, p_0 + p_1. \tag{34}$$

Using the standard symmetric functions:

$$\begin{aligned}
&S_0 = 1, \\
&S_1 = b_1 + b_2 + \ldots + b_{p_0+p_1}, \\
&S_2 = b_1\, b_2 + b_1\, b_3 + \ldots b_{p_0+p_1-1}\, b_{p_0+p_1}, \\
&\ldots \\
&S_m = \sum_{1 \le i_1 < i_2 < \ldots i_m \le p_0+p_1} b_{i_1}\, b_{i_2}\, \ldots b_{i_m}, \\
&\ldots \\
&S_{p_0+p_1} = b_1\, b_2 \ldots b_{p_0+p_1},
\end{aligned} \tag{35}$$

we can rewrite (34) as

$$\sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} (q^m + q^{p_0+p_1-m}) = 0. \tag{36}$$

Adding this last equation to the identity

$$q^{-p_0-p_1} \prod_{j=1}^{p_0+p_1} (b_i - b_j) = \sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} q^{-p_0-p_1} = 0, \tag{37}$$

we obtain

$$\sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} (q^m + q^{p_0+p_1-m} + q^{-p_0-p_1}) = 0. \tag{38}$$

For $q = \exp(2i\pi/3)$ we have the following possibilities

$$\begin{aligned}
q^m + q^{p_0+p_1-m} + q^{-p_0-p_1} &= 3q^{-p_0-p_1} \text{ for } p_0 + p_1 + m = 3n, \\
q^m + q^{p_0+p_1-m} + q^{-p_0-p_1} &= 0 \text{ for } p_0 + p_1 + m \neq 3n.
\end{aligned} \tag{39}$$

Let $p_0 + p_1 = 3k + r$, where k is an integer and $r \in \{-1, 0, 1\}$. Inserting (39) into (38) we obtain

$$\begin{aligned}
&\sum_{\mu=0}^{k} (-1)^\mu S_{3\mu} b_i^{-3\mu} = 0 \text{ for } r = 0, \\
&\sum_{\mu=0}^{k-1} (-1)^\mu S_{3\mu+2} b_i^{-3\mu} = 0 \text{ for } r = 1, \\
&\sum_{\mu=0}^{k-1} (-1)^\mu S_{3\mu+1} b_i^{-3\mu} = 0 \text{ for } r = -1,
\end{aligned} \tag{40}$$

where $i = 1, 2, ..., p_0 + p_1$. We then see that the subsystem (30) can be satisfied if we impose

$$S_{3\nu+\rho} = 0, \;\; \nu = 0, 1, ..., k-1, \tag{41}$$

where $\rho = 0$ for $r = 0$, $\rho = 2$ for $r = 1$ and $\rho = 1$ for $r = -1$. Since $S_0 = 1$ it is not possible to obtain solutions of this type for $L = 3k$ ($\rho = 0$) and we have to limit ourselves to the cases where $p_0 + p_1 \neq 3k$.

Let us fix now p_1 variables u_i, satisfying the FFC (31). Variables v_i, $i = 1, 2, ..., p_0$ can, in principle, be found from the system (41). In order to do that we use the decomposition of the symmetric functions depending on two set of variables, namely

$$\begin{aligned} &S_0 = 1, \\ &S_1 = s_1 + \sigma_1, \\ &S_2 = s_2 + s_1\sigma_1 + \sigma_2, \\ &... \\ &S_m = \sum_{k=max\{0,m-p_0\}}^{min\{m,p_1\}} s_{m-k}\sigma_k, \\ &... \\ &S_{p_0+p_1} = s_{p_0}\sigma_{p_1}, \end{aligned} \tag{42}$$

where σ_i are the symmetric combination of the known variables u_i and s_i are the symmetric combinations of the unknown variables v_j. Consequently the system (41) can be reduced to a linear system for symmetric functions s_i, $i = 1, 2, ..., p_0$. This system can be solved if the number of variables p_0 is greater or equal to the number of equations k. Then we have the system

$$\begin{aligned} &p_0 = n_0, \quad p_1 = n_0 + n_1, \quad n_0 \leq n_1, \\ &p_0 + p_1 = 3k \pm 1, \quad k \leq p_0. \end{aligned} \tag{43}$$

These relations give us the constraint $n_0 = k$ and $n_1 = k + 1$, and consequently $p_0 + p_1 = 2n_0 + n_1 = 3k + 1$, implying that solutions of (41) exist only for $\rho = 2$. These solutions happen in the sector $(k, k+1, L-2k-1)$. Consider for illustration the case $k = 2$ ($p_0 = 2$ and $p_1 = 5$). We have 5 variables $u_1, ..., u_5$, which we fix with the FFC (31), and 2 unknown variables v_1, v_2. Using (42) we can write the system (41) as follows:

$$\begin{aligned} &S_2 = s_2 + s_1\sigma_1 + \sigma_2 = 0, \\ &S_5 = s_5 + s_4\sigma_1 + s_3\sigma_2 = 0. \end{aligned} \tag{44}$$

The functions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 we know, so we have 2 linear equations for $s_1 = v_1 + v_2$ and $s_2 = v_1 v_2$.

This conclude our proof of conjecture 1. The extension of this conjecture to the free boundary case can be done along the same lines [6].

4. Free-fermion solutions of the NBAE for generic q

In this section we explain the conjecture 9 of §3. Differently from the other conjectures, where the free-fermion solutions were found for a specific value of q, we are going present free-fermion solutions of the NBAE that are valid for arbitrary values of q. These solutions are valid only for free boundary conditions and will happen in the special sectors of the $SU(N)_q$ model where the number of particles of each distinct specie is at most one, except for one of the species, that can be considered as the background (holes for example). The $S(U)_q$ symmetry ensures that the general sector containing all these solutions is the special sector where $(n_0, \ldots, n_{N-1}) = (1, \ldots, 1, L-N+1)$, that gives from the definition (4) the values

$$p_0 = 1, p_1 = 2, ..., p_{N-2} = N-1, p_{N-1} = L. \tag{45}$$

For this case the NBAE (3) consist of $N-1$ subsets ($k = 0, 1, \ldots, N-1$) of equations where the kth subset has precisely $k+1$ equations.

To illustrate our procedure let us consider for simplicity the $SU(4)$ model in the sector $(1, 1, 1, L-3)$. In this case we have three subsets ($k = 0, 1, 2$) and three groups of variables $(u^{(0)}, u^{(1)}, u^{(2)})$:

$$1 = f(w, v_1) f(w, v_2), \tag{46}$$

$$\begin{aligned} F(v_1, v_2) &= f(v_1, w) \times f(v_1, u_1) f(v_1, u_2) f(v_1, u_3), \\ F(v_2, v_1) &= f(v_2, w) \times f(v_2, u_1) f(v_2, u_2) f(v_2, u_3), \end{aligned} \tag{47}$$

$$\begin{aligned} F(u_1, u_2) F(u_1, u_3) &= f(u_1, v_1)\, f(u_1, v_2), \\ F(u_2, u_1) F(u_2, u_3) &= f(u_2, v_1)\, f(u_2, v_2), \\ F(u_3, u_1) F(u_3, u_2) &= f(u_3, v_1)\, f(u_3, v_2), \end{aligned} \tag{48}$$

where we write w instead of $u^{(0)}$, v_i instead of $u_i^{(1)}$ ($i = 1, 2$) and u_i instead of $u_i^{(2)}$ ($i = 1, 2, 3$). Inserting (6) into the first equation (46) we obtain promptly two possibilities:

$$\sin 2w = 0, \tag{49}$$

$$\cos 2v_1 + \cos 2v_2 - 2\cos\eta \cos 2w = 0. \tag{50}$$

We do not intend to consider for the moment the first possibility (49). Imposing the condition (50) we can check (eliminating, for example the variable w) that for (v_1, v_2, w) satisfying (50), besides (46) we have also two additional equalities, namely

$$\begin{aligned} F(v_1, v_2) &= f(v_1, w), \\ F(v_2, v_1) &= f(v_2, w). \end{aligned} \tag{51}$$

Consequently the second subsystem becomes:

$$\begin{aligned} 1 &= f(v_1, u_1)\, f(v_1, u_2)\, f(v_1, u_3), \\ 1 &= f(v_2, u_1)\, f(v_2, u_2)\, f(v_2, u_3). \end{aligned} \tag{52}$$

We will show bellow that the set of 5 equations formed by the second (52) and the third subsets (48) contain only two independent equations, and consequently we may fix the variables $u_i (i = 1, 2, 3)$ by imposing the FFC (31 and find the two variables v_1 and v_2 from these equations. The remaining variable w is then obtained from (50). Consequently we find the free-fermion eigenspectra (9) for arbitrary values of q or η.

The previous analysis can be extended for the sector $(1, \ldots, 1, L-N+1)$ for general $SU(N)_q$ by exploring the general theorem:

For any k fixed numbers $u_1, u_2, ...u_k$ one can find $k-1$ numbers $v_1, v_2, ..., v_{k-1}$ satisfying the two set of equations

$$\prod_{j=1}^{k} f(v_i, u_j) = 1 \quad (i = 1, 2, ..., k-1), \tag{53}$$

$$\prod_{j=1, j\neq i}^{k} F(u_i, u_j) = \prod_{j=1}^{k-1} f(u_i, v_j) \quad (i = 1, 2, ..., k). \tag{54}$$

Leaving the proof of the above theorem for the moment, we can see that applying this theorem to $(k-1)$ known numbers $v_1, \ldots, v_{k-1}$, one can find the numbers $w_1, \ldots, w_{k-2}$ satisfying the equations

$$\prod_{j=1}^{k-1} f(w_i, v_j) = 1 \quad (i = 1, 2, ..., k-2), \tag{55}$$

$$\prod_{j=1, j\neq i}^{k-1} F(v_i, v_j) = \prod_{j=1}^{k-2} f(v_i, w_j) \quad (i = 1, 2, ..., k-1). \tag{56}$$

Multiplying the ith $(i = 1, \ldots, k-1)$ equation of the sets (53) and (56) we obtain literally one of the subset of the NBAE:

$$\prod_{j=1, j\neq i}^{k-1} F(v_i, v_j) = \prod_{j=1}^{k-2} f(v_i, w_j) \prod_{j=1}^{k} f(v_i, u_j) \quad (i = 1, 2, ..., k). \tag{57}$$

Applying the above theorem $k-1$ times we obtain a tower of numbers:

$$\begin{array}{c} u_1, u_2, u_3 ......u_{k-2}, u_{k-1}, u_k \\ v_1, v_2 .....v_{k-2}, v_{k-1} \\ w_1, ...w_{k-2} \\ .... \\ .... \\ y_1, y_2 \\ z \end{array}$$

This imply that if we begin by fixing the $k = N-1$ variables u_i $(i = 1, \ldots, k)$, satisfying the FFC (31) we obtain by using recursively the construction (57) the solution of the NBAE of the $SU(N)$ Perk-Schultz model, with free boundaries, in the sector $(n_0, n_1, \ldots, n_{N-1}) = (1, 1, \ldots, 1, L-N+1)$. The free-fermion-like energies are given by (9) for arbitrary values of η. The previous results (48) and (52) are just consequences of the particular case where $k=3$.

Let us now prove the announced theorem. Let us fix $\{u_j\}, j = 1, 2, ..., k$. The equation (53) can then be written as follows

$$P(v_i) = 0, \quad (i = 1, 2, ..., k) \tag{58}$$

where

$$P(v) \equiv \prod_{j=1}^{k} \Big(\cos 2u_j - \cos(2v-\eta) \Big) - \prod_{j=1}^{k} \Big(\cos 2u_j - \cos(2v+\eta) \Big). \tag{59}$$

The use of the symmetric functions

$$\begin{aligned} S_0 &= 1 \\ S_1 &= \cos 2u_1 + \cos 2u_2 + \cdots + \cos 2u_{p_0+p_1} \\ &\cdots \\ S_{p_0+p_1} &= \cos 2u_1 \cos 2u_2 \cdots \cos 2u_{p_0+p_1} \end{aligned} \tag{60}$$

allow us to write

$$P(v) \equiv \sum_{m=1}^{k} (-1)^{m+1} S_{k-m} \Big(\cos^m(2v-\eta) - \cos^m(2v+\eta) \Big). \tag{61}$$

Since $\cos(2v-\eta) - \cos(2v+\eta) = 2\sin\eta \sin 2v$ and $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + ... + b^{m-1})$ we have

$$P(v) \equiv \sin 2v \; p(\cos 2v), \tag{62}$$

where $p(t)$ is a polynomial of degree $k-1$. We can factorize this polynomial, and apart form a multiplicative constant (Ω)we can write

$$P(v) = \Omega \sin 2v \prod_{i=1}^{k-1} (\cos 2v - \cos 2b_i). \tag{63}$$

Now (58) is easily solved: $v_i = b_i, \;\; i = 1, 2, ..., k-1$.

Consider now the right side of the second equation (54). The relation (63) allow us to write

$$\begin{aligned} \prod_{i=1}^{k-1} f(u_j, v_i) &= \prod_{i=1}^{k-1} f(u_j, b_i) = \prod_{i=1}^{k-1} \frac{\cos(2u_i - \eta) - \cos 2b_i}{\cos(2u_i + \eta) - \cos 2b_i} = \\ &= \frac{P(2u_j - \eta)}{\sin(2u_j - \eta)} \frac{\sin(2u_j + \eta)}{P(2u_j + \eta)}. \end{aligned} \tag{64}$$

Using the expression (59) for $P(v)$ we obtain:

$$\begin{aligned} \prod_{i=1}^{k-1} f(u_j, v_i) &= \frac{\sin(2u_j + \eta)}{\sin(2u_j - \eta)} \\ &\times \frac{\prod_{i=1}^{k} (\cos 2u_i - \cos(2u_j - 2\eta)) - \prod_{i=1}^{k} (\cos 2u_i - \cos 2u_j)}{\prod_{i=1}^{k} (\cos 2u_i - \cos 2u_j) - \prod_{i=1}^{k} (\cos 2u_i - \cos(2u_j + 2\eta))}. \end{aligned} \tag{65}$$

Since $\prod_{i=1}^{k}(\cos 2u_i - \cos 2u_j) = 0$, we obtain

$$\prod_{i=1}^{k-1} f(u_j, v_i) = -\frac{\sin(2u_j+\eta)}{\sin(2u_j-\eta)} \prod_{i=1}^{k} \frac{\cos 2u_i - \cos(2u_j - 2\eta)}{\cos 2u_i - \cos(2u_j + 2\eta)}. \tag{66}$$

One can also check that

$$\frac{\sin(2u_j+\eta)}{\sin(2u_j-\eta)} = -\frac{\cos 2u_j - \cos(2u_j + 2\eta)}{\cos 2u_j - \cos(2u_j - 2\eta)}, \tag{67}$$

so that (66) can be written as

$$\prod_{i=1}^{k-1} f(u_j, v_i) = \prod_{i=1, i\neq j}^{k} F(u_j, u_i), \tag{68}$$

concluding the proof of the theorem.

We can also calculate [6] the eigenfunction corresponding to a given free-fermion energy

$$E = \sum_{j=1}^{N-1} \epsilon_j = \sum_{j=1}^{N-1} (q + \frac{1}{q} - x_j - \frac{1}{x_j}), \tag{69}$$

where

$$x_j = \exp(i\frac{\pi k_j}{L}), \quad 1 \leq k_j \leq L-1. \tag{70}$$

The wavefunction is given by

$$|\psi_{\{x_1,\ldots,x_{N-1}\}} >= \sum_{m_1,\ldots,m_{N-1}=1}^{L} q^{-f(m_1,\ldots,m_{N-1})}$$

$$\times \det \begin{vmatrix} \Psi_1(m_1) & \Psi_1(m_2) & \cdots & \Psi_1(m_{N-1}) \\ \Psi_2(m_1) & \Psi_2(m_2) & \cdots & \Psi_2(m_{N-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \Psi_{N-1}(m_1) & \Psi_{N-1}(m_2) & \cdots & \Psi_{N-1}(m_{N-1}) \end{vmatrix} |m_1, \ldots, m_{N-1} >,$$

where $|m_1, \ldots, m_{N-1} >$ are the vector basis representing the configuration where the ith particle is located at site m_i ($i = 1, \ldots, N-1$, $1 \leq m_i \leq L$). The Slater determinants entering in (71) depend upon the set of one-particle amplitudes $\Psi_j(m)$, that apart from a normalization factor are given by

$$\Psi_j(m) = \left(1 - \frac{q}{x_j}\right) x_j^m - (1 - q x_j)/x_j^m, \tag{71}$$

and the factor $f(m_1, \ldots, m_{N-1})$ is the minimum number of pair permutations $(m_i, m_j) \to (m_j, m_i)$ necessary to put them in an increasing order $m_1 < m_2 < \cdots < m_{N-1}$.

7. Conclusions

We have found many solutions of the NBAE for the $SU(N)$ Perk-Schultz model. However despite the eigenenergies being free-fermion like, the corresponding eigenvectors are not simple to derive. Actually we are able to show [11] that the appearance of these special free-fermion-like energies in the eigenspectra of the $SU(N)$ quantum chain with free boundaries is due to existence of a factorizable eigenvalue of the associated transfer matrix of the inhomogeneous $SU_q(N)$ model at $q = \exp(i\frac{\pi(N-1)}{N})$. The components of this special eigenvector provide us with the recurrence relations necessary for the calculation of the wave vector corresponding to the free-fermion solutions [11]. We believe that as a byproduct, these observations will provide a road for the calculation of the components of the ground state eigenfunction of the XXZ chain with anisotropy $\Delta = -\frac{1}{2}$, where many interesting conjectures were raised in the literature [3].

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