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Workshop on Integrable Theories, Solitons and Duality

Extensions of Soliton equations to non-commutative $(2+1)$ dimensions*

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Abstract: We report a strong method to generate various equations which have Lax representations on noncommutative $(1+1)$ and $(2+1)$-dimensional spaces. The generated equations contain noncommutative integrable equations obtained by using bicomplex method and by reductions of noncommutative anti-self-dual Yang-Mills equation. This suggests that noncommutative Lax equations would be integrable.

1. Introduction

Non-Commutative (NC) gauge theory has been studied intensively for the last several years and succeeded in revealing various aspects of the gauge theory in the presence of background magnetic field [1]. The noncommutativity is characterized by the commutation relations

$$[x^i,\ x^j] = i\theta^{ij}, \tag{1.1}$$

of the space-time coordinates, where θ^{ij} are real constants. This commutation relation can be extended to the associative $\star$-product

$$f \star g(x) \equiv \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x}, \tag{1.2}$$

of functions f and g on the space-time, where $\partial_i \equiv \dfrac{\partial}{\partial x_i}$ and $\partial_j \equiv \dfrac{\partial}{\partial x_j}$. The $\star$-product reduces to the ordinary product in the limit $\theta^{ij} \to 0$, which is called *the commutative limit*. And this relation (1.1) looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation". Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the distinguished

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features of NC theories and gives rise to various new physical objects. For example, even when the gauge group is $U(1)$, instanton solutions exist [2] because of the resolution of the small instanton singularities of the complete instanton moduli space [3].

NC gauge theories are naively realized from ordinary commutative theories just by replacing all products of the fields with the $\star$-product (1.2). In this context, NC theories are considered to be a deformed theories from commutative ones and look very close to commutative ones. Under the deformation, anti-self-dual Yang-Mills (ASDYM) equation could be considered to preserve the integrability in the same sense as commutative case [4]. On the other hand, with regard to typical integrable equations such as Kadomtsev-Petviasfvili (KP) equation [5], the naive NC extension generally destroys the integrability. There is known to be a method, *bicomplex method*, to yield NC integrable equations which have many conserved quantities [6].

On the other hand, a NC extension method of wider class of integrable equations has been discussed in [7]. The generated equations on NC spaces are expected to preserve their integrability. And it is very easy to propose higher dimensional equations by the method. In this paper, we will report the construction of NC equations by the method. All the results here are consistent and we can expect that the NC equations would be integrable.

This paper is organized as follows. In Sec.2, we review a strong method to give rise to NC Lax pairs and construct equations on NC $(1+1)$ dimensions. Next we extend the associated NC Lax pair for the NC KdV equation to $(2+1)$ dimensions and propose equations on NC $(2+1)$ dimensions. In Sec.4, conclusion and discussion are given.

2. On Lax-Pair Generating Technique

On commutative spaces, Lax representations are common in many of known integrable equations. In this section we briefly introduce how to find Lax representations on NC spaces [7].

2.1 Lax-Pair Generating Technique and NC KdV equation

An integrable equation on commutative spaces which has the Lax representation can be rewritten as the following equation

$$[L,\ T] \equiv LT - TL = 0, \tag{2.1}$$

where $\partial_t \equiv \dfrac{\partial}{\partial t}$. This equation and the pair of operators (L, T) are called *the Lax equation* and *the Lax pair*, respectively. The NC version of the Lax equation (2.1), or *the NC Lax equation*, is easily defined just by replacing the product of the operators L and T with the $\star$-product (1.2). Namely it is given by

$$[L,\ T]_\star \equiv L \star T - T \star L = 0. \tag{2.2}$$

The modification of the product makes the ordinary coordinate "noncommutative," which means

$$[x^i,\ x^j]_\star = x^i \star x^j - x^j \star x^i = i\theta^{ij}. \tag{2.3}$$

In this paper, we shall look for NC Lax pairs whose the operators L are differential operators. In order to make this study systematic, we set up the following problem :

Problem : For a given operator L, find the corresponding operator T which satisfies the NC Lax equation (2.2).

It is in general very difficult to solve the problem. However if we put an ansatz on the operator T, then we can get the answer for wide class of the Lax pair including NC case. The ansatz for the operator T is of the following type:

Ansatz for the operator T :

$$T = \partial_i^n L + T' + \partial_t, \tag{2.4}$$

where $n = 1, 2, \quad \cdots$ **.**

Then the problem for an associated operator T reduces to that for an unknown operator T'. This ansatz is very simple, however, very strong to determine the unknown operator T', which is called *Lax-pair generating technique* in this paper.

In order to explain it more concretely, let us consider in detail the Korteweg-de Vries (KdV) equation on $(1+1)$ dimensional NC spaces, referred as the NC KdV equation [8]. In this section the noncommutativity relation is defined by $[t,\ x] = i\theta$. The operator L is given by

$$L_{\rm KdV} = \partial_x^2 + u(t,x). \tag{2.5}$$

The ansatz for the operator T is given by

$$T_{\rm KdV} = \partial_x L_{\rm KdV} + T'_{\rm KdV} + \partial_t, \tag{2.6}$$

which is the case for $n = 1$ and $\partial_i = \partial_x$ in the general ansatz (2.4). Here is

$$\partial_x L_{\rm KdV} = \partial_x^3 + u\partial_x + u_x. \tag{2.7}$$

The ansatz (2.4) is very simple and, however, becomes very strong. Let us show the power of it.

The NC Lax equation (2.2) gives an equation for the unknown operator $T'_{\rm KdV}$

$$[L_{\rm KdV},\ T_{\rm KdV}]_\star = -u_x\partial_x^2 - (u_t + u_x \star u) + [L_{\rm KdV},\ T'_{\rm KdV}]_\star = 0, \tag{2.8}$$

where $u_x \equiv \dfrac{\partial u}{\partial x}$ and so on. Here we want to delete the term $u_x\partial_x^2$ in the RHS of (2.8) so that this equation finally reduces to a differential equation. Therefore the operator $T'_{\rm KdV}$ could be taken as

$$T'_{\rm KdV} = A\partial_x + B, \tag{2.9}$$

where A and B are polynomials of u, u_t, u_x, u_{xx} etc. Hence equation (2.8) becomes

$$[L_{\text{NEW}},\ T'_{\text{NEW}}]_\star = 2A_x\partial_x^2 + \Big(A_{xx} + [u,\ A]_\star + 2B_x\Big)\partial_x - \Big(A\star u_x - B_{xx} - [u,\ B]_\star\Big). \tag{2.10}$$

Now one can obtain A, B and C of the unknown operator T'_{KdV} (2.9) from equation (2.8) as follows. The coefficients of ∂_x^2 and ∂_x give, respectively,

$$A = \frac{1}{2}u, \quad B = -\frac{1}{4}u_x, \tag{2.11}$$

where two integral constants can be taken zero without loss of generality[1] [7]. So we get the operator T_{KdV}

$$T_{\text{KdV}} = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x. \tag{2.12}$$

Then the NC Lax equation (2.2) reads the NC KdV equation

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u\star u)_x = 0, \tag{2.13}$$

which was derived in [8].

We have reviewed the Lax-Pair Generating Technique taking the NC KdV equation as an example. The KdV hierarchy on NC spaces can be also constructed by the method. Next let us show that.

2.2 NC KdV hierarchy –NC 5th-KdV equation–

In this subsection we shall propose the KdV hierarchy on NC spaces. The operator L is $L_{\text{KdV-h}} = L_{\text{KdV}}$, and the ansatz (2.4) is the following

$$T_{\text{KdV-h}} = \partial_x^{2n-1}L_{\text{KdV-h}} + T'_{\text{KdV-h}} + \partial_t. \tag{2.14}$$

Then the NC Lax equation (2.2) gives the $(2n+1)$th-KdV equation on NC spaces, namely NC KdV hierarchy. We report lower two equations of the NC KdV hierarchy as follows.

- For $n = 1$, the NC Lax equation (2.2) gives the NC (3th-)KdV equation (2.13).

- For $n = 2$, the NC Lax equation (2.2) gives the NC 5th-KdV equation. The NC Lax pair is given by

$$L_{\text{5th}} = L_{\text{KdV}} \quad \text{and} \quad T_{\text{5th}} = \partial_x^3 L_{\text{5th}} + T'_{\text{5th}} + \partial_t, \tag{2.15}$$

[1] We will always take integral constants zero in this paper.

where

$$\partial_x^3 L_{\rm 5th} = \partial_x^5 + u\partial_x^3 + 3u_x\partial_x^2 + 3u_{xx}\partial_x + u_{xxx}. \tag{2.16}$$

Then the NC Lax equation (2.2) gives an operator $T'_{\rm 5th}$

$$\begin{aligned} T'_{\rm 5th} = & \frac{3}{2}u\partial_x^3 + \frac{3}{4}u_x\partial_x^2 + \left(\frac{1}{2}u_{xx} + \frac{15}{8}u\star u\right)\partial_x \\ & -\left(\frac{1}{4}u_{xxx} - \frac{15}{16}(u\star u)_x - \frac{1}{8}[u,u_x]_\star\right), \end{aligned} \tag{2.17}$$

and the NC 5th-KdV equation

$$\begin{aligned} & u_t + \frac{1}{4}u_{xxxxx} + \frac{1}{16}(u\star u)_{xx} + \frac{1}{2}u\star u_{xxx} - \frac{15}{8}u\star u_x\star u + \frac{1}{8}[u,u_{xxx}]_\star \\ & -\frac{13}{16}\Big[u,[u,u_x]_\star\Big]_\star = 0. \end{aligned} \tag{2.18}$$

- For $n \geq 3$, we can obtain the NC KdV hierarchy (7th, 9th, $\cdots$), similarly.

We can generate many other NC Lax pairs in the same method and construct NC $(1+1)$ dimensional equations[2]. The NC modified KdV equation can be also constructed [7].

On commutative case, it is widely known that the strong and useful method reported here can be applied to find more wider class of the Lax pairs in higher dimensions [9]. Next we will reported extensions of this method to higher dimensions on NC case.

3. Extensions to higher dimensions

On commutative spaces, the Lax pair plays a key role in studying higher dimensional integrable equations from lower dimensional integrable equations. That is, we can generate wide class of Lax pairs including higher dimensional integrable equations [9]. For example

$$L = L_{\rm KdV} + \partial_y, \tag{3.1}$$

gives rise to the KP equation for $u = u(t,x,y)$ with an additional spacial coordinate y, respectively by the same ansatz (2.6) for T. If we take $L = L_{\rm KdV}$ and the ansatz for $\partial_i = \partial_z$, that is,

$$T = \partial_z L_{\rm KdV} + T' + \partial_t, \tag{3.2}$$

then we get the Bogoyavlenskii-Calogero-Schiff (BCS) equation for $u = u(t,x,z)$ with an additional spacial coordinate z [10]. The KP and BCS equations are of $(2+1)$ dimensional

[2]See Appendix. We will display our results there.

KdV equations. Good news here is that this method is also applicable to other soliton equations [11].

In this section we shall present three results on NC case by using the Lax-pair generating technique as same as $(1+1)$ dimensions. We focus on three extensions of the NC Lax pair for the NC KdV equation (2.13) to $(2+1)$ dimensions . In this section, let us suppose that the noncommutativity should be introduced only in the spacial directions.

3.1 NC KP equation

The noncommutativity relation here is defined by $[x,\ y] = i\theta$. Taking the operator L

$$L_{\rm KP} = \partial_x^2 + u(t,x,y) + \partial_y \equiv \widetilde{L}_{\rm KP} + \partial_y, \tag{3.3}$$

and the ansatz for the operator T (for $\partial_i = \partial_x$):

$$T_{\rm KP} = \partial_x \widetilde{L}_{\rm KP} + T'_{\rm KP} + \partial_t, \tag{3.4}$$

as the Lax pair for the NC KP equation, the NC Lax equation (2.2) reads

$$[L_{\rm KP},\ T_{\rm KP}]_\star = -u_x\partial_x^2 + u_y\partial_x - (u_t - u_{xy} + u_x \star u) + [L_{\rm KP},\ T'_{\rm KP}]_\star = 0. \tag{3.5}$$

Then we find

$$T'_{\rm KP} = \frac{1}{2}u\partial_x - \frac{1}{4}u_x - \frac{3}{4}\partial_x^{-1}u_y, \tag{3.6}$$

and the NC KP equation [12]

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u\star u)_x + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_y]_\star = 0, \tag{3.7}$$

where $\partial_x^{-1}f(x) := \int^x dx' f(x')$. There is seen to be a nontrivial deformed term $[u, \partial_x^{-1}u_y]_\star$ in the equation (3.7), which vanishes in the commutative limit. In [12] the multi-soliton solution was found by the first order to small θ expansion, which suggested that this equation would be considered as an integrable equation. And the NC KP hierarchy can be constructed. However they have many terms. So we do not display explicitly. The NC modified KP equation can be also derived [7].

3.2 NC BCS equation

The noncommutativity relation here is defined by $[x,\ z] = i\theta$. We take the operator L

$$L_{\rm BCS} = \partial_x^2 + u(t,x,z), \tag{3.8}$$

and the ansatz for the operator T (for $\partial_i = \partial_z$)

$$T_{\rm BCS} = \partial_z L_{\rm BCS} + T'_{\rm BCS} + \partial_t, \tag{3.9}$$

where

$$\partial_z L_{\rm BCS} = \partial_x^2\partial_z + u\partial_z + u_z. \tag{3.10}$$

Then the NC Lax equation (2.2) yields

$$[L_{\rm BCS},\ T_{\rm BCS}]_\star = -u_z\partial_x^2 - (u_t + u_z \star u) + [L_{\rm BCS},\ T'_{\rm BCS}]_\star = 0. \tag{3.11}$$

After calculation, we find

$$T'_{\rm BCS} = \frac{1}{2}(\partial_x^{-1}u_z)\partial_x - \frac{1}{4}u_z - \frac{1}{4}\partial_x^{-1}\Big([u, \partial_x^{-1}u_z]_\star\Big), \tag{3.12}$$

and obtain a new NC equation

$$\begin{aligned} u_t + \frac{1}{4}u_{xxz} + \frac{1}{2}(u\star u)_z + \frac{1}{4}\Big(u\star(\partial_x^{-1}u_z) + (\partial_x^{-1}u_z)\star u\Big)_x \\ +\frac{1}{4}\Big[u, \partial_x^{-1}\Big([u, \partial_x^{-1}u_z]_\star\Big)\Big]_\star = 0, \end{aligned} \tag{3.13}$$

which is called here the NC BCS equation. Note that a non-trivial term $\partial_x^{-1}\Big([u, \partial_x^{-1}u_z]_\star\Big)$ is found even in the operator T. And the NC BCS hierarchy can be constructed. However they have many terms. So we do not display explicitly.

3.3 NC extended BCS equation

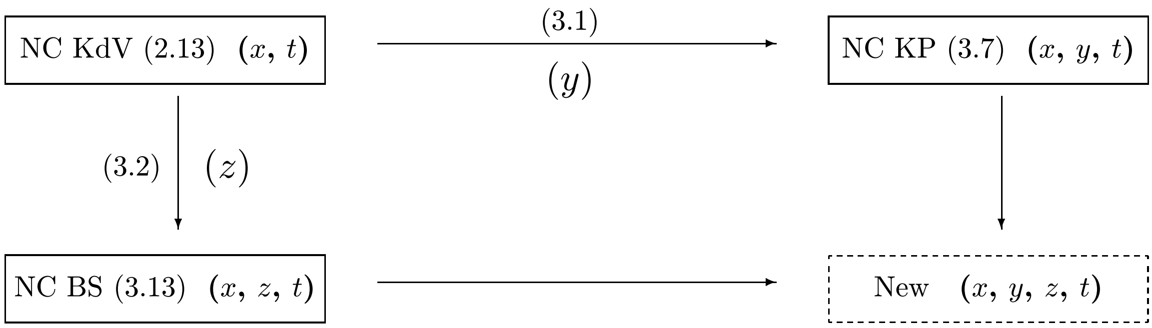

Figure 1: There are two directional routes of the extensions for the Lax pair. One leads us to the NC KP equation with an additional spacial coordinate y and the other does to the NC BCS equation with an additional spacial coordinate z. By the combination of the extensions, "NEW" equation seems to be possible a NC equation in $(3+1)$ dimensions.

On commutative case, it is known that $(2+1)$ dimensional integrable equations is given by combinating the extension (3.1) with the extension (3.2) [13, 14, 15]. We would like to study here the combination on NC case. In this subsection, $[x,\ y] = i\theta_1$ and $[x,\ z] = i\theta_2$ give the noncommutativity. Figure 1 means the schemes of the dimensional extension of the NC KdV equation to $(2+1)$ dimensions. Here we take the operator L as

$$L_{\rm NEW} = \partial_x^2 + u(t,x,y,z) + \partial_y \equiv \widetilde{L}_{\rm NEW} + \partial_y. \tag{3.14}$$

And by the ansatz (2.4), an associated operator T is taken as

$$T_{\rm NEW} = \partial_z \widetilde{L}_{\rm NEW} + T'_{\rm NEW} + \partial_t, \tag{3.15}$$

where

$$\partial_z \widetilde{L}_{\rm NEW} = \partial_x^2 \partial_z + u\partial_z + u_z. \tag{3.16}$$

The operators $L_{\rm NEW}$ and $T_{\rm NEW}$ seem to give an equation in $(3+1)$ dimensions. However $(3+1)$ dimensions have to reduce to $(2+1)$ dimensions as we shall see soon. The NC Lax equation (2.2) for the operators $L_{\rm NEW}$ and $T_{\rm NEW}$ reads

$$[L_{\rm NEW},\ T_{\rm NEW}]_\star = -u_z\partial_x^2 + u_y\partial_z - (u_t - u_{yz} + u_z \star u) + [L_{\rm NEW},\ T'_{\rm NEW}]_\star = 0. \tag{3.17}$$

Now the operator T' could be taken as

$$T'_{\rm NEW} = A\partial_z^2 + B\partial_x + C, \tag{3.18}$$

where A, B and C are polynomials of u, u_x, u_{xy}, etc. Then we get

$$\begin{aligned}
[L_{\rm NEW},\ T'_{\rm NEW}]_\star = {} & 2A_x\partial_x\partial_z^2 + 2B_x\partial_x^2 + \Big(A_{xx} + A_y + [u,\ A]_\star\Big)\partial_z^2 \\
& + \Big(B_{xx} + B_y + [u,\ B]_\star + 2C_x\Big)\partial_x - 2A \star u_z\partial_z \\
& - \Big(A \star u_{zz} + B \star u_x - C_{xx} - C_y - [u,\ C]_\star\Big).
\end{aligned} \tag{3.19}$$

By the relations (3.17) and (3.19), we can obtain A, B and C as follows. The coefficients of $\partial_x\partial_z^2,\ \partial_x^2,\ \partial_z^2$ and ∂_x give, respectively,

$$A = \alpha,\ B = \frac{1}{2}\partial_x^{-1}u_z \ \text{ and }\ C = -\left(\frac{1}{2}u_z + \frac{1}{4}\partial_x^{-2}u_{yz} + \frac{1}{2}\partial_x^{-1}[u, \partial_x^{-1}u_z]_\star\right), \tag{3.20}$$

where α is an arbitrary constant. And then the coefficient of ∂_z yields

$$u_y = 2\alpha u_z, \tag{3.21}$$

that is,

$$y = \frac{1}{2\alpha}z. \tag{3.22}$$

This relation (3.22) means $(3+1)$ dimensions reduce to $(2+1)$ dimensions. Now the field u is a function of three coordinates t, x and z. So the unknown operator $T'_{\rm NEW}$ is expressed as

$$T'_{\rm NEW} = \alpha\partial_z^2 + \frac{1}{2}(\partial_x^{-1}u_z)\partial_x - \left(\frac{1}{4}u_z + \frac{\alpha}{2}\partial_x^{-2}u_{zz} + \frac{1}{2}\partial_x^{-1}[u, \partial_x^{-1}u_z]_\star\right). \tag{3.23}$$

Therefor we propose a new equation on NC spaces

$$u_t + \frac{1}{4}u_{xxz} + u \star u_z + \frac{1}{2}u_x \star (\partial_x^{-1} u_z) + \alpha^2 \partial_x^{-2} u_{zzz} + \partial_x^{-1} \Big([u, \partial^{-1} u_z]_\star \Big)_z$$
$$+ \frac{1}{2}\Big[u, u_z + \alpha \partial_x^{-2} u_{zz} + \partial_x^{-1} [u, \partial_x^{-1} u_z]_\star \Big]_\star = 0. \qquad (3.24)$$

In the commutative limit, equation (3.24) reads the extended BCS equation, which has soliton solutions [13]. So we refer to equation (3.24) as *the NC extended BCS equation.*

All of the NC integrable equations derived from bicomplex method are also obtained by our method. Bicomplex method guarantees the existence of the many conserved topological quantities. These results suggest that NC equations presented in his paper would possess the integrability.

4. Conclusion and Discussion

A natural problem in the integrable systems is whether we can find new higher dimensional integrable equations from already known $(1+1)$ dimensional integrable ones. On commutative spaces, it is well known that the Lax representation is a powerful tool to do so. In the present paper, we derived $(1+1)$ and $(2+1)$ dimensional equations on NC spaces using a powerful method or *Lax-Pair Generating Technique* to find the NC Lax pairs. There are expected to be integrable. The simple, but mysterious, ansatz (2.4) plays an important role and actually gives rise to various new NC Lax pairs.

Let us note here about reduction of ASDYM equation on NC spaces. In commutative case, it is widely known that many of integrable equations could be derived from symmetry reduction of $(2+2)$ dimensional ASDYM equation, which is first conjectured by R.Ward [16]. And it is also known that ASDYM equation on NC spaces is given integrable equations on NC spaces [7, 17].

Now there would be mainly three methods to gives NC integrable equations

- Lax-pair generating technique,
- Bicomplex method,
- Reduction of ASDYM equation on NC spaces.

An interesting point is that all the results are consistent at least for the known Lax pairs, which suggests the existence and the uniqueness of NC deformations of integrable equations which preserve the integrability.

Though we can get many new NC Lax pairs, there need to be more discussions so that such study should be fruitful as integrable systems. First, we have to clarify whether the

NC Lax pairs are really good equations in the sense of integrability, that is, the existence of many conserved quantities, of multi-soliton solutions and so on. All of the previous studies including our works strongly suggest that this would be true. Second, we have to reveal the physical meaning of such equations. If such integrable theories can be embedded in string theory, there would be fruitful interaction between the both theories, just as between ASDYM equation on NC spaces and D0-D4 brane system (in the background of NS-NS B field).

It has been shown in this paper that the Lax-Pair Generating Technique is also a very powerful tool for introducing higher dimensional equations, which we believe integrable. We believe that any higher dimensional integrable equations on Commutative spaces and Non-commutative ones can be obtained from several lower dimensional integrable ones by extending the Lax pairs to higher dimensions. We have a dream such as constructing (3+1) dimensional integrable equations on Commutative spaces and Non-commutative ones (if there exist). Further study on this topic is continued.

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A. Some results in $(1+1)$ dimensions

In this Appendix, we will propose some results for $u = u(t, x)$ in $(1+1)$ dimensions by the Lax-Pair Generating Technique. The noncommutativity relation here is defined by $[t,\ x] = i\theta$.

A.1 NC Burgers equation

The Lax pair is given by

$$L = \partial_x + u, \tag{A.1}$$

and

$$T = \partial_x^2 + 2u\partial_x + (\alpha + 1)u_x + \beta u^2 + \partial_t. \tag{A.2}$$

Then the NC Lax equation (2.2) gives the NC Burgers equation

$$u_t - \alpha u_{xx} + (1 - \alpha - \beta)u \star u_x + (1 + \alpha - \beta)u_x \star u = 0, \tag{A.3}$$

where α and β are arbitrary constants. In the commutative limit, equation (A.3) is reduced to the Burgers equation [18]. Equation (A.3) and exact solutions will be reported in [19].

A.2 NC shallow water-wave equation

The Lax pair is given by

$$L = \partial_x^2 + u, \tag{A.4}$$

and

$$T = \partial_x^2 \partial_t + (u+1)\partial_t + \frac{1}{2}(\partial_x^{-1} u_t)\partial_x + \frac{3}{4}u_t - \frac{1}{4}\partial_x^{-1}\Big([u, \partial_x^{-1} u_t]_\star\Big). \tag{A.5}$$

Then the NC Lax equation (2.2) gives a NC equation

$$\begin{aligned} &u_t + \frac{1}{4}u_{xxt} + \frac{1}{2}(u \star u)_t + \frac{1}{4}\Big(u \star (\partial_x^{-1} u_t) + (\partial_x^{-1} u_t) \star u\Big)_x \\ &+ \frac{1}{4}\Big[u, \partial_x^{-1}\Big([u, \partial_x^{-1} u_t]_\star\Big)\Big]_\star = 0. \end{aligned} \tag{A.6}$$

In the commutative limit, equation (A.6) is reduced to the Shallow Water-Wave equation [20]. Hence we would like to call equation (A.6) *NC Shallow Water-Wave equation.*

A.3 NC Boussinesq equation

The Lax pair is given by

$$L = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1} u_t, \tag{A.7}$$

and

$$T = \partial_x^2 + u + \partial_t. \tag{A.8}$$

Then the NC Lax equation (2.2) gives a NC equation

$$u_{tt} \; + \; \Big([u, \; \partial_x^{-1} u_t]_\star\Big)_x + (u \star u)_{xx} + \frac{1}{3}u_{xxxx} = 0. \tag{A.9}$$

In the commutative limit, equation (A.9) is reduced to the two directional KdV or Boussinesq equation [21]. Hence we would like to call equation (A.9) *NC Boussinesq equation.*

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