PROCEEDINGS

Workshop on Integrable Theories, Solitons and Duality

WDVV Equations, Darboux-Egoroff Metric and the Dressing Method

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Abstract: Dressing technique is used to construct commuting Lax operators which provide an integrable (canonical) structure behind Witten–Dijkgraaf–Verlinde–Verlinde equations. The commuting flows are related to the isomonodromic flows. Examples of the canonical integrable structure are given in two- and three-dimensional cases. The three-dimensional example is associated with the rational Landau-Ginzburg potentials.

Keywords: WDVV Equations, Darboux-Egoroff metric, tau function.

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1. Introduction, WDVV Equation in Flat Coordinates

In this talk we describe the commuting structure behind Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations based on the dressing approach. The solutions to the WDVV equations [1, 2, 3, 4], expressed in terms of the so-called flat coordinates $x^1, x^2, \ldots, x^N$,

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are provided by the prepotential $F(x^1, x^2, \ldots, x^N)$ and the Euler operator :

$$E = \sum_{\alpha=1}^{N} (d_\alpha x^\alpha + r_\alpha) \frac{\partial}{\partial x^\alpha} \tag{1.1}$$

with $d_\alpha r_\alpha = 0$ and $d_\alpha = 1 + \mu_1 - \mu_\alpha$ [3], with constants $\mu_\alpha, \alpha = 1, \ldots, N$ defined below (see (4.21)).

The pair (F, E) satisfies the WDVV equations if the following three conditions are satisfied.

- Associativity :

$$\sum_{\delta,\gamma=1}^{N} \frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\beta \partial x^\delta} \eta^{\delta\gamma} \frac{\partial^3 F(x)}{\partial x^\gamma \partial x^\omega \partial x^\rho} = \sum_{\delta,\gamma=1}^{N} \frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\omega \partial x^\delta} \eta^{\delta\gamma} \frac{\partial^3 F(x)}{\partial x^\gamma \partial x^\beta \partial x^\rho}$$

- Normalization :

$$\frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\beta \partial x^1} = \eta_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ defines a constant non-degenerate metric: $g = \sum_{\alpha\beta=1}^{N} \eta_{\alpha\beta} dx^\alpha dx^\beta$.

- Quasi-Homogeneity condition : The quasi-homogeneity condition states that :

$$E(F) = d_F F + \text{quadratic terms} \tag{1.2}$$

where the number d_F denotes the degree (or homogeneity) of the prepotential F.

As an example consider the three-dimensional space with three flat coordinates x^1, x^2, x^3 and with (E, F) :

$$F = \frac{1}{6} x^3 (x^2)^3 + \frac{1}{6} (x^1)^3 + x^1 x^2 x^3 + \frac{1}{2} (x^3)^2 \left(\log x^3 - \frac{3}{2} \right)$$
$$E = x^1 \frac{\partial}{\partial x^1} + \frac{1}{2} x^2 \frac{\partial}{\partial x^2} + \frac{3}{2} x^3 \frac{\partial}{\partial x^3}$$

such that $E(F) = 3F +$ quadratic terms. More details on this example will be given in Section 8.

The content of this talk is as follows. In Section 2, we define the Darboux-Egoroff equations which characterize the metric behind WDVV solutions when expressed in terms of the curvlinear orthogonal coordinates referred to as canonical coordinates. The canonical integrable structure behind the WDVV equations is presented in Section 3 emphasizing its connection to the Darboux-Egoroff metric. We use the setting of the Riemann-Hilbert problem augmented by an extra twisting condition [5]. The tau function appears naturally in this formalism. The dressing matrix entering the Riemann-Hilbert problem generates the dressing procedure which is used to construct the commuting structure behind the WDVV equations. The dressing procedure is developed in Section 4 and used to provide relation between the canonical integrable structure and the flat coordinates, structure constants and associativity equations. Section 5 establishes a connection between the commuting flows of the canonical integrable structure behind the WDVV equations and isomodromic deformations related to the Schlesinger equation. Another evidence of such connection is provided by the fact that the tau function of the Riemann-Hilbert problem turns into the isomonodromic tau function once the conformal condition on the integrable structure is imposed as explained in Section 4 (see [6, 7]).

In case of two-dimensions, solutions to the Darboux-Egoroff equations, the tau functions and the corresponding prepotential satisfying WDVV equations can be found explicitly. This is described in Section 6. More difficult is the case of three dimensions presented in Section 7 where the Darboux-Egoroff equations are shown to take the form of the classical Euler equations of free rotations of a rigid body. In three-dimensions the scaling dimension of the tau function is found to be related to the integral of the Euler equations.

Section 8 shows how to derive the canonical integrable structures for a class of rational Lax functions associated with a particular reduction of the dispersionless KP hierarchy. This derivation generalizes the well-known construction of the monic polynomials [3, 8]. An example of the three-dimensional canonical integrable model derived from the rational potentials is given in subsection 8.1. The three-dimensional example shown in this subsection provides solutions to the Painlevé VI equation. Given that the flows of the canonical integrable models can essentially be reformulated as isomonodromic deformations, as shown in Section 5, the connection to the sixth Painlevé equation is not surprising. The tau function of the three-dimensional example has a scaling dimension of $R^2 = 1/4$ and the corresponding prepotential contains logarithmic terms.

For the scaling dimensions, $R^2 = n^2$ such that n is an integer, the multi-component KP hierarchy provides a framework for the construction of canonical integrable hierarchies [9]. It would be of interest to find a universal approach to the formulation of the canonical integrable models which would include models with fractional scaling dimensions as the ones encountered in example of subsection 8.1 based on the rational potentials of Section 8.

2. Darboux–Egoroff metric

Massive topological field theories can be classified locally by the Darboux–Egoroff metric given in terms of the canonical coordinates $u_1, \ldots, u_N$ [3] :

$$g = \sum_{\alpha\beta=1}^{N} \eta_{\alpha\beta} dx^\alpha dx^\beta = \sum_{i=1}^{N} h_i^2(\mathbf{u})(du_i)^2 \tag{2.1}$$

with Lamé coefficients $h_i^2(\mathbf{u}) = \partial\phi/\partial u_i$. The fact that $h_i^2(u)$ is a gradient ensures that the so-called "rotation coefficients"

$$\beta_{ij} = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j}, \;\; i \neq j, \;\; 1 \leq i, j \leq N, \tag{2.2}$$

are symmetric $\beta_{ij} = \beta_{ji}$ and therefore the metric becomes the Darboux-Egoroff metric when expressed in terms of the curvlinear orthogonal or canonical coordinates u_i. The Darboux-Egoroff equations for the rotation coefficients are:

$$\frac{\partial}{\partial u_k}\beta_{ij} = \beta_{ik}\beta_{kj}, \quad \text{distinct} \;\; i, j, k \tag{2.3}$$

$$\sum_{k=1}^{N} \frac{\partial}{\partial u_k} \beta_{ij} = 0, \quad i \neq j\,. \tag{2.4}$$

In addition one also assumes the conformal condition :

$$\sum_{k=1}^{N} u_k \frac{\partial}{\partial u_k} \beta_{ij} = -\beta_{ij}\,. \tag{2.5}$$

3. Integrable structure behind WDVV equations

Consider a loop group element $g(z) : S^1 \to GL(N, \mathbb{C})$ which decomposes as

$$g(z) = g_-(z) g_+(z)$$

w.r.t. two subgroups of the Lie loop group $G = \widehat{GL}(N, \mathbb{C})$ consisting of all such maps g:

$$\begin{aligned} G_- &= \left\{ g \in G |\; g(z) = 1 + \sum_{i>0} g^{(-i)} z^{-i} \right\} \\ G_+ &= \left\{ g \in G |\; g(z) = \sum_{i\geq 0} g^{(i)} z^{i} \right\} \end{aligned}$$

Assume from now on that $g(z)$ satisfies twisting condition $g^{-1}(z) = g^T(-z)$ [5]. Let the un-dressed wave matrix be :

$$\Psi_0(\mathbf{u}, z) = \exp\left(\sum_{j=1}^{N} z E_{jj} u_j \right) = \exp\left(zU\right) \tag{3.1}$$

$$U = \mathrm{diag}\left(u_1, \ldots, u_N\right) = \sum_{i=1}^{N} u_i E_{ii}\,.$$

Our notation is : $(\mathbf{u}) = (u_1, \ldots, u_N)$, $\partial_j = \partial/\partial u_j$ and E_{ij} is an elementary matrix with matrix elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

The Riemann-Hilbert problem is here defined as :

$$\Psi_0(\mathbf{u}, z)\, g(z) = \Theta^{-1}(\mathbf{u}, z)\, M(\mathbf{u}, z) \tag{3.2}$$

with the dressing matrices :

$$\Theta(\mathbf{u}, z) \in G_-, \quad \Theta = 1 + \theta^{(-1)} z^{-1} + \theta^{(-2)} z^{-2} + \ldots \tag{3.3}$$

$$M(\mathbf{u}, z) \in G_+, \quad M = M_0 + M_1 z + M_2 z^2 + \ldots \tag{3.4}$$

satisfying the twisting conditions [5, 10]

$$\Theta^{-1}(\mathbf{u}, z) = \Theta^T(\mathbf{u}, -z)\,, \qquad M^{-1}(\mathbf{u}, z) = M^T(\mathbf{u}, -z)\,. \tag{3.5}$$

The twisting conditions imply, in particular, that

$$M_0^T = M_0^{-1} \quad ; \quad \theta^{(-1)} = \theta^{(-1)\,T}$$

The Riemann-Hilbert problem (3.2) gives rise to the commuting symmetry flows :

$$\frac{\partial}{\partial u_j}\Theta(\mathbf{u},z) = -\left(\Theta z E_{jj}\Theta^{-1}\right)_-\Theta(\mathbf{u},z) \tag{3.6}$$

$$\frac{\partial}{\partial u_j}M(\mathbf{u},z) = \left(\Theta z E_{jj}\Theta^{-1}\right)_+ M(\mathbf{u},z)\,. \tag{3.7}$$

Equations (3.6) imply that the following tracelessness condition

$$I\left(\Theta(\mathbf{u},z)\right) = \sum_{j=1}^{N}\frac{\partial}{\partial u_j}\,\Theta(\mathbf{u},z) = 0 \tag{3.8}$$

holds for the so-called identity vector-field

$$I = \sum_{j=1}^{N}\partial/\partial u_j\,. \tag{3.9}$$

We also define the so-called Euler vector field in terms of canonical coordinates as

$$E = \sum_{i=1}^{N} u_i\frac{\partial}{\partial u_i}\,. \tag{3.10}$$

One finds from equations (3.6) that

$$E(\Theta)\Theta^{-1} = -\left(\Theta z U\Theta^{-1}\right)_-\,. \tag{3.11}$$

The tau function can be associated with the Riemann-Hilbert problem (3.2) through relations :

$$\frac{\partial\log\tau}{\partial u_j} = \mathrm{Res}_z\left(\mathrm{tr}\left(E_{jj}\Theta^{-1}z\frac{d\Theta}{dz}\right)\right),\;\; j=1,\ldots,N\,. \tag{3.12}$$

Accordingly, we introduce the following parametrization for the symmetric $\theta^{(-1)}$ matrix :

$$\theta^{(-1)}_{ij} = \begin{cases} -\partial\log\tau/\partial u_i & i=j \\ \beta_{ij} & i\neq j \end{cases} \tag{3.13}$$

with off-diagonal elements of $\theta^{(-1)}$-matrix defining the so-called rotation coefficients β_{ij} satisfying the Darboux-Egoroff equations (2.3) and (2.4) as follows from expressions (3.6) and (3.8).

4. Dressing

The un-dressed structure is given by operators :

$$\delta_j = \frac{\partial}{\partial u_j} - zE_{jj}, \qquad j=1,\ldots,N \tag{4.1}$$

and

$$L_k = -z^{k+1}\frac{d}{dz} + Uz^{k+1}, \tag{4.2}$$

which both annihilate the un-dressed wave function $\Psi_0(\mathbf{u}, z)$ (3.1)

$$\delta_j \Psi_0(\mathbf{u}, z) = 0, \quad L_k \Psi_0(\mathbf{u}, z) = 0\,. \tag{4.3}$$

These operators satisfy the commutation relations :

$$[\delta_j \,,\, \delta_i] = 0, \quad [L_k, L_r] = (k-r)L_{k+r}, \quad [L_k, \delta_j] = 0\,. \tag{4.4}$$

The so called dressing procedure maps the un-dressed wave function $\Psi_0(\mathbf{u}, z)$ to :

$$\Psi_0(\mathbf{u}, z) \;\rightarrow\; \Psi(\mathbf{u}, z) = \Theta(\mathbf{u}, z)\Psi_0(\mathbf{u}, z)$$

while the operators δ_j and L_k are mapped into the dressed operators:

$$\delta_j \;\rightarrow\; \mathcal{D}_j = \Theta\delta_j\Theta^{-1}, \quad L_k \;\rightarrow\; \mathcal{L}_k = \Theta L_k \Theta^{-1}\,. \tag{4.5}$$

By construction these operators annihilate the "dressed" wave (matrix) function $\Psi(\mathbf{u}, z)$:

$$\mathcal{L}_k \Psi = 0, \quad \mathcal{D}_j \Psi = 0, \quad j = 1, \ldots, N\,. \tag{4.6}$$

Dressing preserves the commutation relations and so it holds that $[\mathcal{D}_j \,,\, \mathcal{D}_i] = 0$ and $[\mathcal{L}_k, \mathcal{L}_r] = (k-r)\mathcal{L}_{k+r}$, $[\mathcal{L}_k, \mathcal{D}_j] = 0$.

Since

$$\Theta\partial_j\Theta^{-1} = \partial_j + \Theta(\partial_j\Theta^{-1}) = \partial_j + \left(\Theta z E_{jj}\Theta^{-1}\right)_- \tag{4.7}$$

we obtain

$$\mathcal{D}_j = \partial_j - \left(\Theta z E_{jj}\Theta^{-1}\right)_+ = \partial_j - zE_{jj} - V_j\,. \tag{4.8}$$

Also,

$$\mathcal{L}_0 = \Theta L_0 \Theta^{-1} = -z\frac{d}{dz} + zU + V + \left(z\frac{d\Theta}{dz} - E(\Theta)\right)\Theta^{-1} \tag{4.9}$$

with

$$V_j \equiv [\theta^{(-1)} \,,\, E_{jj}], \quad (V_j)_{kl} = (\delta_{lj} - \delta_{kj})\,\beta_{kl} \tag{4.10}$$

$$V \equiv [\theta^{(-1)} \,,\, U], \quad V_{ij} = (u_j - u_i)\beta_{ij}\,. \tag{4.11}$$

Note, that the first three terms on the right hand side of equation (4.9) contain only terms of positive grade in z while the remaining term contains terms of the negative grade in z (terms with z^k , $k \leq -1$). We will now impose the so-called conformal condition which amounts to $(\mathcal{L}_0)_+ = \mathcal{L}_0$ or

$$E(\Theta) = z\frac{d\Theta}{dz}\,. \tag{4.12}$$

As shown in reference [7] this condition is compatible with flows from equation (3.6)

The following equations follow now from relations (4.6)

$$\mathcal{D}_j\Psi = 0 \;\rightarrow\; \frac{\partial\Psi}{\partial u_j} = (V_j + zE_{jj})\Psi \tag{4.13}$$

$$\mathcal{L}_0\Psi = 0 \;\rightarrow\; z\frac{d\Psi}{dz} = (zU + V)\,.\Psi \tag{4.14}$$

From (4.13) one can easily calculate the action of the Euler vector field on the wave-function Ψ:

$$E(\Psi) = \sum_{j=1}^{N} u_j \frac{\partial \Psi}{\partial u_j} = (V + zU)\Psi \tag{4.15}$$

as follows from relations $\sum_{j=1}^{N} u_j V_j = V$ and $\sum_{j=1}^{N} u_j E_{jj} = U$. Comparing this with equation (4.14) we find that :

$$E(\Psi) = z \frac{d\Psi}{dz} \tag{4.16}$$

and so E and zd/dz coincide when applied on the wave function Ψ in the framework of the dressing formalism. Plugging relation (4.12) into the formula (3.12) for the τ-function one obtains [11]:

$$\partial_j \log \tau = \mathrm{Res}_z \left(\mathrm{tr} \left(\Theta^{-1} E(\Theta) E_{jj} \right) \right) = \frac{1}{2} \, \mathrm{tr} \left(V_j V \right) \,. \tag{4.17}$$

Also, $I(\Psi) = z\Psi$, which together with relation (4.12) yield

$$E(\beta_{ij}) = -\beta_{ij} \; ; \qquad I(\beta_{ij}) = 0 \tag{4.18}$$

$$E(\tau) = R^2 \, \tau \; ; \quad I(\tau) = 0 \tag{4.19}$$

where in relation (4.19) we introduced constant R^2 defining the scaling dimension (also called homogeneity) of the tau function.

Commutation relations yield :

$$\partial_j V = [V_j \, , \, V], \qquad [V \, , \, E_{jj}] = [V_j \, , \, U] \tag{4.20}$$

and so $I(V) = 0$ since $\sum_{j=1}^{N} V_j = 0$.

The similarity transformation $V \to \mathcal{V} = M_0^{-1} V M_0$ transforms V to the constant matrix $\mathcal{V}$ ($\partial_j \mathcal{V} = 0$) due to the flow equations $\partial_j M_0 = V_j M_0$, which follow from relation (3.7), and the above equation (4.20). Assume, now that there exists an invertible matrix S which diagonalizes $\mathcal{V}$ [3]:

$$S^{-1} \mathcal{V} S = \mu = \sum_{j=1}^{N} \mu_j E_{jj} \tag{4.21}$$

where μ is a constant diagonal matrix $\mu = \mathrm{diag}(\mu_1, \ldots, \mu_N)$.

Next, define a matrix

$$M(u) = M_0(u) S = (m_{ij}(u))_{1 \le i,j \le N} \,. \tag{4.22}$$

M satisfies

$$(\partial_j - V_j)(M) = 0 \;\; \to \;\; M^{-1}(\partial_j M) = M^{-1} V_j M \tag{4.23}$$

and

$$E(M) = \sum_{j=1}^{N} u_j V_j M = VM = M_0 \mathcal{V} S = M\mu \,. \tag{4.24}$$

Define a constant non-degenerate metric to be :

$$\eta = (\eta_{\alpha\beta})_{1 \le \alpha,\beta \le N} = M^T M = S^T S, \quad \text{and denote} \quad \eta^{-1} = (\eta^{\alpha\beta})_{1 \le \alpha,\beta \le N} \,. \tag{4.25}$$

Hence $\eta_{\alpha\beta} = \sum_{i=1}^{N} m_{i\alpha} m_{i\beta}$.

We have :

$$(\frac{\partial}{\partial u_j} - V_j - zE_{jj})\Phi = 0 \tag{4.26}$$

where

$$\Phi(\mathbf{u}, z) = \Psi(\mathbf{u}, z) g(z) S = M(\mathbf{u}, z) S = M(\mathbf{u}) + O(z)\,. \tag{4.27}$$

Under a similarity transformation generated by the $M(u)$ matrix the Lax operators $\mathcal{D}_j = \partial_j - zE_{jj} - V_j$ transform to:

$$\widetilde{\mathcal{D}}_j \equiv M^{-1}\mathcal{D}_j M = \partial_j - zM^{-1}E_{jj}M = \partial_j - zC_j\,, \tag{4.28}$$

where in (4.28) we have introduced :

$$C_j = M^{-1}E_{jj}M = \sum_{\alpha=1,\beta=1}^{N} (C_j)^{\alpha}_{\beta} E_{\alpha\beta}, \quad (C_j)^{\alpha}_{\beta} = \sum_{\gamma} \eta^{\alpha\gamma} m_{j\gamma} m_{j\beta} \tag{4.29}$$

such that

$$C_i C_j = C_i \delta_{ij}, \quad \sum_{i=1}^{N} C_i = I_N\,. \tag{4.30}$$

The matrix $\Xi(u, z)$ defined as

$$\Xi(u, z) = M^{-1}\Phi(u, z) = I + \sum_{n=1}^{\infty} z^n \Xi^{(n)}(u) = I + z\Xi^{(1)} + z^2\Xi^{(2)} + \cdots \tag{4.31}$$

is annihilated by the transformed $\widetilde{\mathcal{D}}_j$ operators:

$$\widetilde{\mathcal{D}}_j(\Xi) = (\partial_j - zC_j)(\Xi) = 0\,. \tag{4.32}$$

Due to $\left[\widetilde{\mathcal{D}}_i\,,\, \widetilde{\mathcal{D}}_j\right] = 0$ it holds that:

$$\partial_i C_j - \partial_j C_i = 0\,. \tag{4.33}$$

From (4.33), it follows that we can define a matrix C

$$C = \sum_{\alpha=1,\beta=1}^{N} C^{\alpha}_{\beta} E_{\alpha\beta} \tag{4.34}$$

such that :

$$C_j = \partial_j C\,. \tag{4.35}$$

Accordingly, plugging the expansion (4.31) into equation (4.32) yields :

$$\partial_j \Xi^{(n)} = C_j \Xi^{(n-1)} = (\partial_j C)\, \Xi^{(n-1)}, \quad n \geq 1, \;\; \Xi^{(0)} = I \tag{4.36}$$

and therefore we can choose C to be equal to $\Xi^{(1)}$:

$$\Xi^{(1)} = C \tag{4.37}$$

while $\partial_j \Xi^{(2)} = (\partial_j C)C$, etc. Also, by summing over j in (4.36) we get

$$\sum_{j=1}^{N} \partial_j \Xi^{(n)} = \Xi^{(n-1)} \quad \text{or} \quad \sum_{j=1}^{N} \partial_j \Xi = I(\Xi) = z\Xi \,. \tag{4.38}$$

Note, that the relation (4.16) yields :

$$z\frac{d}{dz}\Xi = z\mathcal{U}\Xi + [\,\mu\,,\,\Xi\,] \;. \tag{4.39}$$

In components it gives

$$n\Xi^{(n)} - \Big[\,\mu\,,\,\Xi^{(n)}\Big] = \mathcal{U}\Xi^{(n-1)} \,. \tag{4.40}$$

Comparing with (4.78) we see that $E = nI - \mathrm{ad}_\mu$ when applied on $\Xi^{(n)}$. Equation (4.36) determines $\Xi^{(n)}$ recursively up to a constant. Note, that we can add to $\Xi^{(n)}$ a constant A_n such that $nA_n - [\,\mu\,,\,A_n\,] = 0$ without changing the right hand side of (4.40). Hence, $\Xi^{(n)}$ which is a solution of equation (4.40) as well equation (4.36) is determined up to a constant A_n of grade n with respect to the the grading defined by the semisimple element μ according to equation $nA_n - [\,\mu\,,\,A_n\,] = 0$.

The above ambiguity amounts to the fact that Ξ is determined upto a constant power series $A(z)$ such that

$$A(z) = I + A_1 z + A_2 z^2 + \cdots, \qquad A(z)\eta A(-z)^T = \eta.$$

with A_n satisfying $nA_n - [\,\mu\,,\,A_n\,] = 0$.

The matrix C is crucial for the whole theory and we will now study its properties. According to (4.29) its matrix elements C^α_β must satisfy :

$$\partial_j C^\alpha_\beta = \sum_\gamma \eta^{\alpha\gamma} m_{j\gamma} m_{j\beta} \;\;\to\;\; \partial_j C_{\beta\alpha} = m_{j\alpha} m_{j\beta} \tag{4.41}$$

where $C_{\beta\alpha} = \sum_{\gamma=1}^N C^\gamma_\beta \eta_{\alpha\gamma}$. Therefore from equation (4.41):

$$C_{\alpha\beta} = C_{\beta\alpha} \quad \text{or} \quad \sum_{\gamma=1}^{N} C^\gamma_\alpha \eta_{\beta\gamma} = \sum_{\gamma=1}^{N} C^\gamma_\beta \eta_{\alpha\gamma} \,. \tag{4.42}$$

Next, define flat coordinates as the first column of the C matrix :

$$x^\alpha \equiv C^\alpha_1 \,. \tag{4.43}$$

The Jacobian of the change of variables from u_i to x^α is according to (4.29) given by :

$$\frac{\partial x^\alpha}{\partial u_i} = \sum_{\beta=1}^{N} \eta^{\alpha\beta} m_{i1} m_{i\beta} \,. \tag{4.44}$$

Also,

$$\frac{\partial u_i}{\partial x^\alpha} = \frac{m_{i\alpha}}{m_{i1}} \tag{4.45}$$

as follows from the identity: $\delta_{ij} = \sum_{\alpha\beta} m_{i\alpha} \eta^{\alpha\beta} m_{j\beta}$.

In terms of x^α-coordinates the Lax operators $\widetilde{\mathcal{D}}_j$ become:

$$\widetilde{\mathcal{D}}_\alpha = \sum_{j=1}^{N} \frac{\partial u_j}{\partial x^a} \widetilde{\mathcal{D}}_j = \frac{\partial}{\partial x^\alpha} - zC_\alpha \,. \tag{4.46}$$

The commutation relations :

$$\left[\frac{\partial}{\partial x^\alpha} - zC_\alpha \, , \, \frac{\partial}{\partial x^\beta} - zC_\beta \right] = 0 \tag{4.47}$$

yield associativity

$$[C_\alpha \, , \, C_\beta] = 0 \; \rightarrow \; \sum_{\gamma=1}^{N} \left(c^\gamma_{\alpha\delta} c^\omega_{\beta\gamma} - c^\gamma_{\beta\delta} c^\omega_{\alpha\gamma} \right) = 0 \tag{4.48}$$

and integrability

$$\frac{\partial}{\partial x^\alpha} C_\beta - \frac{\partial}{\partial x^\beta} C_\alpha = 0 \tag{4.49}$$

relations. The matrix C_α :

$$C_\alpha \equiv \sum_{\gamma=1 \beta=1}^{N} c^\gamma_{\alpha\beta} E_{\gamma\beta} \tag{4.50}$$

is equal to

$$C_\alpha = \sum_{j=1}^{N} \frac{\partial u_j}{\partial x^a} C_j = \frac{\partial}{\partial x^a} C \tag{4.51}$$

The structure constant $c_{\alpha\beta\gamma} = \sum_{\delta=1}^{N} \eta_{\alpha\delta} c^\delta_{\beta\gamma}$ becomes therefore equal to

$$c_{\alpha\beta\gamma} = \frac{\partial C_{\beta\gamma}}{\partial x^\alpha} \tag{4.52}$$

which according to (4.44) and (4.41) is equal to :

$$c_{\alpha\beta\gamma} = \frac{\partial C_{\beta\gamma}}{\partial x^\alpha} = \sum_{j=1}^{N} \frac{m_{j\alpha}}{m_{j1}} \partial_j C_{\beta\gamma} = \sum_{j=1}^{N} \frac{m_{j\alpha} m_{j\beta} m_{j\gamma}}{m_{j1}} \,. \tag{4.53}$$

Hence the structure constants $c_{\alpha\beta\gamma}$ are symmetric in all three indices. Also, $c_{1\beta\gamma} = \eta_{\beta\gamma}$.

It follows from (4.53) that

$$\frac{\partial C_{\beta\gamma}}{\partial x^\alpha} = \frac{\partial C_{\beta\alpha}}{\partial x^\gamma} \tag{4.54}$$

and therefore

$$C_{\alpha\beta} = \frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} \quad \text{or} \quad c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \tag{4.55}$$

where F is called the prepotential.

Let us go back to the linear problem :

$$\left(\frac{\partial}{\partial x^\alpha} - zC_\alpha \right) (M^{-1}\Phi) = \left(\frac{\partial}{\partial x^\alpha} - zC_\alpha \right) (\Xi) = 0 \,. \tag{4.56}$$

Introduce

$$\phi_\beta \equiv \sum_{\alpha=1}^{N} \eta_{1\alpha} (\Xi)^\alpha_\beta = 1 + z\phi^{(1)}_\beta + z^2 \phi^{(2)}_\beta + z^3 \phi^{(3)}_\beta + \ldots \tag{4.57}$$

then as we will show below, the prepotential is given by a closed expression :

$$F = -\frac{1}{2}\phi^{(3)}_1 + \frac{1}{2}\sum_{\delta=1}^{N} x^\delta \phi^{(2)}_\delta \,. \tag{4.58}$$

To show this, multiply equation (4.56) by $\sum_{\alpha=1}^N \eta_{1\alpha}$ which yields :

$$\frac{\partial \phi_\beta}{\partial x^\alpha} - z\left(\eta M^{-1}\Phi\right)_{\alpha\beta} = 0 \tag{4.59}$$

where we introduced:

$$\phi_\beta \equiv \sum_{\alpha=1}^{N} \eta_{1\alpha} (M^{-1}\Phi)^\alpha_\beta = \sum_{\alpha=1}^{N} \eta_{1\alpha} \sum_{\gamma,\delta=1}^{N} \eta^{\alpha\gamma} m_{\delta\gamma} \Phi_{\delta\beta} = \sum_{\delta=1}^{N} m_{\delta 1}\Phi_{\delta\beta} \tag{4.60}$$

and where use was made of the identity :

$$\sum_{\gamma=1}^{N} \eta_{1\gamma} (C_\alpha)^\gamma_\beta = c_{1\alpha\beta} = \eta_{\alpha\beta} \tag{4.61}$$

leading to

$$\sum_{\gamma,\delta=1}^{N} \eta_{1\gamma} (C_\alpha)^\gamma_\delta (M^{-1}\Phi)^\delta_\beta = (\eta M^{-1}\Phi)_{\alpha\beta} \,. \tag{4.62}$$

Applying $\partial/\partial x^\gamma$ on (4.59) and using repeatedly (4.59) one gets :

$$\begin{aligned} &\frac{\partial^2 \phi_\beta}{\partial x^\alpha x^\gamma} - z\left(\eta z C_\gamma M^{-1}\Phi\right)_{\alpha\beta} = \frac{\partial^2 \phi_\beta}{\partial x^\alpha x^\gamma} - z^2\left(\eta C_\gamma \eta^{-1}\eta M^{-1}\Phi\right)_{\alpha\beta} = \\ &\frac{\partial^2 \phi_\beta}{\partial x^\alpha x^\gamma} - z\sum_{\delta=1}^{N} c^\delta_{\alpha\gamma}\left(z\eta M^{-1}\Phi\right)_{\alpha\beta} = \frac{\partial^2 \phi_\beta}{\partial x^\alpha x^\gamma} - z\sum_{\delta=1}^{N} c^\delta_{\alpha\gamma}\frac{\partial \phi_\beta}{\partial x^\delta} = 0 \end{aligned} \tag{4.63}$$

where we used $\sum_{\delta\omega=1}^N \eta_{\alpha\delta}(C_\gamma)^\delta_\omega \eta^{\omega\beta} = c^\beta_{\alpha\gamma}$.

From (4.31) we find that ϕ_β as defined in (4.60) can be expanded in z as follows :

$$\begin{aligned} \phi_\beta &= \sum_{n=0}^{\infty} \phi^{(n)}_\beta z^n = \phi^{(0)}_\beta + z\phi^{(1)}_\beta + z^2\phi^{(2)}_\beta + z^3\phi^{(3)}_\beta + \cdots = \sum_{\gamma=1}^{N} \eta_{1\gamma}\left(e^{zC}\right)^\gamma_\beta \\ &= \eta_{1\beta} + z\sum_{\alpha=1}^{N} \eta_{1\alpha} C^\alpha_\beta + z^2 \sum_{\gamma=1}^{N} \eta_{1\gamma} (\Xi^{(2)})^\gamma_\beta + z^3 \sum_{\gamma=1}^{N} \eta_{1\gamma} (\Xi^{(3)})^\gamma_\beta + \cdots \end{aligned} \tag{4.64}$$

where we used relation (4.37).

From that we can read:

$$\phi^{(0)}_\beta = \eta_{1\beta}, \qquad \phi^{(1)}_\beta = \sum_{\alpha=1}^{N} \eta_{1\alpha} C^\alpha_\beta = \sum_{\alpha=1}^{N} \eta_{\beta\alpha} x^\alpha \tag{4.65}$$

where in the last equation we have used (4.42) and (4.43). Also,

$$\phi_\beta^{(2)} = \sum_{\gamma=1}^{N} \eta_{1\gamma} (\Xi^{(2)})^\gamma_\beta \quad , \quad \phi_\beta^{(3)} = \sum_{\gamma=1}^{N} \eta_{1\gamma} (\Xi^{(3)})^\gamma_\beta \, . \tag{4.66}$$

Next, we have from (4.59) :

$$\Xi^\alpha_\beta(u, z) = z^{-1} \sum_{\delta=1}^{N} \eta^{\alpha\delta} \frac{\partial \phi_\beta(u, z)}{\partial x^\delta} \, . \tag{4.67}$$

Combining the above two equations we find

$$C^\alpha_\beta(u) = \sum_{\delta=1}^{N} \eta^{\alpha\delta} \frac{\partial \phi^{(2)}_\beta(u)}{\partial x^\delta} \quad \to \quad C_{\alpha\beta}(u) = \frac{\partial \phi^{(2)}_\beta(u)}{\partial x^\alpha} = \frac{\partial \phi^{(2)}_\alpha(u)}{\partial x^\beta} \, . \tag{4.68}$$

Comparing with (4.55) we see that :

$$\phi^{(2)}_\beta = \frac{\partial F}{\partial x^\beta} \, . \tag{4.69}$$

Also, from (4.31) it follows that

$$(\Xi^{(2)})^\alpha_\beta(u) = \sum_{\delta=1}^{N} \eta^{\alpha\delta} \frac{\partial \phi^{(3)}_\beta(u)}{} . \partial x^\delta \tag{4.70}$$

The condition $\Xi(z)\eta^{-1}\Xi(-z)^T = \eta^{-1}$ holds due to the twisting condition. Expanding in z gives on level of z relation $C_{\alpha\beta} = C_{\alpha\beta}$ which is already known. On next, z^2 level we get: $(\Xi^{(2)})^T\eta + \eta\Xi^{(2)} = C^T\eta C = 0$ or

$$\sum_{\gamma=1}^{N} \eta_{\beta\gamma} (\Xi^{(2)})^\gamma_\beta + \sum_{\gamma=1}^{N} \eta_{\alpha\gamma} (\Xi^{(2)})^\gamma_\beta = \sum_{\gamma=1}^{N} \eta_{\alpha\gamma} C^\gamma_\delta C^\delta_\beta \, . \tag{4.71}$$

For $\beta = 1$ one gets from (4.70) and (4.68) that (4.71) is equivalent to

$$\phi^{(2)}_\alpha + \frac{\partial \phi^{(3)}_1}{\partial x^\alpha} = \sum_{\delta=1}^{N} C^\delta_1 \frac{\partial \phi^{(2)}_\alpha}{\partial x^\delta} = \sum_{\delta=1}^{N} x^\delta \frac{\partial \phi^{(2)}_\alpha}{\partial x^\delta} \, . \tag{4.72}$$

Hence as in [12] we find :

$$\phi^{(2)}_\alpha = \frac{\partial F}{\partial x^\alpha} = -\frac{\partial \phi^{(3)}_1}{\partial x^\alpha} + \sum_{\delta=1}^{N} x^\delta \frac{\partial \phi^{(2)}_\alpha}{\partial x^\delta} \quad \to \quad F = -\frac{1}{2}\phi^{(3)}_1 + \frac{1}{2} \sum_{\delta=1}^{N} x^\delta \phi^{(2)}_\delta \, . \tag{4.73}$$

We will now derive an expression for the Euler vector field E in terms of the flat coordinates.

Define

$$\mathcal{U} = M^{-1} U M = \sum_{\alpha,\beta=1}^{N} \mathcal{U}^\alpha_\beta E_{\alpha\beta} \tag{4.74}$$

and notice, that

$$\mathcal{U} = \sum_{i=1}^{N} u_i M^{-1} E_{ii} M = \sum_{i=1}^{N} u_i \partial_i C = E(C) \,. \tag{4.75}$$

From (4.75) and the fact that the first column of matrix C defines the flat variables as in (4.43) we find that

$$E(x^\alpha) = E(C_1^\alpha) = \mathcal{U}_1^\alpha \tag{4.76}$$

from which follows an expression for the Euler operator in terms of flat variables x^α :

$$E = \sum_\alpha \mathcal{U}_1^\alpha \frac{\partial}{\partial x^\alpha} \,. \tag{4.77}$$

Furthermore, from (4.36)

$$E(\Xi^{(n)}) = \sum_j u_j \partial_j \Xi^{(n)} = \sum_j u_j C_j \Xi^{(n-1)} = \mathcal{U}\Xi^{(n-1)}, \quad n \geq 1, \;\; \Xi^{(0)} = I \,. \tag{4.78}$$

Put $n = 1$ in (4.40), it gives :

$$\Xi^{(1)} - \left[\mu \, , \, \Xi^{(1)} \right] = C - [\, \mu \, , \, C \,] = \mathcal{U} \,. \tag{4.79}$$

Hence :

$$\mathcal{U}_\beta^\alpha = (1 - \mu_\alpha + \mu_\beta) C_\beta^\alpha \tag{4.80}$$

and

$$\mathcal{U}_1^\alpha = (1 + \mu_1 - \mu_\alpha)\, x^\alpha \,. \tag{4.81}$$

Accordingly, the Euler vectorfield $E = \sum_i u_i \partial_i$ becomes in terms of the flat coordinates

$$E = \sum_{\alpha=1}^{N} \mathcal{U}_1^\alpha \frac{\partial}{\partial x^\alpha} = \sum_{\alpha=1}^{N} (1 + \mu_1 - \mu_\alpha) x^\alpha \frac{\partial}{\partial x^\alpha} \,. \tag{4.82}$$

Similarly, for the identity vector filed I we find

$$I(x^\alpha) = \sum_{i=1}^{n} \frac{\partial x^\alpha}{\partial u_i} = \sum_{i,\beta=1}^{N} \eta^{\alpha\beta} m_{i1} m_{i\beta} = \sum_{\beta=1}^{N} \eta^{\alpha\beta} \eta_{1\beta} = \delta_{\alpha 1} \tag{4.83}$$

and therefore $I = \partial / \partial x^1$ in terms of the flat coordinates.

5. Monodromy and Frobenius manifold

Let us first introduce notion of monodromy. The notion of monodromy preserving deformations for linear differential equations in the complex plane was first studied by Schlesinger [13]. Consider a linear differential equation with rational coefficients:

$$\frac{dW}{dz} = A(z)\, W \tag{5.1}$$

where $A(z)$ is an $N \times N$ matrix valued function with rational entries. In the case that $A(z)$ has only simple poles in the finite plane one can write:

$$A(z) = \sum_{\nu=1}^{N} \frac{A_\nu}{z - a_\nu} \tag{5.2}$$

In a neighborhood of any regular point for the differential eq. (5.1) one can find a fundamental set of solutions $\{y_1(z), \ldots, y_N(z)\}$. If one analytically continues such a solution $y_j(z)$ around a singular point a_ν it does not in general return to the solution y_j but to a linear combination $\sum_k M^\nu_{kj} y_k$. The matrices M^ν are called monodromy matrices.

A question investigated by Schlesinger was as follows. How must the coefficient matrices A_ν depend on the poles $\{a_1, \ldots, a_n\}$ so that the monodromy matrices M^ν do not depend on the location of the poles. The conditon for this takes a form of a non-linear system of equations

$$\sum_\mu \frac{\partial A_\nu}{\partial a_\mu} da_\mu = - \sum_{\nu \neq \mu} [A_\nu \, , \, A_\mu] \frac{da_\nu - da_\mu}{a_\nu - a_\mu} \, . \tag{5.3}$$

These equations are known as Schlesinger equations. In reference [14] it was shown that for solutions $A_\nu(a)$ to the Schlesinger equations (5.3) the right hand side of this equation is exact:

$$\sum_\mu \frac{\partial \log\left(\tau(a)\right)}{\partial a_\mu} da_\mu = \frac{1}{2} \sum_{\nu \neq \mu} \mathrm{Tr}\left(A_\nu A_\mu\right) \frac{da_\nu - da_\mu}{a_\nu - a_\mu} \, . \tag{5.4}$$

We will now show that that the canonical flows of the Frobenius manifolds reproduce the structure of the Schlesinger equations.

Define

$$S_i = M^{-1} E_{ii} (V - \alpha I) M \tag{5.5}$$

where α is an arbitrary constant.

Recalling relations (4.20) and (4.23) one finds

$$\partial_j S_i = M^{-1} [E_{jj} \, , \, V_i](V - \alpha I) M = \partial_i S_j \tag{5.6}$$

due to the fact that $[E_{jj} \, , \, V_i] = [E_{ii} \, , \, V_j]$.

Thus, locally there exists a function S such that $S_i = \partial_i S$. A calculation based on relations (4.20) and (4.23) yields

$$\frac{\partial}{\partial u_i} \, S_i = \sum_{j=1, \; j \neq i}^{N} \frac{[S_i, S_j]}{u_i - u_j} \; , \tag{5.7}$$

$$\frac{\partial}{\partial u_j} \, S_i = \frac{[S_i, S_j]}{u_j - u_i} \; , \qquad i \neq j \, . \tag{5.8}$$

These are the Schlesinger equations. They can be rewritten in a more compact form as :

$$dS_i = \sum_{j=1, \; j \neq i}^{N} [S_i, S_j] \frac{du_j - du_i}{u_j - u_i}, \qquad d = \sum_j^N \partial_j du_j \, . \tag{5.9}$$

It follows from the Schlesinger equations that

$$S_\infty = \sum_{j=1}^{N} S_j = M^{-1}(V - \alpha I)M = \mathcal{V} - \alpha I \tag{5.10}$$

is constant as already established in (4.21).

The Schlesinger equations can be obtained as compatibility equations of the following linear equations :

$$\frac{d}{d\lambda}\, X = \; - \sum_{i=1}^{N} \frac{S_i}{\lambda - u_i}\, X \;, \qquad \partial_i \, X = \frac{S_i}{\lambda - u_i}\, X \,. \tag{5.11}$$

The compatibility equations reproduce equations (5.7)-(5.8) by evaluating residues at $\{u_i\}_{i=1,\ldots,N}$

The system of first-order differential equations (5.11) can be rewritten as

$$(D \; + \; \mathcal{S})\, X = 0 \;, \tag{5.12}$$

where D :

$$D = \frac{\partial}{\partial\lambda} d\lambda + \sum_{i=1}^{N} \frac{\partial}{\partial u_i} du_i \tag{5.13}$$

is the exterior derivative and $\mathcal{S}$ the flat connection :

$$\mathcal{S} = \sum_{j=1}^{N} \; S_j(\mathbf{u}) \; D \log(\lambda - u_j) = \sum_{j=1}^{N} \; S_j \frac{d\lambda - du_j}{\lambda - u_j} \tag{5.14}$$

which satisfies the zero-curvature condition :

$$D \, \mathcal{S} + \mathcal{S} \wedge \mathcal{S} = 0 \;, \tag{5.15}$$

due to (5.12). Let, $dU = \sum_{j=1}^{N} E_{jj} du_j$. One finds then that

$$D \log(\lambda - U) = (\lambda - U)^{-1}(d\lambda - dU) = \sum_{j=1}^{N} \frac{E_{jj}}{\lambda - u_j}(d\lambda - dU) \tag{5.16}$$

and the connection $\mathcal{S}$ can be rewritten as :

$$\mathcal{S} = M^{-1}(\lambda - U)^{-1}(d\lambda - dU)(V - \alpha I)M = M^{-1}BM \tag{5.17}$$

where in the last identity we defined

$$B = (\lambda - U)^{-1}(d\lambda - dU)(V - \alpha I) = \left(D \log(\lambda - U)\right)(V - \alpha I)\,. \tag{5.18}$$

Let

$$\mathcal{D} = D + \mathcal{S} = D + M^{-1}BM \tag{5.19}$$

then the zero-curvature condition (5.15) implies that $\mathcal{D}^2 = 0$.

It follows that

$$\widetilde{\mathcal{D}} = M\mathcal{D}M^{-1} = D - dMM^{-1} + B = D - A + B \tag{5.20}$$

with

$$A = dMM^{-1} = \sum_{j=1}^{N} V_j du_j \tag{5.21}$$

will also define the flat connection, in agreement with [8, 15]. In components :

$$\begin{aligned}\widetilde{\mathcal{D}} &= \sum_{i=1}^{N}\left(\partial_i - V_i - \frac{E_{ii}(V-\alpha I)}{\lambda - u_i}\right) du_i + \left(\frac{\partial}{\partial\lambda} + \sum_{i=1}^{N}\frac{E_{ii}(V-\alpha I)}{\lambda - u_i}\right) d\lambda \\ &= \sum_{i=1}^{N}\left(\partial_i - V_i - \frac{E_{ii}(V-\alpha I)}{\lambda - U}\right) du_i + \left(\frac{\partial}{\partial\lambda} + \frac{(V-\alpha I)}{\lambda - U}\right) d\lambda\,. \end{aligned} \tag{5.22}$$

Let $\Omega(\mathbf{u},\lambda)$ be such that the condition $\widetilde{\mathcal{D}}\Omega(\mathbf{u},\lambda) = 0$ holds. In components this condition takes a form of

$$\frac{\partial}{\partial\lambda}\Omega(\mathbf{u},\lambda) = \frac{(V-\alpha I)}{U-\lambda}\Omega(\mathbf{u},\lambda) \tag{5.23}$$

$$\partial_i\Omega(\mathbf{u},\lambda) = \left(V_i + \frac{E_{ii}(V-\alpha I)}{\lambda - u_i}\right)\Omega(\mathbf{u},\lambda)\,. \tag{5.24}$$

We will now show that $\Omega(\mathbf{u},\lambda)$ defined as :

$$\Omega(\mathbf{u},\lambda) = \mathrm{Res}_z\left(\Psi(\mathbf{u},z)z^{-1-\alpha}e^{-\lambda z}\right) \tag{5.25}$$

with the wave function $\Psi(\mathbf{u},z)$ which satisfies equations (4.13)-(4.14), will satisfy eqns. (5.23)-(5.24).

It follows that :

$$\begin{aligned}(U-\lambda I)\frac{\partial\Omega(\mathbf{u},\lambda)}{\partial\lambda} &= -\mathrm{Res}_z(U-\lambda I)z^{-\alpha}\Psi(\mathbf{u},z)e^{-\lambda z} \\ = -\mathrm{Res}_z&\left(U\Psi(\mathbf{u},z)z^{-\alpha}e^{-\lambda z} + \Psi(\mathbf{u},z)z^{-\alpha}\frac{\partial e^{-\lambda z}}{\partial z}\right) \\ = -\mathrm{Res}_z&\left(U\Psi(\mathbf{u},z)z^{-\alpha}e^{-\lambda z} - \frac{\partial\Psi(\mathbf{u},z)}{\partial z}z^{-\alpha}e^{-\lambda z} + \alpha z^{-1-\alpha}\Psi(\mathbf{u},z)e^{-\lambda z}\right) \\ = \mathrm{Res}_z&(V-\alpha I)(\mathbf{u})z^{-1-\alpha}\Psi(\mathbf{u},z)e^{-\lambda z} = (V-\alpha I)(\mathbf{u})\Omega(\mathbf{u},\lambda)\,. \end{aligned} \tag{5.26}$$

Equation (4.14) was used in this derivation. Hence $\Omega(\mathbf{u},\lambda)$ indeed satisfies eq. (5.23). Furthermore, due to equation (4.13) it also holds that

$$\begin{aligned}\partial_i\Omega(\mathbf{u},\lambda) &= V_i\Omega(\mathbf{u},\lambda) + E_{ii}\mathrm{Res}_z\left(z^{-\alpha}\Psi(\mathbf{u},z)e^{-\lambda z}\right) = V_i\Omega(\mathbf{u},\lambda) - E_{ii}\frac{\partial}{\partial\lambda}\Omega(\mathbf{u},\lambda) \\ &= V_i\Omega(\mathbf{u},\lambda) + E_{ii}\frac{(V-\alpha I)}{\lambda - U}\Omega(\mathbf{u},\lambda)\end{aligned} \tag{5.27}$$

which agrees with (5.24). Hence W from eqs.(5.11) is given by $W(\mathbf{u},\lambda) = M\Omega(\mathbf{u},\lambda)$.

Note, that due to (4.13)-(4.14) $\Psi(\mathbf{u}, z)$ satisfies $(zd/dz - \sum_{i=1} u_i\partial_i)\Psi(\mathbf{u}, z) = 0$. This leads to equation $(\lambda d/d\lambda + \sum_{i=1} u_i\partial_i)\Omega(\mathbf{u}, \lambda) = 0$.

Plugging S_i into (5.4) we get

$$\partial_j \log \tau = \sum_{k=1,\ k\neq j}^{N} \frac{\mathrm{tr}(S_j S_k)}{u_j - u_k} = \sum_{k=1,\ k\neq j}^{N} \frac{\mathrm{tr}(E_{jj} V E_{kk} V)}{u_j - u_k} = \frac{1}{2}\,\mathrm{tr}(V_j V) \tag{5.28}$$

which reproduces the well-known result for the isomonodromic tau function (4.17). The isomonodromic tau function τ is related to Dubrovin's [3] isomonodromic tau function τ_I as follows: $\tau_I = 1/\sqrt{\tau}$ [6, 7].

6. Darboux-Egoroff Metric, the Two-dimensional Case

For $N = 2$ there are only two canonical coordinates from which one can construct function $\tau_0 = u_1 - u_2$ such that $I(\tau_0) = 0, E(\tau_0) = \tau_0$. Then the tau function $\tau = \tau_0^{R^2}$ satisfies $I(\tau) = 0, E(\tau) = R^2\tau$. In order to satisfy equation (4.13) we take $\beta_{21} = \beta_{12} = iR/\tau_0$ and we find in terms of the Pauli matrices :

$$V_j = [\,\beta\,,\, E_{jj}\,] = \partial_j (R \log \tau_0 \sigma_2) \qquad V = [\,\beta\,,\, U\,] = R\sigma_2\,. \tag{6.1}$$

Solution to equation $(\partial_j - V_j)M_0 = 0$ can be calculated explicitly in $N = 2$ and is given by $M_0 = \exp(\sigma_2 R \log \tau_0)$. Let

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}, \quad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}, \ \eta = S^T S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

Then $\mu = M^{-1} V M = R\sigma_3$ for $M = M_0 S$. Also,

$$\mathcal{U} = M^{-1} U M = \frac{1}{2}\begin{pmatrix} u_1 + u_2 & \tau_0^{1-2R} \\ \tau_0^{1+2R} & u_1 + u_2 \end{pmatrix} \tag{6.3}$$

and

$$\partial_1 C = M^{-1} E_{11} M = \frac{1}{2}\begin{pmatrix} 1 & \tau_0^{-2R} \\ \tau_0^{2R} & 1 \end{pmatrix}, \quad \partial_2 C = M^{-1} E_{22} M = \frac{1}{2}\begin{pmatrix} 1 & -\tau_0^{-2R} \\ -\tau_0^{2R} & 1 \end{pmatrix}. \tag{6.4}$$

For the matrix $\Xi^{(n)}$ we have :

$$E\,(\Xi^{(n)}) = (nI - \mathrm{ad}_\mu)\,\Xi^{(n)} = \begin{pmatrix} n\Xi_{11}^{(n)} & (n-2R)\Xi_{12}^{(n)} \\ (n+2R)\Xi_{21}^{(n)} & n\Xi_{22}^{(n)} \end{pmatrix} = \mathcal{U}\Xi^{(n-1)} \tag{6.5}$$

and therefore from $E(\Xi^{(1)}) = E(C) = (I - \mathrm{ad}_\mu)C = \mathcal{U}$ and (6.3) we derive

$$C = \frac{1}{2}\begin{pmatrix} u_1 + u_2 & \frac{\tau_0^{1-2R}}{1-2R} \\ \frac{\tau_0^{1+2R}}{1+2R} & u_1 + u_2 \end{pmatrix} = \begin{pmatrix} x^1 & \frac{1}{2(1-2R)}(2(1+2R)x^2)^{\frac{1-2R}{1+2R}} \\ x^2 & x^1 \end{pmatrix}, \tag{6.6}$$

valid for $R \neq \pm\frac{1}{2}$. Note, that $E = x^1 \frac{\partial}{\partial x^1} + (1+2R)x^2 \frac{\partial}{\partial x^2}$ and

$$\mathcal{U} = \begin{pmatrix} x^1 & \frac{1}{2}(2(1+2R)x^2)^{\frac{1-2R}{1+2R}} \\ (1+2R)x^2 & x^1 \end{pmatrix} . \tag{6.7}$$

Using expression (6.2) we find :

$$\phi_i^{(2)} = \Xi_{2\,i}^{(2)},\ i = 1, 2 \quad \phi_1^{(3)} = \Xi_{2\,1}^{(3)} \tag{6.8}$$

which when plugged into expression (4.58) or $2F = -\phi_1^{(3)} + \sum_\delta x^\delta \phi_\delta^{(2)}$ yields

$$2F = -\Xi_{2\,1}^{(3)} + x^1 \Xi_{2\,1}^{(2)} + x^2 \Xi_{2\,2}^{(2)} . \tag{6.9}$$

We will use equation (6.5), which for $n = 2$ reads

$$(2I - \mathrm{ad}_\mu)\, \Xi^{(2)} = \begin{pmatrix} 2\Xi_{11}^{(2)} & (2-2R)\Xi_{12}^{(2)} \\ (2+2R)\Xi_{21}^{(2)} & 2\Xi_{22}^{(2)} \end{pmatrix} = \mathcal{U}C \tag{6.10}$$

and plugging $\mathcal{U}$ from (6.3) into the above relation yields

$$\Xi_{1\,1}^{(2)} = \frac{1}{8}\left((u_1+u_2)^2 + \frac{\tau_0^2}{1+2R}\right) = \frac{1}{2}(x^1)^2 + \frac{(2(1+2R)x^2)^{\frac{2}{2R+1}}}{8(1+2R)} \tag{6.11}$$

$$\Xi_{2\,2}^{(2)} = \frac{1}{8}\left((u_1+u_2)^2 + \frac{\tau_0^2}{1-2R}\right) = \frac{1}{2}(x^1)^2 + \frac{(2(1+2R)x^2)^{\frac{2}{2R+1}}}{8(1-2R)} \tag{6.12}$$

$$\Xi_{21}^{(2)} = \frac{1}{4(2+2R)}\left((u_1+u_2)\tau_0^{1+2R}(\frac{1}{1+2R}+1)\right) = x^1 x^2 \tag{6.13}$$

where we used identification between x^1, x^2 and the first column of the C matrix in (6.6). Furthermore, for (6.5) with $n = 3$ we find

$$\Xi_{2\,1}^{(3)} = \frac{1}{2(3+2R)}\left(\tau_0^{2R+1}\Xi_{1\,1}^{(2)} + (u_1+u_2)\Xi_{2\,1}^{(2)}\right) = \frac{1}{2(3+2R)}\left(2(1+2R)x^2\Xi_{1\,1}^{(2)} + 2x^1\Xi_{2\,1}^{(2)}\right) \tag{6.14}$$

or

$$\Xi_{2\,1}^{(3)} = \frac{1}{2(3+2R)}\left((1+2R)(x^1)^2x^2 + 2(x^1)^2x^2 + \frac{(2(1+2R)x^2)^{\frac{3+2R}{2R+1}}}{8(1+2R)}\right) . \tag{6.15}$$

Plugging it into (6.9) gives

$$F = \frac{1}{2}(x^1)^2x^2 + \frac{(2(1+2R)x^2)^{\frac{3+2R}{2R+1}}}{16(3+2R)(1-2R)} \tag{6.16}$$

valid for $R \neq -3/2, R \neq \pm 1/2$. The remaining special cases of $R = -3/2, \pm 1/2$ must be considered separately.

R=-3/2

The problem arises for $R = -3/2$ due to the fact that the matrix operator $(nI - \mathrm{ad}_\mu)$ in equation (6.5) does not have an inverse for $n = 3$. So, instead of using the matrix equation (6.5) we will use $E(\Xi^{(3)}) = \mathcal{U}\Xi^{(2)}$ with the Euler operator $E = x^1 \frac{\partial}{\partial x^1} + (1+2R)x^2 \frac{\partial}{\partial x^2}$ being equal for $R = -3/2$ with :

$$E = x^1 \frac{\partial}{\partial x^1} - 2x^2 \frac{\partial}{\partial x^2} \tag{6.17}$$

Recall from relation (6.9) that in order to calculate the superpotential F we need to find the matrix element $\Xi^{(3)}_{21}$. The relevant recursion relation is

$$E(\Xi^{(3)}_{21}) = (\mathcal{U}\Xi^{(2)})_{21} = -2x^2\Xi^{(2)}_{11} + x^1\Xi^{(2)}_{21} = -\frac{1}{2^5} \tag{6.18}$$

where in the product $(\mathcal{U}\Xi^{(2)})_{21}$ we used $\mathcal{U}$, $\Xi^{(2)}_{11}$ and $\Xi^{(2)}_{21}$ as given in equations (6.7), (6.11) and (6.13) with $R = -3/2$. Solution to the differential equation (6.18) is given by

$$\Xi^{(3)}_{21} = \frac{1}{2}(x^1)^2 x^2 + \frac{1}{2^6}\log(x^2)\,. \tag{6.19}$$

Note, that the first term on the right hand side is annihilated by the Euler vector field (6.17) $E\left((x^1)^2x^2\right) = 0$ and therefore it can not be obtained from the relation (6.18) alone. To obtain this term we used as additional information relation (4.38) which, in view of (4.83), implies $\partial(\Xi^{(3)}_{21})/\partial x^1 = \Xi^{(2)}_{21}$.

The prepotential according to (6.9) is then :

$$2F = -\Xi^{(3)}_{2\,1} + x^1\Xi^{(2)}_{2\,1} + x^2\Xi^{(2)}_{2\,2} = (x^1)^2x^2 - \frac{1}{2^6}\log(x^2) - \frac{1}{2^7}, \quad R = -\frac{3}{2}\,. \tag{6.20}$$

The last term being a constant can be droped.

R=1/2

From (6.4) we derive

$$C = \frac{1}{2}\begin{pmatrix} u_1+u_2 & \log\tau_0 \\ \frac{\tau_0^2}{2} & u_1+u_2 \end{pmatrix} = \begin{pmatrix} x^1 & \frac{1}{2}\log(2\sqrt{x^2}) \\ x^2 & x^1 \end{pmatrix}, \; R = \frac{1}{2} \tag{6.21}$$

or, since C is defined up to a constant :

$$C = \begin{pmatrix} x^1 & \frac{1}{4}\log(x^2) \\ x^2 & x^1 \end{pmatrix}, \; R = \frac{1}{2} \tag{6.22}$$

where $x^1 = \frac{1}{2}(u_1+u_2)$, $x^2 = \tau_0^2/4$ and therefore the Euler operator is

$$E = x^1\frac{\partial}{\partial x^1} + 2x^2\frac{\partial}{\partial x^2}\,. \tag{6.23}$$

Equation (6.3) gives in this case

$$\mathcal{U} = \frac{1}{2}\begin{pmatrix} u_1+u_2 & 1 \\ \tau_0^2 & u_1+u_2 \end{pmatrix} = \begin{pmatrix} x^1 & \frac{1}{2} \\ 2x^2 & x^1 \end{pmatrix} = E(C) \tag{6.24}$$

From $E(\Xi^{(2)}) = \mathcal{U}C$ and (6.23) we find :

$$\Xi^{(2)} = \begin{pmatrix} \frac{1}{2}(x^1)^2 + \frac{1}{4}x^2 & \frac{x^1}{4}\log(x^2) \\ x^1x^2 & \frac{(x^1)^2}{2} + \frac{x^2}{4}\log(x^2) - \frac{x^2}{4} \end{pmatrix} . \tag{6.25}$$

Next

$$\begin{aligned} E(\Xi^{(3)}_{21}) &= (\mathcal{U}\Xi^{(2)})_{21} = 2x^2\Xi^{(2)}_{11} + x^1\Xi^{(2)}_{21} = 2(x^1)^2x^2 + \frac{1}{2}(x^2)^2 \\ \to \Xi^{(3)}_{21} &= \frac{1}{2}(x^1)^2x^2 + \frac{1}{8}(x^2)^2 \end{aligned} \tag{6.26}$$

and the prepotential according to (6.9) is

$$2F = -\Xi^{(3)}_{2\,1} + x^1\Xi^{(2)}_{2\,1} + x^2\Xi^{(2)}_{2\,2} = (x^1)^2x^2 + \frac{(x^2)^2}{4}\left(\log(x^2) - \frac{3}{2}\right), \quad R = \frac{1}{2} . \tag{6.27}$$

R=-1/2

From (6.4) we derive

$$C = \frac{1}{2}\begin{pmatrix} u_1 + u_2 & \frac{\tau_0^2}{2} \\ \log\tau_0 & u_1 + u_2 \end{pmatrix} = \begin{pmatrix} x^1 & \frac{1}{4}e^{4x^2} \\ x^2 & x^1 \end{pmatrix}, \; R = -\frac{1}{2} \tag{6.28}$$

where $x^1 = \frac{1}{2}(u_1 + u_2)$, $x^2 = \frac{1}{2}\log\tau_0$ and therefore the Euler operator is :

$$E = x^1\frac{\partial}{\partial x^1} + \frac{1}{2}\frac{\partial}{\partial x^2} . \tag{6.29}$$

Equation (6.3) gives in this case

$$\mathcal{U} = \frac{1}{2}\begin{pmatrix} u_1 + u_2 & \tau_0^2 \\ 1 & u_1 + u_2 \end{pmatrix} = \begin{pmatrix} x^1 & \frac{1}{2}e^{4x^2} \\ \frac{1}{2} & x^1 \end{pmatrix} = E(C) . \tag{6.30}$$

As in the case $R = 1/2$ we can determine $\Xi^{(2)}$ from $E(\Xi^{(2)}) = \mathcal{U}C$ and equations (6.28)-(6.30) and $\Xi^{(3)}_{21}$ from $E(\Xi^{(3)}_{21}) = (\mathcal{U}\Xi^{(2)})_{21}$. This leads according to (6.9) to

$$2F = -\Xi^{(3)}_{2\,1} + x^1\Xi^{(2)}_{2\,1} + x^2\Xi^{(2)}_{2\,2} = (x^1)^2x^2 + \frac{1}{32}e^{4x^2} = (x^1)^2x^2 + \frac{1}{2^5}e^{4x^2}, \quad R = -\frac{1}{2} \tag{6.31}$$

7. Darboux-Egoroff Metric, the Three-dimensional Case

Let us now consider the three-dimensional manifolds. In this case, we can rewrite the antisymmetric matrix V as:

$$V = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{7.1}$$

or $(V)_{ij} = (u_j - u_i)\beta_{ij} = \epsilon_{ijk}\omega_k$. From (4.18) and (4.19) we see that ω_k vanishes when acted on by the vectorfields E and I. That makes ω_k effectively a function of one variable s such that $E(s) = I(s) = 0$. Let us choose

$$s = \frac{u_2 - u_1}{u_3 - u_1} . \tag{7.2}$$

Then equation (4.20) takes a form equivalent to the Euler top equations:

$$\frac{d\omega_1}{ds} = \frac{\omega_2\omega_3}{s}, \qquad \frac{d\omega_2}{ds} = \frac{\omega_1\omega_3}{s(s-1)}, \qquad \frac{d\omega_3}{ds} = \frac{\omega_1\omega_2}{1-s}\,. \tag{7.3}$$

One verifies that $d(\sum_{k=1}^{3}\omega_k^2)/ds = 0$. Consequently,

$$\sum_{k=1}^{3}\omega_k^2 = -R^2 \tag{7.4}$$

where a constant R^2 is an integral of equations (7.3). The same constant R^2 characterizes the homogeneity of the tau function. Indeed, starting from expression (4.17) one finds for the scaling dimension [16]

$$E(\log\tau) = \frac{1}{2}\sum_{j=1}^{3} u_j \,\mathrm{tr}\,(V_j V) = \frac{1}{2}\,\mathrm{tr}\,\left(V^2\right) = \frac{1}{2}\,\mathrm{tr}\,\left(\mu^2\right) = \frac{1}{2}\sum_{\alpha=1}^{3}\mu_\alpha^2\,. \tag{7.5}$$

Recalling that $(V)_{ij} = \epsilon_{ijk}\omega_k$ we can rewrite the above as :

$$E(\log\tau) = \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}(\epsilon_{ijk}\omega_k)^2 = -\sum_{k=1}^{3}\omega_k^2 = R^2\,. \tag{7.6}$$

As shown in [17], for η_{11} different from zero the homogeneity of the Lamé coefficients h_i must vanish. In such case, the Lamé coefficients h_i depend only on one variable s due to the fact that $I(h_i) = E(h_i) = 0$. The relations $\partial_j h_i^2 = \partial_i h_j^2$ translate for the function $h_i^2(s)$ to

$$s\frac{dh_1^2}{ds} = (s-1)s\frac{dh_2^2}{ds} = (1-s)\frac{dh_3^2}{ds}\,. \tag{7.7}$$

Also, since

$$\omega_k = \frac{u_j - u_i}{2h_i h_j}\frac{\partial h_i^2}{\partial u_j}, \quad i,j,k \text{ cyclic} \tag{7.8}$$

we find e.g.

$$\omega_3 = \frac{s}{2h_1h_2}\frac{dh_1^2}{ds}, \;\; \omega_2 = \frac{s}{2h_1h_3}\frac{dh_1^2}{ds} \tag{7.9}$$

and so $h_3\omega_2 = h_2\omega_3$ and similarly $h_1\omega_2 = h_2\omega_1$. We conclude that

$$\omega_i^2 = -\frac{R^2}{\eta_{11}}h_i^2, \quad i = 1,2,3 \tag{7.10}$$

and comparing equations (7.3) with equation (7.7) we obtain like in [18] :

$$s\frac{dh_1^2}{ds} = (s-1)s\frac{dh_2^2}{ds} = (1-s)\frac{dh_3^2}{ds} = -2i\frac{R}{\sqrt{\eta_{11}}}h_1h_2h_3\,. \tag{7.11}$$

8. Rational Landau - Ginsburg Models

In this section we will show how to associate the canonical Darboux-Egoroff structure to the rational Landau - Ginsburg models. Following Aoyama and Kodama [19] we study a rational potential :

$$\begin{aligned} W(z) = & \frac{1}{n+1} z^{n+1} + a_{n-1} z^{n-1} + \ldots + a_0 + \frac{v_1}{z - v_{m+1}} + \frac{v_2}{2(z-v_{m+1})^2} + \ldots \\ & + \frac{v_m}{m(z-v_{m+1})^m} \end{aligned} \tag{8.1}$$

which is known to characterize the topological Landau-Ginzburg (LG) theory. The rational potential in this form can be regarded as the Lax operator of a particular reduction of the dispersionless KP hierarchy [19, 20, 21]

The space of rational potentials from (8.1) is naturally endowed with the metric :

$$g(\partial_t W, \partial_{t'} W) = \mathrm{Res}_{z \in \mathrm{Ker} W'} \left(\frac{\partial_t W \partial_{t'} W}{W'} \right) dz \tag{8.2}$$

where $\partial_t W = \partial_t a_{n-1} z^{n-1} + \ldots + \partial_t a_0 + \frac{\partial_t v_1}{z - v_{m+1}} + \ldots$ describes a tangent vector to the space of rational potentials obtained by taking derivative of all coefficients with respect to their argument. $W'(z)$ is a derivative with respect to z of the rational potential W :

$$W'(z) = z^n + (n-1) a_{n-1} z^{n-2} + \ldots - \frac{v_m}{(z-v_{m+1})^{m+1}} \,. \tag{8.3}$$

Next, we find the flat coordinates $x_\alpha, \alpha = 1, \ldots, m+1$ and $\tilde{x}_\gamma, \gamma = 1, .., n$ such that

$$g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}) = \eta_{\alpha\beta}, \quad g(\frac{\partial W}{\partial \tilde{x}_\gamma}, \frac{\partial W}{\partial \tilde{x}_\delta}) = \tilde{\eta}_{\gamma\delta}, \quad g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial \tilde{x}_\gamma}) = 0 \tag{8.4}$$

with constant and non-degenerate matrices $\eta_{\alpha\beta}$ and $\tilde{\eta}_{\gamma\delta}$.

Consider first the function $w = w(W, z)$ such that $W(z) = w^{-m}/m$ and $z = x_{m+1} + x_m w + \ldots + x_1 w^m = \sum_{\alpha=1}^{m+1} x_\alpha w^{m+1-\alpha}$. We take $z \sim x_{m+1}$ or $|w| \ll 1$. It follows that

$$W' dz = -\frac{1}{w^{m+1}} dw, \qquad \frac{\partial W}{\partial x_\alpha} = W' \frac{\partial z}{\partial x_\alpha} = W' w^{m+1-\alpha} \tag{8.5}$$

Consequently:

$$\begin{aligned} g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}) & = -\mathrm{Res}_{z=\infty} \left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) dz \\ = -\mathrm{Res}_{z=\infty} \left(W' w^{m+1-\alpha} w^{m+1-\beta} \right) dz & = \mathrm{Res}_{w=\infty} \left(\frac{w^{m+1-\alpha} w^{m+1-\beta}}{w^{m+1}} \right) dw = \delta_{\alpha+\beta=m+2} \,. \end{aligned} \tag{8.6}$$

Hence x_α are flat coordinates with the metric $\eta_{\alpha\beta} = \delta_{\alpha+\beta=m+2}$. The coefficients v_j, $j = 1, \ldots, m+1$ of $W(z)$ are given in terms of the flat coordinates as [19]:

$$\begin{aligned} v_k & = \sum_{\alpha_1 + \ldots + \alpha_k = (k-1)m + k} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}, \quad k = 1, \ldots, m \\ v_{m+1} & = x_{m+1} \,. \end{aligned} \tag{8.7}$$

Examples are :

$$v_m = (x_m)^m,\ v_{m-1} = (m-1)x_{m-1}(x_m)^{m-2},\ \ldots,\ v_1 = x_1\,. \tag{8.8}$$

To represent the remaining coefficients of $a_i, i = 1, \ldots, n$ of W in terms of the flat coordinates we consider a relation:

$$z = w + \frac{\tilde{x}_1}{w} + \frac{\tilde{x}_2}{w^2} + \ldots + \frac{\tilde{x}_n}{w^n} \tag{8.9}$$

valid for large z and $|w| \gg 1$. In this limit we impose a relation $W = w^{n+1}/(n+1)$ from which it follows that

$$W' dz = w^n dw, \qquad \frac{\partial W}{\partial \tilde{x}_\gamma} = W' \frac{\partial z}{\partial \tilde{x}_\gamma} = W' w^{-\gamma}\,. \tag{8.10}$$

We find

$$\begin{aligned} g(\frac{\partial W}{\partial \tilde{x}_\gamma}, \frac{\partial W}{\partial \tilde{x}_\delta}) &= \mathrm{Res}_{z \in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial \tilde{x}_\gamma)(\partial W/\partial \tilde{x}_\delta)}{W'} \right) dz \\ = \mathrm{Res}_{z \in \mathrm{Ker} W'} \Big(W' w^{-\gamma} w^{-\delta} \Big)\, dz &= \mathrm{Res}_{w=0} w^{n-\gamma-\delta} dw = \delta_{\gamma+\delta=n+1}\,. \end{aligned} \tag{8.11}$$

Hence $\tilde{x}_\gamma$ are flat coordinates with the metric $\tilde{\eta}_{\gamma\delta} = \delta_{\gamma+\delta=n+1}$. By similar considerations $\eta_{\alpha\gamma} = 0$ for $\alpha = 1, \ldots, m+1, \gamma = 1, \ldots, n$.

From expression (8.9) and $W(z) = w^{n+1}/(n+1)$ one can find relations between coefficients a_γ and $\tilde{x}_\gamma$ [19] starting with $a_{n-1} = -\tilde{x}_1$ and so on.

We will now show how to associate to the rational potentials W canonical coordinates $u_i, i = 1, \ldots, n+m+1$ for which the metric (8.2) becomes a Darboux-Egoroff metric.

Let α_i, $i = 1, \ldots, n+m+1$ be roots of the rational potential $W'(z)$ in (8.3). Equivalently, $W'(\alpha_i) = 0$ for all $i = 1, \ldots, n+m+1$. Thus $W'(z)$ can be rewritten as

$$W'(z) = \frac{\prod_{j=1}^{n+m+1} (z - \alpha_j)}{(z - v_{m+1})^{m+1}}\,. \tag{8.12}$$

Next, define the canonical coordinates as

$$u_i = W(\alpha_i), \quad i = 1, \ldots, n+m+1\,. \tag{8.13}$$

The identity :

$$\begin{aligned} \delta^i_j = \frac{\partial u_i}{\partial u_j} &= \frac{\partial W(\alpha_i)}{\partial u_j} \\ &= W'(\alpha_i) \frac{\partial \alpha_i}{\partial u_j} + \frac{\partial W}{\partial u_j}(\alpha_i) = \frac{\partial W}{\partial u_j}(\alpha_i) \end{aligned} \tag{8.14}$$

implies that

$$\frac{\partial W}{\partial u_j}(z) = \frac{\partial a_{n-1}}{\partial u_j} z^{n-1} + \ldots + \frac{\partial a_0}{\partial u_j} + \frac{\partial v_1/\partial u_j}{z - v_{m+1}} + \ldots + \frac{v_m}{(z - v_{m+1})^{m+1}} \frac{\partial v_{m+1}}{\partial u_j} \tag{8.15}$$

can be rewritten as

$$\frac{\partial W}{\partial u_j}(z) = \frac{\prod_{k=1,j\neq k}^{n+m+1}(z-\alpha_k)}{(z-v_{m+1})^{m+1}} \frac{(\alpha_j - v_{m+1})^{m+1}}{\prod_{k=1,j\neq k}^{n+m+1}(\alpha_j-\alpha_k)} . \tag{8.16}$$

Consider

$$g(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_j}) = \mathrm{Res}_{z\in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial u_i)(\partial W \partial u_j)}{W'} \right) dz . \tag{8.17}$$

Recalling (8.12) and (8.16) we find that $g(\partial W/\partial u_i, \partial W/\partial u_j) = 0$ for $i \neq j$. For $i = j$, we find

$$\begin{aligned} g(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_i}) &= \mathrm{Res}_{z\in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial u_i)^2}{W'} \right) dz \\ &= \frac{(\alpha_i - v_{m+1})^{m+1}}{\prod_{j=1,j\neq i}^{n+m+1}(\alpha_i-\alpha_j)} = \frac{\partial a_{n-1}}{\partial u_i} \end{aligned} \tag{8.18}$$

where the last identity was obtained by comparing coefficients of the z^{n-1} term in (8.15) and (8.16).

Hence, in terms of the coordinates u_i the metric can be rewritten as $g = \sum_{i=1}^N h_i^2(u)(du_i)^2$ with the Lamé coefficients :

$$h_i^2(u) = \frac{\partial a_{n-1}}{\partial u_i} . \tag{8.19}$$

8.1 N=3 Model, Example of Rational Landau - Ginsburg models

Consider the model with $n = m = 1$ in (8.1):

$$W(z) = \frac{1}{2}z^2 + x_1 + \frac{x_2}{z - x_3} \tag{8.20}$$

where as coefficients we used the flat coordinates $x_1 = -\tilde{x}_1$ and x_2, x_3 corresponding to x_1, x_2 of the previous section. The flat coordinates $x_\alpha, \alpha = 1, 2, 3$ are related to the flat metric :

$$\eta^{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{Res}_{z\in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) dz = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{8.21}$$

The metric tensor can be derived from the more general expression involving the structure constants

$$c^{\alpha\beta\gamma} = \mathrm{Res}_{z\in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)(\partial W/\partial x_\gamma)}{W'} \right) dz \tag{8.22}$$

through relation $\eta^{\alpha\beta} = c^{\alpha\beta 1}$. The non-zero values of the components of $c_{\alpha\beta\gamma}$ are found from (8.22) to be :

$$c^{111} = 1, \; c^{123} = 1, c^{222} = 1/x_2, \; c^{233} = x_3, \; c^{333} = x_2 \tag{8.23}$$

the other values can be derived using that $c^{\alpha\beta\gamma}$ is symmetric in all three indices. These values can be reproduced from the formula (4.55) with the prepotential :

$$F(x_1, x_2, x_3) = \frac{1}{6}x_2(x_3)^3 + \frac{1}{6}(x_1)^3 + x_1x_2x_3 + \frac{1}{2}(x_2)^2 \left(\log x_2 - \frac{3}{2} \right) . \tag{8.24}$$

The prepotential satisfies the quasi-homogeneity relation (1.2) with $d_F = 3$ with respect to the Euler vectorfield :

$$E = x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + \frac{1}{2} x_3 \frac{\partial}{\partial x_3} = x^1 \frac{\partial}{\partial x^1} + \frac{1}{2} x^2 \frac{\partial}{\partial x^2} + \frac{3}{2} x^3 \frac{\partial}{\partial x^3} . \tag{8.25}$$

We now adopt a general discussion of canonical coordinates from Section 7 to the case of $n = m = 1$. Let α_i, $i = 1, 2, 3$ be roots of the polynomial $W'(z) = z - x_2/(z - x_3)^2$. So, α_i satisfy $W'(\alpha_i) = 0$ or $\alpha_i(\alpha_i - x_3)^2 - x_2 = 0$ for all $i = 1, 2, 3$.

Then, it follows by taking derivatives of $\alpha_i(\alpha_i - x_3)^2 = x_2$ with respect to x_2, x_3 that

$$\frac{\partial \alpha_i}{\partial x_3} = \frac{2\alpha_i}{3\alpha_i - x_3}, \qquad \frac{\partial \alpha_i}{\partial x_2} = \frac{1}{(\alpha_i - x_3)(3\alpha_i - x_3)} \tag{8.26}$$

and further that

$$\frac{\partial u_i}{\partial x_3} = \frac{x_2}{(\alpha_i - x_3)^2} = \alpha_i, \qquad \frac{\partial u_i}{\partial x_2} = \frac{1}{\alpha_i - x_3} \tag{8.27}$$

for the canonical coordinates $u_i = W(\alpha_i) = \frac{1}{2}\alpha_i^2 + x_1 + x_2/(\alpha_i - x_3)$. We now present a method of inverting the derivatives in (8.27) or alternatively to find the matrix elements m_{ij} of the matrix M from relation (4.22). The sum of the canonical coordinates is equal to $\sum_{i=1}^3 u_i = 3x_1 + x_3^2$ and therefore

$$1 = 3 \frac{\partial x_1}{\partial u_i} + 2x_3 \frac{\partial x_3}{\partial u_i} = h_i^2 \left(3 + 2x_3 \frac{\partial u_i}{\partial x_2} \right) \tag{8.28}$$

where we used the fact that

$$\frac{\partial x_3}{\partial u_i} = m_{i1}^2 \frac{\partial u_i}{\partial x_2} \tag{8.29}$$

because of

$$\frac{\partial x_\alpha}{\partial u_i} = m_{i1} m_{i\alpha}, \quad \frac{\partial u_i}{\partial x_\alpha} = \eta_{\alpha\beta} \frac{m_{i\beta}}{m_{i1}}, \quad h_i^2 = m_{i1}^2 \tag{8.30}$$

Hence, from relation (8.28) it holds that $h_i^2 = \left(3 + 2x_3 \frac{\partial u_i}{\partial x_2} \right)^{-1}$ or by using equation (8.27) that

$$\frac{\partial x_1}{\partial u_i} = h_i^2 = \frac{\alpha_i - x_3}{3\alpha_i - x_3} . \tag{8.31}$$

Plugging the last equation into equation (8.29) and using relation (8.27) we obtain

$$\frac{\partial x_3}{\partial u_i} = \frac{1}{3\alpha_i - x_3} . \tag{8.32}$$

Similarly, from

$$\frac{\partial x_2}{\partial u_i} = m_{i1}^2 \frac{\partial u_i}{\partial x_3} \tag{8.33}$$

we obtain

$$\frac{\partial x_2}{\partial u_i} = \frac{x_2}{(\alpha_i - x_3)(3\alpha_i - x_3)} = \frac{\alpha_i(\alpha_i - x_3)}{(3\alpha_i - x_3)} . \tag{8.34}$$

Furthermore,

$$\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_i}{\partial x_2} \frac{\partial x_2}{\partial u_j} + \frac{\partial \alpha_i}{\partial x_3} \frac{\partial x_3}{\partial u_j} \tag{8.35}$$

gives for $i \neq j$:

$$\frac{\partial \alpha_i}{\partial u_j} = \frac{1}{(3\alpha_i - x_3)(3\alpha_j - x_3)} \left(\frac{\alpha_j(\alpha_j - x_3)}{(3\alpha_i - x_3)} + 2\alpha_i \right) \tag{8.36}$$

and for $i = j$:

$$\frac{\partial \alpha_i}{\partial u_i} = \frac{3\alpha_i}{(3\alpha_i - x_3)^2} \, . \tag{8.37}$$

Using (8.36) we can take a derivative of h_i^2 in (8.31) and find the rotation coefficients defined in (2.2) to be

$$\beta_{ij} = -\frac{(\alpha_k - x_3)(3\alpha_k - x_3)}{(3\alpha_i - x_3)(3\alpha_j - x_3)} \frac{1}{\sqrt{(\alpha_i - x_3)(3\alpha_i - x_3)(\alpha_j - x_3)(3\alpha_j - x_3)}} \, . \tag{8.38}$$

Its square is then

$$\beta_{ij}^2 = -\frac{1}{(\alpha_i - \alpha_j)^2} \frac{1}{(4x_3 - 3\alpha_k)^2} \frac{\partial x_1}{\partial u_k}, \tag{8.39}$$

where i, j, k are cyclic. Recall that in equation (7.1) we have introduced the functions $\omega_k = (u_j - u_i)\beta_{ij}$, where again we used the cyclic indices i, j, k. The difference of canonical coordinates can be written as : $u_j - u_i = (\alpha_i - \alpha_j)(3\alpha_k - 4x_3)/2$ which together with equation (8.38) yields:

$$\omega_k^2 = -\frac{1}{4} h_k^2 = -\frac{1}{4} \frac{\partial x_1}{\partial u_k} = -\frac{1}{4} \frac{\alpha_k - x_3}{3\alpha_k - x_3} \, . \tag{8.40}$$

Since $I = \sum_{i=1}^{3} \partial/\partial u_i = \partial/\partial x_1$ then

$$\sum_{k=1}^{3} \omega_k = -\frac{1}{4}, \quad E(\log \tau) = \frac{1}{4} \, . \tag{8.41}$$

The explicit form of the roots α_i is needed to find expressions for ω_k and its dependence on the parameter s. It is convenient to introduce $q = x_2/(x_3)^3$ and $a_i = \alpha_i/x_3$ which satisfy equation $a_i(a_i - 1)^2 = q$. Let us furthermore introduce a parameter ω such that $q = 4(\omega^2 - 1)^2/(\omega^2 + 3)^3$. This parametrization makes it possible to obtain the compact expressions for ω_k. The three solutions to the algebraic equation

$$a(a-1)^2 = q = 4 \frac{(\omega^2 - 1)^2}{(\omega^2 + 3)^3} \tag{8.42}$$

are:

$$a_1 = \frac{4}{\omega^2 + 3}, \quad a_2 = \frac{(\omega + 1)^2}{\omega^2 + 3}, \quad a_3 = \frac{(\omega - 1)^2}{\omega^2 + 3} \, . \tag{8.43}$$

Note, that $a_2 \leftrightarrow a_3$ under $\omega \leftrightarrow -\omega$ transformation, which shows that ω is a purely imaginary variable. First, we find that the variable s from (7.2) can be expressed as :

$$s = \frac{(a_2 - a_1)}{(a_3 - a_1)} \frac{(3a_3 - 4)}{(3a_2 - 4)} = \frac{(\omega - 3)^3(\omega + 1)}{(\omega + 3)^3(\omega - 1)} \tag{8.44}$$

Next, from relations $h_i^2 = (a_i - 1)/(3a_i - 1)$ and equation (8.40) we derive :

$$\omega_1^2 = -\frac{1}{4} \frac{(\omega^2 - 1)}{(\omega^2 - 9)}, \quad \omega_2^2 = \frac{1}{4} \frac{(\omega + 1)}{\omega(\omega - 3)}, \quad \omega_3^2 = -\frac{1}{4} \frac{(\omega - 1)}{\omega(\omega + 3)} \, . \tag{8.45}$$

They provide solutions to the Euler top equations (7.3). The corresponding function [22, 18]

$$y(\omega) = \frac{(\omega-3)^2(\omega+1)}{(\omega+3)(\omega^2+3)} \tag{8.46}$$

connected with ω_k's through relations [23, 24, 25]:

$$\begin{aligned}
\omega_1^2 &= -\frac{(y-s)y^2(y-1)}{s}\left(v-\frac{1}{2(y-s)}\right)\left(v-\frac{1}{2(y-1)}\right) \,, \\
\omega_2^2 &= \frac{(y-s)^2y(y-1)}{s(1-s)}\left(v-\frac{1}{2(y-1)}\right)\left(v-\frac{1}{2y}\right) \,, \\
\omega_3^2 &= -\frac{(y-s)y(y-1)^2}{(1-s)}\left(v-\frac{1}{2y}\right)\left(v-\frac{1}{2(y-s)}\right)
\end{aligned} \tag{8.47}$$

with the auxiliary variable v defined by equation

$$\frac{dy}{ds} = \frac{y(y-1)(y-s)}{s(s-1)}\left(2v-\frac{1}{2y}-\frac{1}{2(y-1)}+\frac{1}{2(y-s)}\right) \tag{8.48}$$

is a solution of the Painlevé VI equation [23, 24, 25]:

$$\begin{aligned}
\frac{d^2y}{ds^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-s}\right)(\frac{dy}{ds})^2-\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{y-s}\right)\frac{dy}{ds} \\
&+ \frac{y(y-1)(y-s)}{s^2(s-1)^2}\left[\frac{1}{8}-\frac{s}{8y^2}+\frac{s-1}{8(y-1)^2}+\frac{3s(s-1)}{8(y-s)^2}\right] .
\end{aligned} \tag{8.49}$$

Introducing parameter $x=(\omega-3)/(\omega+3)$ one can rewrite expressions (8.46) and (8.44) as :

$$y = \frac{x^2(x+2)}{x^2+x+1} \,, \qquad s = \frac{x^3(x+2)}{2x+1} \,, \tag{8.50}$$

which reproduces the $k=3$ Poncelet polygon solution of Hitchin [24, 25].

We now proceed to calculate the underlying τ-function. Our knowledge of the τ-function is based on equation (4.17) from which we derive that

$$\partial_j \log\tau = \sum_{i=1}^{3} \beta_{ij}^2 (u_i-u_j) \,. \tag{8.51}$$

The identity $I(\log\tau)=0$, shows that $\tau=\tau(x_2,x_3)$ is a function of two variables x_2, x_3. Furthermore, it satisfies :

$$E(\log\tau) = \left(\frac{3}{2}x_2\frac{\partial}{\partial x_2}+\frac{1}{2}x_3\frac{\partial}{\partial x_3}\right)\log\tau = \frac{1}{4} \,. \tag{8.52}$$

A solution to the above equation is

$$\log\tau = \frac{1}{4}\left(\frac{1}{3}\log x_2+\log x_3\right)+f\left(\frac{1}{3}\log x_2-\log x_3\right) \tag{8.53}$$

where $f(\cdot)$ is an arbitrary function of it's argument. In order to determine the function f we use equation (8.51) to calculate the derivative

$$\frac{\partial \log \tau}{\partial x_3} = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_3} \partial_j \log \tau = \sum_{i,j=1}^{3} \alpha_j \beta_{ij}^2 (u_i - u_j)\,. \tag{8.54}$$

A calculation based on equation (8.39) yields:

$$x_3 \frac{\partial}{\partial x_3} \log \tau = \frac{1}{8} \frac{1}{1 - \frac{27}{4} q} = \frac{1}{4} - f' \left(\frac{1}{3} \log x_2 - \log x_3 \right) \tag{8.55}$$

where the last equality was obtained by comparing with equation (8.53) (recall that $q = x_2/(x_3)^3$). Integration gives (ignoring an inessential integration constant) :

$$f \left(\frac{1}{3} \log x_2 - \log x_3 \right) = \frac{1}{24} \left(\log q + \log(-4 + 27q) \right) \,. \tag{8.56}$$

Using that $x_2 = q x_3^3$ we can now rewrite $\log \tau$ as

$$\log \tau = \frac{1}{4} \log x_3^2 + \frac{1}{24} \log \left(q^3 (-4 + 27q) \right) \,. \tag{8.57}$$

Inserting parametrization of q from (8.42) and using relation $u_2 - u_3 = 8 x_3^2 \omega^3 (\omega^2 + 3)^{-2}$ we obtain the following expression for $\log \tau$:

$$\log \tau = \log(u_2 - u_3)^{\frac{1}{4}} + \frac{1}{24} \log \left((\omega - 1)^6 (\omega + 1)^6 (\omega - 3)^2 (\omega + 3)^2 \omega^{-16} \right) \,. \tag{8.58}$$

It is easy to confirm $I(\log \tau) = 0$ and $E(\log \tau) = 1/4$ based on this expression.

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