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String Theory on AdS_3

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This lecture is a short review of string theory on three dimensional anti-de Sitter spacetime. We critically examine the important progress achieved in recent years, emphasizing the relevance of the AdS/CFT correspondence in our comprehension of the theory, and pointing out some issues which remain unclear. We discuss various aspects of the Feigin-Fuchs construction of this non-rational conformal field theory.

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1. Introduction

This lecture is about string propagation on three dimensional anti de Sitter spacetime. This has been a subject of intensive investigation since the early nineties. The original motivation can be traced to the exact conformal invariance of the theory which can be formulated as a WZW model on the $SL(2,\mathbb{R})$ group manifold. Actually the non linear sigma model describing strings on a background AdS_3 geometry plus antisymmetric NS tensor field is the simplest example of exactly solvable string theory with a single non-trivial timelike direction. Interest in this model is enhanced by its close connection to two and three dimensional black holes (which can be constructed as cosets and orbifolds of AdS_3 respectively), so that understanding this string theory allows to address many important conceptual problems of black hole physics. See [1] for a list of references of this initial period.

A more recent incentive for studying this theory arises from the AdS/CFT correspondence [2]. This is a concrete realization of the holographic idea which establishes a duality between gravity or string/M theory in the bulk of anti-de Sitter space and a quantum field theory on the boundary of spacetime. In this context also AdS_3 is special. Both the theory on the boundary and the non-linear sigma model describing the worldsheet of the string in the bulk of AdS_3 have an infinite dimensional local algebra of conformal transformations. This allows on the one hand to verify the correspondence beyond the supergravity approximation, *i.e*. at the full string theory level, and on the other hand one can get important information on the string theory using the conjecture. Reference [3] presents a profound investigation of this concrete $\mathrm{AdS}_3/\mathrm{CFT}_2$ example (previous work can be found in [4]).

Lately there has been a revival of noncritical string theories in two dimensions. This is largely due to the recent extension of the Matrix model to include fermions as well as to include a variety of backgrounds such as D-branes and black holes. These non-critical strings come in several varieties: Type 0A, Type 0B, Type I, Type IIA, Type IIB, Heterotic $E_8\times SO(8)$ and Heterotic $SO(24)$ and therefore one is interested in constructing three dimensional non-critical M theory to obtain a unified model of non-critial string theories in the same fashion as one proceeds with the ten dimensional parents. Non-critical three dimensional M theory is expected to have solutions of three dimensional black holes, AdS_3 and $AdS_2\times S^1$. This 3D M theory may give important clues about the 11 dimensional M theory.

The status of the perturbative world-sheet theory on AdS_3 can be summarized as follows. The spectrum of physical states has been definitely established in reference [5] where a unitary Hilbert space was revealed after fully appreciating the role of the spectral flow symmetry. The construction of a modular invariant partition function in [6, 7] supplied an independent check on the field content of the model and further confirmed the well-defined structure of the free theory. However in order to establish the complete consistency of the model one has to consider interactions and verify the closure of the operator product expansion. But the fusion rules are difficult to find in the non-compact worldsheet CFT that defines string theory on AdS_3 because of the non-rational structure of the model. There are generically no null vectors in the relevant current algebra representations, so that most of the techniques from rational conformal field theories are not available and consequently the factorization properties of the model have not been fully determined yet. Nevertheless, important progress has been achieved in recent years and this will be the subject of the forthcoming pages.

The outline of this short review is as follows. In Section 2 a brief introduction to free string propagation on AdS$_3$ and the spectrum is presented. The correlation functions on the sphere are the subject of Section 3 and an analysis of the partition function is performed in Section 4. Conclusions are put forward in Section 5.

2. Review of free string theory on AdS$_3$

AdS_3 is a maximally symmetric solution of Einstein's equations with negative cosmological constant. A constructive definition of this manifold can be given by embedding it into a four dimensional space with signature $(-,-,+,+)$ satisfying a quadratic constraint

$$\begin{aligned} ds^2 &= -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2 \\ & -X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = l^2 \end{aligned} \tag{2.1}$$

where l is the radius of the manifold, related to the scalar curvature as $R = -\frac{2}{l^2}$.

There are several possible parametrizations of AdS$_3$. In global coordinates the metric takes the form

$$ds^2 = l^2(-cosh\rho\ dt^2 + d\rho^2 + sinh\rho\ d\theta) \tag{2.2}$$

where $t \in [0,2\pi), \rho \in [0,\infty)$ and $\theta \in [0,2\pi)$. In order to avoid closed timelike geodesics and the causality problems associated with an angular time coordinate, one works in the universal covering of AdS_3. This amounts to decompactifying t and considering $t \in \mathbb{R}$. Note that θ is an angular coordinate and ρ takes real positive values, *i.e.* it is like a *radial* coordinate so one refers to $\rho = 0$ as the center of the manifold and $\rho \to \infty$ as the boundary of AdS$_3$ (actually the time a light ray takes to get arbitrarily far from any point is finite).

An alternative parametrization is given by Poincaré coordinates, leading to the metric

$$ds^2 = l^2(d\phi^2 - e^{2\phi} d\gamma^+ d\gamma^-) \tag{2.3}$$

where $\{\phi, \gamma^+, \gamma^-\} \in \mathbb{R}^3$. It is easy to verify that these coordinates only cover one half of spacetime.

Analytically continuing to the complex plane one obtains the euclidean hyperbolic space H_3^+ with metric

$$ds^2 = l^2(d\phi^2 - e^{2\phi} d\gamma d\overline{\gamma}) \quad . \tag{2.4}$$

Writing explicitly the Poincaré coordinates in terms of global coordinates it is easy to see that the limit $\rho \to \infty$ is equivalent to $\phi \to \infty$. Therefore the boundary of AdS_3 is parametrized by $\{\gamma, \overline{\gamma}\}$ and the remote past ($t \to -\infty$) and future ($t \to \infty$) can be mapped to the points 0 and ∞ of the complex plane or to the poles of a sphere through a stereographic projection.

Being a maximally symmetric space, AdS_3 possesses the maximum number of generators. The isometry group can be easily constructed observing the $SO(2,2)$ symmetry of (2.1) which is locally $SO(2,2) \sim SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$ *i.e.* two commuting copies of $SL(2,\mathbb{R})$, namely

$$\left[J^3, J^\pm\right] = \pm J^\pm \quad , \quad \left[J^+, J^-\right] = -2J^3 \quad . \tag{2.5}$$

The classical solutions of this theory were presented in [5]. Timelike geodesics oscilate around the center of AdS_3 whereas spacelike geodesics representing tachyons travel from one side of the boundary to the opposite. Solutions describing string propagation are obtained from the dynamics of pointlike particles. Timelike geodesics give rise to *short strings*, bound states trapped in the gravitational potential of AdS_3. Conversely, *long strings* arising from spacelike geodesics can reach the boundary. Given one classical solution one can generate new solutions by the spectral flow operation which amounts to stretching the geodesics in the time direction and rotating them w times around the center of AdS_3. The spectral flow parameter w, named *winding number*, is an integer representing the number of revolutions in the light-cone direction $t+\theta$ [1]. Different values of w correspond to distinct solutions, even at the classical level (as exhibited, for instance, by the energy spectrum).

Consistent string propagation in the background metric (2.4) requires in addition an antisymmetric rank two tensor background field $B = e^{2\phi} d\gamma \wedge d\bar{\gamma}$. The theory is described by a non linear sigma model with action

$$S = \frac{k}{8\pi}\int d^2z(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}) \quad , \tag{2.6}$$

which is equivalent to a WZW model on the universal cover of $SL(2,\mathbb{R})_k$ (or actually its Euclidean version $SL(2,\mathbb{C})/SU(2)$) with action

$$S_{WZW} = \frac{k}{8\pi}\int_\Sigma \mathrm{d}^2\sigma Tr\,[(\partial_\alpha g)(\partial^\alpha g^{-1})] + \frac{k}{12\pi}\int_V \mathrm{d}^3x \varepsilon^{ijk} Tr\,[(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)] \tag{2.7}$$

where $g \in SL(2,\mathbb{R})$, Σ is the worldsheet with $\partial V = \Sigma$ and k is the level of the Kac-Moody algebra. Actually this action has a larger symmetry than the isometries of the group, namely $g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z})$, Ω being an arbitrary element of $SL(2,\mathbb{R})$. The corresponding currents

$$J(z) = -\frac{k}{2}(\partial g)g^{-1} \quad , \quad \bar{J}(\bar{z}) = -\frac{k}{2}(\bar{\partial} g^{-1})g \tag{2.8}$$

can be expanded in Laurent series as

$$J^a(z) = \sum_{n=-\infty}^{\infty} J_n^a\, z^{-n-1} \quad , \quad \bar{J}^a(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{J}_n^a\, \bar{z}^{-n-1} \tag{2.9}$$

and the coefficients J_n^a ($\bar{J}_n^a$) satisfy a Kac-Moody algebra given by

$$[J_n^a, J_m^b] = i\varepsilon_c^{ab} J_{n+m}^c - \frac{k}{2}\eta^{ab} n\delta_{n+m,0} \quad , \tag{2.10}$$

where the Cartan Killing metric of $SL(2,\mathbb{R})$ is $\eta^{+-} = \eta^{-+} = 2$, $\eta^{33} = -1$ and ε_c^{ab} is the Levi Civita antisymmetric tensor. And similarly for the antiholomorphic currents.

The Sugawara stress-energy tensor is given by

$$T(z) = \frac{\eta_{ab}}{k-2} : J^a(z)J^b(z) : \quad . \tag{2.11}$$

[1] Strictly the spectral flow stretches the geodesic solution in the t-direction and rotates it around the center of AdS_3 (see [5]).

It is related to the Casimir of the group as $C=(k-2)T$ and it leads to the following central charge of the Virasoro algebra

$$c=\frac{3k}{k-2}=2+1+\frac{12}{\alpha_+^2} \quad , \tag{2.12}$$

($\alpha_+=\sqrt{2(k-2)}$). The last equality emphasizes that the central charge can be written as that of two free bosons plus a Liouville field. We will see more indications of this field content as we proceed.

At the quantum level, the building blocks of the Hilbert space $\mathcal{H}$ are unitary hermitic representations of $SL(2,\mathbb{R})$. The states $|j,m>$ satisfy

$$C_0|j,m>=j(j+1)|j,m> \quad , \qquad J_0^3|j,m>=m|j,m>,$$
$$J_0^{\pm}|j,m\rangle \;=\; (m\mp j)|j,m\pm 1\rangle \quad , \tag{2.13}$$

with $\{m\in\mathbb{R}, j\in\mathbb{R}\}$ or $\{m\in\mathbb{R}, j\in -\frac{1}{2}+i\mathbb{R}\}$, as required by hermiticity, and in addition they must be Kac Moody primaries, namely

$$J_n^a|j,m\rangle=0 \quad \forall\, n>0 \quad . \tag{2.14}$$

Unitarity selects the discrete ($\mathcal{D}_j^{\pm}$) and continuous ($\mathcal{C}_j^{\alpha}$) representations, namely:

- $\mathcal{D}_j^+=\{|j,m\rangle;\;\; j\in\mathbb{R};\;\; m=j+1,j+2,j+3,...\}$.
- $\mathcal{D}_j^-=\{|j,m\rangle;\;\; j\in\mathbb{R};\;\; m=-j-1,-j-2,-j-3,...\}$.
- $\mathcal{C}_j^{\alpha}=\{|j,m\rangle;\;\; j=-\frac{1}{2}+i\lambda;\; \lambda\in\mathbb{R};\;\; m=\alpha,\alpha\pm 1,\alpha\pm 2,...;\;\; \alpha\in\mathbb{R}\}$.

Notice that the vectors in $\mathcal{H}$ related by $j\leftrightarrow -1-j$ represent the same physical state and therefore j can be restricted to $j\geq -\frac{1}{2}$. The complete basis of $\mathcal{L}^2(AdS_3)$ is given by $\mathcal{C}^{\alpha}_{j=-1/2+i\lambda}\times\mathcal{C}^{\alpha}_{j=-1/2+i\lambda}$ and $\mathcal{D}_j^{\pm}\times\mathcal{D}_j^{\pm}$ with $j>-1/2$.

The representation space can be enlarged by acting on the primary states in these series with J_n^a, $n<0$. The corresponding representations are denoted by $\widehat{\mathcal{D}}_j^{\pm}, \widehat{\mathcal{C}}_j^{\alpha}$. The states in string theory must satisfy in addition the Virasoro constraints given by

$$L_0|\Psi>=\left(-\frac{j(j+1)}{k-2}+N\right)|\Psi> \quad ; \quad L_n|\Psi>=0 \quad , \quad n>0 \quad . \tag{2.15}$$

Unlike in flat spacetime where these constraints decouple the negative norm states from the spectrum, in string theory on AdS_3 it was noticed long ago that this is not enough and the bound $j<k/2$ was proposed to achieve a unitary Hilbert space [1]. However this limit presents two important problems: i) it implies an unnatural restriction on the excitation level (and consequently on the mass) of the states, and ii) the partition function (which contains the information on the physical spectrum of the theory) is not modular invariant.

The solution to these problems was proposed by J. Maldacena and H. Ooguri in [5]. The full representation space contains the spectral flow images of the standard series mentioned above. Actually the spectral flow operation leads to the following automorphism of the $SL(2,\mathbb{R})$ currents

$$J_n^3 \;\rightarrow\; \widetilde{J}_n^3=J_n^3-\frac{k}{2}w\delta_{n,0} \quad , \tag{2.16}$$
$$J_n^{\pm} \;\rightarrow\; \widetilde{J}_n^{\pm}=J_{n\pm w}^{\pm} \tag{2.17}$$

with $w \in \mathbb{Z}$ and consequently the modes of the Virasoro generators transform as

$$L_n \to \widetilde{L}_n = L_n + wJ^3_n - \frac{k}{4}w^2\delta_{n,0} \quad . \tag{2.18}$$

Unlike the compact $SU(2)$ case, the new operators generate inequivalent representations of $SL(2,\mathbb{R})$ with states $|\tilde{j},\tilde{m},w\rangle$ satisfying

$$L_0|\widetilde{j},\widetilde{m},w\rangle \;=\; \left(-\frac{\widetilde{j}(\widetilde{j}+1)}{k-2} - w\widetilde{m} - \frac{k}{4}w^2\right)|\widetilde{j},\widetilde{m},w\rangle \quad , \tag{2.19}$$

$$J^3_0|\widetilde{j},\widetilde{m},w\rangle \;=\; \left(\widetilde{m} + \frac{k}{2}w\right)|\widetilde{j},\widetilde{m},w\rangle \quad . \tag{2.20}$$

Finally, the complete Hilbert space of string theory on AdS_3 is obtained by applying creation operators $\widetilde{J}^a_n, n<0$ on the primary states of the current algebra and the physical state conditions

$$(L_0-1)|\widetilde{j},\widetilde{m},w,\widetilde{N},h\rangle \;=\; \left(-\frac{\widetilde{j}(\widetilde{j}+1)}{k-2} - w\widetilde{m} - \frac{k}{4}w^2 + \widetilde{N} + h - 1\right)|\widetilde{j},\widetilde{m},w,\widetilde{N},h\rangle = 0 \quad , \tag{2.21}$$

$$L_n|\widetilde{j},\widetilde{m},w,\widetilde{N},h\rangle \;=\; \left(\widetilde{L}_n - w\widetilde{J}^3_n\right)|\widetilde{j},\widetilde{m},w,\widetilde{N},h\rangle = 0 \quad \text{for } n>0 \tag{2.22}$$

where $\widetilde{N}$ is the excitation level of $\widetilde{J}_n$ and h is the conformal weight of the state in the internal theory [2]. Notice that the representations $\widehat{\mathcal{D}}^{\pm,w=\mp1}_{\tilde{j}}$ and $\widehat{\mathcal{D}}^{\mp,w=0}_{\frac{k}{2}-2-\tilde{j}}$ are equivalent. This has an important consequence on the values allowed for j. Indeed, recalling the symmetry $j \leftrightarrow -1-j$ which implies $j \geq -\frac{1}{2}$, j is restricted as required by the no-ghost theorem [5] to

$$-\frac{1}{2} < j < \frac{k-3}{2} \quad . \tag{2.23}$$

Note that if one starts with $\tilde{j} > -1/2$ the representation obtained after the spectral flow satisfies $j = \frac{k}{2} - 2 - \tilde{j} < \frac{k-3}{2}$. Conversely if there were a representation $\hat{\mathcal{D}}^{\pm}_j$ with $j > \frac{k-3}{2}$ in the Hilbert space, the spectral flow would generate a representation $\hat{\mathcal{D}}^{\mp}_j$ with $j < -1/2$, in contradiction with the standard harmonic analysis. The upper bound on j was interpreted in [3] as the condition that only local operators be considered in the dual field theory on the boundary. This interpretation fits in nicely with the AdS/CFT dictionary which assigns states in the bulk to operators in the boundary.

Therefore the Hilbert space of string theory on AdS_3 can be summarized as

$$\mathcal{H} = \oplus_{w=-\infty}^{\infty}\left[\left(\int_{-1/2}^{\frac{k-3}{2}} dj\,\hat{\mathcal{D}}^w_j \otimes \hat{\mathcal{D}}^w_j\right) \oplus \left(\int_{-\frac{1}{2}+i\lambda} dj \int_0^1 d\alpha\,\hat{\mathcal{C}}^w_{j,\alpha} \otimes \hat{\mathcal{C}}^w_{j,\alpha}\right)\right] \quad . \tag{2.24}$$

This proposal for the spectrum was independently verified by the computation of a modular invariant partition function in [6, 7].

[2] We have been considering string theory on AdS_3, but more generally we could take a background $AdS_3 \times \mathcal{N}$, with $\mathcal{N}$ a compact internal manifold.

3. Correlation functions on the sphere

The computation of correlation functions in this theory is rather involved because the underlying CFT is non-rational and very little is known about this kind of non-compact theories in general.

Nevertheless certain expectation values of primary fields have been calculated using different procedures. The path integral method was started in [8] and applied to the computation of two and three point amplitudes using the free field approximation in [9]. A generalization of the bootstrap approach was designed by Teschner to deal with the non-compact nature of the hyperbolic space $H_3^+ \equiv SL(2,\mathbb{C})/SU(2)$ and some two, three and four point functions for this euclidean model were given in [10]. The physical interpretation of these exact results, as correlation functions of the dual CFT$_2$, was performed by Maldacena and Ooguri in [3]. Let us review some of these results.

The states in H_3^+ belong to representations with $j = -\frac{1}{2} + i\lambda$, $\lambda \in \mathbb{R}$ and they can be realized in global coordinates by operators of the form

$$\Phi_j(x,\bar{x};z,\bar{z}) = \frac{1+2j}{\pi}\left(e^{-\phi/\alpha_+} + |\gamma - x|^2 e^{\phi/\alpha_+}\right)^{2j} \tag{3.1}$$

and their current algebra descendants. Here $x,\bar{x}$ keep track of the $SL(2,\mathbb{C})$ quantum numbers and can be identified as the location of the operator in the dual CFT on the boundary of H_3^+. We refer the reader to references [10] for explicit expressions of expectation values of these operators.

For string theory applications one has to consider real values of j and consequently these amplitudes have to be analytically continued. This analysis was performed in [3]. However not all correlators that are necessary for string theory can be obtained by analytic continuation of (3.1) and expectation values of these fields. In order to consider states of the spectral flow representations, it is convenient to use fields with definite $SL(2)$ weights. These are given by

$$\Phi_{jm\bar{m}}(z,\bar{z}) = \int d^2x\, x^{j+m}\bar{x}^{j+\bar{m}}\Phi_{-1-j}(x,\bar{x};z,\bar{z}), \tag{3.2}$$

and operators of the spectral flow representations are obtained acting on (3.2) with the spectral flow operator defined as

$$\mathcal{F}(z,\bar{z}) = e^{i\sqrt{\frac{k}{2}}[\phi(z)+\phi(\bar{z})]} \quad . \tag{3.3}$$

Similarly the correlation functions can be converted to the m basis, then one performs the spectral flow and finally transforms the result back to the x basis. This procedure was applied in [3] to obtain three point functions in arbitrary winding sectors.

The results for the scattering amplitudes of $n-$ string states presented in [3] exhibit several subtleties for $n \geq 3$. On the one hand, expectation values of states belonging to discrete representations are only well defined if the sum of the isospins j of the external operators satisfies $\sum_i j_i < k-3$. Moreover the four point functions on the sphere do not factorize as expected into a sum of products of three point functions with physical intermediate states unless the quantum numbers of the external states verify $j_1 + j_2 < \frac{k-3}{2}$ and $j_3 + j_4 < \frac{k-3}{2}$. The interpretation of these constraints was proposed in [3]: correlation functions violating these bounds do not represent well-defined computations in the dual CFT$_2$ description of the theory. This explanation is similar to the interpretation of the upper bound on the spin of the physical states (*i.e.* $j < \frac{k-3}{2}$) as the condition that only local operators be considered in the boundary CFT. However in the latter case one has a clear understanding of the constraint from the representations of $SL(2,\mathbb{R})$ which define the theory in the bulk. Similarly one would like to better understand this unusual feature of the correlation functions from the worldsheet viewpoint.

On the other hand, a curious aspect of this model is that physical amplitudes of n string states may violate winding number conservation up to $n-2$ units. Again, this fact is well understood from the representation theory of $SL(2,\mathbb{R})$ [3]. But the computation of winding non-preserving scattering amplitudes proposed in [3] involves, as mentioned above, the insertion of spectral flow operators in the correlators and this implies the calculation of expectation values of more vertex operators than the n original ones. This procedure has been applied to three point functions, but four point functions violating winding number conservation by one or two units require the calculation of correlators with five or six operator insertions, with the consequent complications. These amplitudes are needed to definitely settle the question about the unitarity of the theory through the analysis of their factorization properties.

In order to develop techniques that simplify these computations and allow to perform others that would clarify the full structure of the model, the free field description of the theory appears as a powerful tool. This approximation was initially applied in [4] to derive the spacetime CFT and establish the conjectured AdS/CFT correspondence (for related work see [11]). Even though this approach is expected to give a good picture of the theory only near the boundary of AdS$_3$, the computation of two and three point amplitudes of string states in the world-sheet theory using the Coulomb gas formalism has produced results in complete agreement with the exact ones in the bosonic theory [12] and sensible results were also obtained in the supersymmetric case [13]. Furthermore the analysis of unitarity in this approximation should give important information on the consistency of the complete theory. Thus we now discuss the extension of this formalism to higher point functions and to higher Riemann surfaces.

The Coulomb gas formalism developed by Dotsenko and Fateev [14] and by Feigin and Fuchs [15] for conformal minimal models is a more practical way to compute the conformal blocks than the bootstrap programme of Belavin, Polyakov and Zamolodchikov. However the generalization of this method to other models with extended conformal algebras is incomplete in most cases. Indeed, even in the simpler compact SU(2) WZW CFT, the naive Coulomb gas construction of [16] fails to reproduce the well-known fusion rules of the admissible representations. Actually the correct fusion rules and characters were derived in [17] where the structure of zero modes and the embedding of the SU(2) parafermion (PF) modules in the boson Fock space was found to play a crucial role.

The extension of this procedure to the non-rational $SL(2,\mathbb{R})$ CFT however is not straightforward [18] since it has an infinite continuous set of fields. [3] Rational CFTs (RCFT) instead contain a finite number of representations of the modular group. They are not necessarily minimal but all the primary fields (possibly infinite) can be organized into a finite number of blocks corresponding to an extended symmetry algebra linearly transforming into each other under the modular group. The simplest example beyond the minimal $c<1$ models is the $c=1$ bosonic theory compactified on a circle with rational square radius. Similarly in the finite reducible Z_N parafermion CFTs which

[3] Alternative procedures leading to the correct results for the admissible representations were followed in [19] and [20] where it is shown that the four point functions obey crossing symmetry and monodromy invariance. However the unitary representations relevant for string theory do not belong to this class.

describe the coset $SL(2,\mathbb{R})/U(1)$ [21] there is a subset of models with rational central charge (*i.e.* fractional levels k) where all the Virasoro primaries appear as descendants in the (finite number of) PF highest weight modules. Many physically interesting cases correspond to fractional levels k, for example critical bosonic string theory on the coset $SL(2,\mathbb{R})/U(1)$ demands $k = \frac{9}{4}$ and in applications to superstring theory the level k is the number of NS5-branes. The finitely reducible PF models correspond to the continuous and discrete unitary representations of $SL(2,\mathbb{R})$.

Actually there is a tight link between the spectrum of string theory on AdS$_3$ and that of the coset. The states in the coset are those of $SL(2,\mathbb{R})$ that are annihilated by J_n^3 and $\overline{J_n^3}$ with $n > 0$, and have zero mode eigenvalues given by the non-compact parafermionic charges $\tilde{m} = (n - wk)/2$ and $\overline{\tilde{m}} = -(n + wk)/2$, $n \in \mathbb{Z}$. It is then possible to reconstruct $SL(2,\mathbb{R})$ starting from the coset much like in the compact $SU(2)$ case.

The construction in [18] is based on the Wakimoto free field representation of the $SL(2,\mathbb{R})_k$ current algebra which can be expressed in terms of three fields with propagators

$$\langle \rho(z)\rho(0)\rangle = \langle X^1(z)X^1(0)\rangle = -\ln z \quad , \quad \langle X^0(z)X^0(0)\rangle = \ln z \tag{3.4}$$

as

$$J^3 = -\sqrt{\frac{k}{2}} i\partial X^0 \quad , \quad J^{\pm} = \left(-\sqrt{\frac{k}{2}} i\partial X^1 \mp \sqrt{\frac{k-2}{2}}\partial\rho\right) e^{\pm\sqrt{\frac{2}{k}} i(X^0 + X^1)} \tag{3.5}$$

and the stress tensor is

$$T = \frac{1}{2}\left(\partial X^0\right)^2 - \frac{1}{2}\left(\partial X^1\right)^2 - \frac{1}{2}(\partial\rho)^2 - \frac{1}{\alpha_+}\partial^2\rho \quad . \tag{3.6}$$

Thus the current algebra can be realized in terms of two free bosons X^0, X^1 and a Liouville field ρ with background charge $Q \equiv \sqrt{\frac{2}{k-2}}$ (which roughly correspond to the global coordinates of AdS$_3$ [22]) in full agreement with the commutation relations (2.10). Actually it can be verified that these currents satisfy the OPEs

$$J^+(z)J^-(z') \sim \frac{k}{(z-z')^2} - \frac{2J^3(z')}{z-z'} \quad , \tag{3.7}$$

$$J^3(z)J^{\pm}(z') \sim \pm\frac{J^{\pm}(z')}{z-z'} \quad , \tag{3.8}$$

$$J^3(z)J^3(z') \sim -\frac{\frac{k}{2}}{(z-z')^2} \quad . \tag{3.9}$$

In terms of $\mathbb{Z}_k$ PF currents of the coset $SL(2,\mathbb{R})/U(1)$, $J^{\pm}$ can be rewritten as

$$J^+ = \sqrt{k}\Psi_1^+ e^{\sqrt{\frac{2}{k}} iX^0} \quad , \quad J^- = \sqrt{k}\Psi_1^- e^{-\sqrt{\frac{2}{k}} iX^0} \quad . \tag{3.10}$$

These currents are defined through the OPEs

$$\begin{aligned} \Psi_1^+(z)\Psi_1^-(0) &= z^{-2\Delta_1}\left[1 + \frac{2\Delta_1}{c} z^2 T(0) + \ldots\right], \\ T(z)\Psi_1^+(0) &= z^{-2}\left[\Delta_1\Psi_1^+(0) + z\partial\Psi_1^+(0) + \ldots\right] \end{aligned} \tag{3.11}$$

and new currents can be generated through

$$\Psi_1^{\pm}(z)\Psi_p^{\pm}(0) = z^{\Delta_1+\Delta_p+\Delta_{p+1}}\Psi_{p+1}^{\pm}(0) + ... \quad (3.12)$$

In general there are infinite $\Psi_p^{\pm}$ but the requirement of finite reducibility selects a subset of models with a finite number.

The vertex operators creating string states are given in this parametrization by

$$\Phi^w_{j\tilde{m}\overline{\tilde{m}}} = V_{j\tilde{m}\overline{\tilde{m}}} e^{i(\tilde{m}+w\frac{k}{2})\sqrt{\frac{2}{k}}X^0(z)+i(\overline{\tilde{m}}+w\frac{k}{2})\sqrt{\frac{2}{k}}X^0(\bar{z})} = e^{\frac{2j}{\alpha_+}\rho(z)} e^{i\sqrt{\frac{2}{k}}\tilde{m}X^1(z)} e^{i\sqrt{\frac{2}{k}}(\tilde{m}+\frac{k}{2}w)X^0(z)} \times c.c. \quad (3.13)$$

with $\tilde{m}-\overline{\tilde{m}} \in \mathbb{Z}$. The fields $V_{j\tilde{m}\overline{\tilde{m}}}$ belong to the non-compact PF discrete and continuous series; they are Virasoro primaries carrying no charge with respect to $J^3, \overline{J^3}$. The conformal dimension and J_0^3 eigenvalue of $\Phi^w_{j\tilde{m}\overline{\tilde{m}}}$ are given by (2.19) and (2.20) respectively.

These fields have both left and right dimensions non-zero, thus their correlators will not be holomorphic or antiholomorphic in their arguments. But in any CFT, if there is a finite number of primary fields with respect to a holomorphic symmetry group, all n-point functions are the sum of a finite number of terms, with each term the product of a holomorphic and an anti-holomorphic function.

Expectation values are computed in the Coulomb gas formalism inserting screening operators to balance the background charge. In the bosonized theory there are three screening operators, namely

$$\eta^{\pm}(z) = e^{-\frac{\alpha_+}{2}\rho} e^{\pm i\sqrt{\frac{k}{2}}X^1} \quad , \quad S(z) = \partial X^1 e^{-\frac{2}{\alpha_+}\rho}. \quad (3.14)$$

Their positions have to be integrated and the contour choices available correspond to different conformal blocks. Unlike the minimal conformal models here $\eta^{\pm}$ are dimension one fermions and their charge conjugates are given by the following dimension zero fermions

$$\xi^{\pm}(z) = e^{\frac{\alpha_+}{2}\rho} e^{\mp i\sqrt{\frac{k}{2}}X^1} \quad . \quad (3.15)$$

In fact, $(\eta^{\pm}, \xi^{\pm})$ are members of two non-commuting fermion ghost systems with central charge $c = -2$ which are embedded in the boson Fock space. The lack of commutation means that states cannot in general be simultaneously diagonalized in terms of both systems. The $U(1)$ currents are $j^{\pm}(z) = -\eta^{\pm}\xi^{\pm}$ and the operators $j_0^{\pm}$ count the fermion charge as $j_0^{\pm}[V_{j\tilde{m}}] = j \pm \tilde{m}$. Vertex operators with integer or non-negative $j_0^{\pm}$ charge are local with respect to the $(\eta^{\pm}, \xi^{\pm})$ system or independent of the zero mode $\xi_0^{\pm}$ respectively.

All the PF/Virasoro highest weight states correspond to unitary representations of $SL(2,\mathbb{R})$ and the vertex operators creating them are given by (3.13). However the descendants of the discrete series are constructed with the screening charges as $\oint \eta^{\pm} V_{j-\frac{k}{2}+1,\pm(j-\frac{k}{2}-p)}(z)$ with $p = 1, 2, ...$. This embedding of the non-compact PF in the boson Fock space is substantially different from the compact case where all the unitary highest weight representations are independent of both fermion zero modes $\xi_0^{\pm}$. This structure plays a crucial role in closing the operator algebra onto unitary states in the compact case. Moreover there are no null states in $SL(2,\mathbb{R})$. These are extremely significant in the minimal models since they determine the differential equations to be satisfied by the correlators and, in the cases where there are a finite number of fields, they allow a complete solution of the theory. Feigin and Fuchs showed that the null states can be built acting with screening operators on the vertex operators with shifted charges. Therefore, inserting them in the correlators they determine the non-vanishing expectation values in the Coulomb gas formalism and thus the fusion rules.

It would be interesting to include the spectral flow representations in this formulation. Actually the spectral flow operator has a current algebra null vector (and through the Sugawara relation (2.11) a conformal-current null vector), namely $J^-_{-1}|\frac{k}{2},\frac{k}{2}>=0$. Thus the winding sectors might mimic the role of the null states to determine the fusion rules. Actually it is not possible to construct a modular invariant partition function unless the spectral flow representations are considered (see below).

4. The partition function

The one loop partition function for bosonic strings on thermal AdS_3 $-i.e.$ $H_3^+/\mathbb{Z}$, which is equivalent to the euclidean black hole background $-$ was computed in [6] using path integral methods previously introduced by Gawedski [8]. More recently, a modular invariant partition function reproducing the spectrum discussed in Section 2 was found in [7]. Here the starting point is the vacuum amplitude for the coset $SL(2,\mathbb{R})/U(1)$ computed in [23] and the result for $SL(2,\mathbb{R})$ is reconstructed by coupling the coset to a free time-like boson corresponding to J^3 (and $\bar{J}^3$). Actually it was shown in [7] that one can write the modular invariant partition function for $SL(2,\mathbb{R})$ as a $\mathbb{Z}_k$ orbifold of the product $\frac{SL(2,\mathbb{R})}{U(1)}\times U(1)_{\sqrt{2k}}$, namely

$$Z(\tau,\bar{\tau}) = 4\sqrt{\tau_2}(k-2)^{3/2}\int d^2s\; d^2t\,\frac{e^{\frac{2\pi}{\tau_2}[\mathrm{Im}(s_1\tau-s_2)]^2}}{|\vartheta_1(s_1\tau-s_2)|^2}\times \\ \times\sum_{m,w,m',w'}\zeta\begin{bmatrix} w+s_1-t_1 \\ m+s_2-t_2\end{bmatrix}(k)\,\zeta\begin{bmatrix} w'+t_1 \\ m'+t_2\end{bmatrix}(-k) \tag{4.1}$$

where $s_i\in[0,1)$ are part of the holonomy parameters and $\zeta(R^2)$ refer to the $U(1)$ characters of a boson compactified at radius R (imaginary for a time-like boson).

Moreover $Z(\tau,\bar{\tau})$ can be expressed in terms of the following light-cone momenta

$$P^+_{L,R} = \frac{n^+}{\sqrt{2k}}\pm\sqrt{\frac{k}{2}}w_- \quad , \quad P^-_{L,R} = \frac{n^-}{\sqrt{2k}}\pm\sqrt{\frac{k}{2}}w_+, \tag{4.2}$$

corresponding to two compact light-cone bosons $X^\pm = \frac{1}{2}(X^1\pm X^0)$ as [4]

$$Z_{SL(2,\mathbb{R})} = 4\;\sqrt{\tau_2}\;(k-2)^{3/2}\int d^2s\; d^2t\;\frac{e^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau-s_2)^2}}{|\vartheta_1(s_1\tau-s_2|\tau)|^2}\times \\ \times\sum_{n^\pm,w_\pm\in\;\mathbb{Z}} e^{-i\pi\sqrt{\frac{k}{2}}\left((P_L^+ +P_R^+)s_2+(P_L^- +P_R^-)(s_2-2t_2)\right)}\times \\ \times\; q^{\frac{1}{2}\left(P_L^+ +\sqrt{\frac{k}{2}}(s_1-2t_1)\right)\left(P_L^- +\sqrt{\frac{k}{2}}s_1\right)}\,\bar{q}^{\frac{1}{2}\left(P_R^+ -\sqrt{\frac{k}{2}}(s_1-2t_1)\right)\left(P_R^- -\sqrt{\frac{k}{2}}s_1\right)}\quad . \tag{4.3}$$

[4]Here m,m' have been Poisson resummed and traded by $n^\pm = n\pm n'$ and $w_\pm = w\pm w'$.

The spectrum extracted from this partition function corresponds to the discrete and continuous representations of $SL(2,\mathbb{R})$ as discussed in Section 2, consistently with the vertex operators (3.13) written in light-cone coordinates [24] with quantum numbers j, $\tilde{m}+\tilde{\bar{m}}=-k(w-t_1)$ and $\tilde{m}-\tilde{\bar{m}}=n$, namely

$$\exp\left\{\sqrt{\frac{2}{k-2}}j\rho+i\sqrt{\frac{2}{k}}\left[\frac{k}{4}wX^{+}+\left(\tilde{m}-\frac{k}{4}w\right)X^{-}+\frac{k}{4}\bar{X}^{-}+\left(\tilde{\bar{m}}-\frac{k}{4}w\right)\bar{X}^{+}\right]\right\}. \quad (4.4)$$

It is interesting to discuss the possibility of expressing (4.3) as a modular invariant combination of $SL(2,\mathbb{R})$ characters. The characters of the coset were computed in [18] from the embedding of the PF modules in the boson Fock space. Denoting $\mathcal{H}$ the holomorphic subspace of states in the boson Fock space and $\mathcal{H}_{local}$ the subspace which contains states that are relatively local with respect to the $(\eta^{\pm},\xi^{\pm})$ systems, the $\mathcal{D}_j^{+}$ ($\mathcal{D}_j^{-}$) PF modules are in $\mathcal{H}^{+}_{small}$ ($\mathcal{H}^{-}_{small}$) which is the restriction of states in $\mathcal{H}_{local}$ to those independent of the zero modes ξ_0^{+} (ξ_0^{-}). Thus one obtains the characters of the discrete and continuous series with charge $\tilde{m}=j+p$ and $\tilde{m}=\alpha+p$ respectively as

$$\chi^{\mathcal{D}^+}_{jp} = \eta(\tau)^{-2}q^{-\frac{c}{24}+\Delta_{jp}}\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+2p+1)} \quad , \quad (4.5)$$

$$\chi^{\mathcal{C}}_{jp} = \eta(\tau)^{-2}q^{\Delta_{\lambda,\alpha+p}} \quad (4.6)$$

where $\Delta_{j,p}=-\frac{j(j-1)}{k-2}+\frac{(j+p)^2}{k}$. It was shown in [25] that a modular invariant partition function can be obtained from these characters, but the expression computed in [8] performing the path integral has contributions from states which do not fit in the discrete and continuous representations. The equivalent analysis for $SL(2,\mathbb{R})$ has not been performed yet.

5. Conclusions

The unitarity constraints for string theory on AdS$_3$ are well understood at the free level, both for the bulk and boundary theories. However, several extra constraints are necessary to render the theory unitary when interactions are considered. The interpretation of these conditions is clear from the BCFT viewpoint but not from the perspective of the theory in the bulk of AdS$_3$.

I have argued that the free field formulation of the theory might allow a better understanding of the worldsheet theory and discussed some speculative ideas on the possibility of using methods of RCFT in this non-rational CFT. In particular the Coulomb gas formalism reproduces the exact results for two and three point functions and the partition function reflects this free field content. However further work is necessary in order to determine the fusion rules and their relation to the modular transformation properties of the characters, a sort of non-compact version of Verlinde theorem.

As conclusion I try to motivate suggestions for further study. The computation of four point functions in the free field formalism is an interesting problem in itself and it could moreover give information about the factorization properties of the model. Important information could be obtained also from the two point functions on the torus. Analyzing their modular invariance in the factorization limit could illuminate the worldsheet origin of the unitarity constraints.

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