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Non-Commutative Generalization of Born-Infeld Theory

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A new generalization of non-linear Born-Infeld electrodynamics is proposed. It is inspired by the non-commutative geometry and a new interpretation of gauge theories. The variational principle introduced here leads to quite complicated non-linear equations, which can be solved numerically in certain cases. The spherically-symmetric ansatz is analyzed, and static finite energy solutions are obtained via numerical integration. Then pure Higgs sector lagrangian is introduced by analogy with the non-abelian Born-Infeld generalization. A spherically-symmetric configuration and a time-dependent homogeneous field are investigated and qualitative behavior of solutions are discussed.

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1. Introduction

The present article is based on the results research published recently in several papers, [18], [19]. It deals with possible generalizations of Born-Infeld non-linear theory of electromagnetism, whose non-abelian version quite unexpectedly appears as an effective low-energy field limit in certain versions of string Lagrangians.

Our aim is to investigate further possibilities of generalization offered by the non-commutative geometry. In this version of discretization of space-time structure, non-linear Lagrangians do appear, whose structure is similar to the original Born-Infeld Lagrangian. The extra advantage is the unification of usual gauge fields with the Higgs-type scalar multiplets, appearing in the Standard Model. We believe that this type of effective Lagrangians may be useful in future developments of field theory and its cosmological applications.

Since Coulomb's law has been formulated in XVIII-th century, it was clear that the electric forces become infinite for point-like particles. Later on, when Maxwell found his final and elegant mathematical formulation of electrodynamics, with the introduction of the energy-momentum tensor of electromagnetic field, the energy remained infinite for point-like charges. After the discovery of the electron, physicists started to look for models able to represent it as an extended, finite-dimensional particle, endowed with finite distribution of charge and energy densities. The model proposed by G. Mie [1] could be considered as the most successful one at the time it was published. It was based on the idea that Maxwell's electrodynamics should be considered as a linear approximation of certain non-linear theory; as long as the field strength is not too high, the linear theory describes almost perfectly its behavior far away from the source, which can be considered point-like as seen from great distance; the non-linear effects should become dominant at small distances, where the extended nature of elementary charges must be taken into account.

To this end, Mie introduced the notion of *maximal field strength*, $\mathbf{E}_0$, and in order to make it impossible for any electric field to go beyond this value, he modified Maxwell's theory by introducing the following non-linear lagrangian density for pure electric field

$$\mathcal{L} = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{E}_0^2}}\,. \tag{1.1}$$

Although the non-linear theory derived from this lagrangian enabled Mie to obtain a non-singular, finite energy solution, it was clear that such a lagrangian can not represent a Lorentz-invariant theory, especially that the magnetic field contribution was absent. This is why Born and Infeld [2] have introduced a Lorentz-invariant lagrangian density, defined as follows

$$\mathcal{L}_{BI}(g,F) = L_{BI}(g,F)\sqrt{|g|} = \beta^2\left(\sqrt{|\det(g_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu} + \beta^{-1}F_{\mu\nu})|}\right)$$
$$= \beta^2\left(1 - \sqrt{1 + \frac{1}{\beta^2}(\mathbf{B}^2 - \mathbf{E}^2) - \frac{1}{\beta^4}(\mathbf{E}\cdot\mathbf{B})^2}\right)\sqrt{|g|}\,. \tag{1.2}$$

The constant β appears for dimensional reasons, and plays the same role here as the limiting value of the electric field in Mie's non-linear electrodynamics. Defining

$$P = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad S = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \quad \text{with} \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}\,,$$

we can write

$$L_{BI} = \beta^2 \left[1 - \sqrt{1 + 2P - S^2} \right] . \tag{1.3}$$

Since Dirac introduced his equation for the electron, the interest in classical models of charged particles has considerably faded. Only in 1970 G. Boillat [3] considered Born-Infeld electrodynamics in order to study its propagation properties. Investigating general non-linear theories derived from a Lagrangian depending on two Lorentz invariants $\mathcal{L}(P,S)$, he discovered that the Born-Infeld electrodynamics is the only one ensuring the absence of bi-refringence, i.e. propagation along a single light-cone, and the absence of shock waves. In this respect the Born-Infeld theory is unique (except for another a singular Lagrangian $\mathcal{L} = P/S$). A beautiful discussion of these properties can be found in I. Bialynicki-Birula's paper [4]; see also [5].

An unexpected new impulse for the revival of interest in Born-Infeld (BI) electrodynamics and in its non-abelian generalizations came from string theories, in which Born-Infeld type lagrangians appear in effective action for membranes.

Another motivation for studying BI-types theories is the possibility of existence of solitonic solutions in non-linear field theories. In a pure Yang-Mills theory in flat space-time, with the usual Lagrangian density $\mathcal{L}_{YM} = -\frac{1}{4} g_{AB} F^A_{\mu\nu} F^{B\mu\nu}$ there are no finite energy static non-singular solutions. This fact can be explained qualitatively by the conformal invariance of the theory and the tracelesness of the energy-momentum tensor,

$$T^{\mu}{}_{\mu} = -T_{00} + \sum_{i=1}^{3} T_{ii} = 0 \, . \tag{1.4}$$

Given the positivity of energy, i.e.$T_{00} > 0$, this means that the sum of principal pressures is positive: $\sum T_{ii} > 0$, leading to the conclusion that the Yang-Mills "matter" is naturally subjected to repulsive forces only.

In the presence of Higgs' field, the conformal invariance is broken, and this leads to the existence of solitonic solutions like 't Hooft's [6] and Prasad-Sommerfield's [7] magnetic monopoles. In what follows, we are looking for soliton-like solutions arising in other non-linear theories, including non-abelian versions of Yang-Mills theory, which are not conformally invariant, as well as the pure Higgs-field model with a generalized Born-Infeld type lagrangian.

2. Non-abelian generalizations of Born-Infeld theory

In their original paper [2] Born and Infeld considered the now famous least action principle

$$S_{BI}[g,F] = \int_{R^4} \beta^2 \left(\sqrt{|\det(g_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu} + \beta^{-1} F_{\mu\nu})|} \right) d^4x \, .$$

This action can be defined not only on the Minkowskian space-time but also on any locally Lorentzian curved manifold.

It is useful to recall here the three basic properties of this lagrangian which we want to maintain in the case of non-abelian generalization, suited for arbitrary finite dimension of space-time.

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- Maxwell's theory (respectively, usual gauge theory with quadratic lagrangian density) is found in the limit $\beta \to \infty$

$$S_{BI} = -\int_{R^4} \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2)\sqrt{|g|}d^4x + o(\frac{1}{\beta^2})\,. \tag{2.1}$$

- There exists an upper limit for the electric field intensity, equal to β when the magnetic components of the field vanish

$$L_{BI}|_{B=0} = \beta^2(1 - \sqrt{1 - \beta^{-2}\mathbf{E}^2})\,. \tag{2.2}$$

  Due to this fact, the energy of a point-like charge is finite, and the field remains finite even at the origin. This was the main goal pursued by Mie, [1] suggesting the choice of non-linear generalization of Maxwell's theory. Indeed, for a point charge e one has

$$\mathbf{E} = \frac{e\mathbf{n}}{\sqrt{e^2 + r^4}} \ \ ,\ \ (\mathbf{n} = \mathbf{r}/r), \quad \text{Mass} = \int_0^\infty (\sqrt{e^2 + r^4} - r^2)dr\,. \tag{2.3}$$

- The Born-Infeld action principle is invariant under the diffeomorphisms of R^4. In this respect, this theory can be viewed as a covariant generalization (in the sense of General Relativity) of Mie's theory, as well as an extension of the usual volume element $\sqrt{|g|}d^4x$.

It is also well known that the Born-Infeld electromagnetism has good causality properties as well as interesting dual symmetries (electric-magnetic duality, Legendre duality) . Here we shall not consider these properties, our main interest being focused on static solutions.

The idea of non-abelian generalisation of Born-Infeld theory lagrangian has been in the air already at the end of the seventies. Hagiwara has discussed various possibilities in [8], however, he did not try to find soliton-like solutions. In 1997 Tseytlin [9] argued in favor of the symmetrized trace prescription which reproduced in the first 4 orders the string effective action for gauge potential. Finally Park [10] introduced yet another non-abelian generalisation and investigated qualitative behavior of instanton-like solutions. Also super-symmetric generalisation has been proposed for abelian and non-abelian versions [11, 12, 13].

Only instanton like solutions were discussed in aforementioned papers. First solitons in Minkowskian space-time were found in [14], which we shortly recall in this section.

A straightforward generalisation of BI theory in 4 dimensions can be achieved by replacing the quadratic invariants of U(1) theory by the non-abelian ones

$$F_{\mu\nu}F^{\mu\nu} \to F^a_{\mu\nu}F^{\mu\nu}_a \text{ and, } F_{\lambda\rho}F^{\lambda\rho} \to {}^*F^a_{\lambda\rho}F^{\lambda\rho}_a\,. \tag{2.4}$$

The corresponding action is

$$S = \frac{\beta^2}{4\pi}\int (1 - \mathcal{R})\, d^4x, \text{ where } \ \mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{16\beta^4}(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a)^2}. \tag{2.5}$$

It is easy to see that the BI non-linearity breaks the conformal symmetry ensuring the non-zero trace of the stress–energy tensor

$$T^\mu_\mu = \mathcal{R}^{-1}\left[4\beta^2(1 - \mathcal{R}) - F^a_{\mu\nu}F^{\mu\nu}_a\right] \neq 0. \tag{2.6}$$

This quantity vanishes in the limit $\beta \to \infty$ when the theory reduces to the standard one. For the Yang-Mills field we assume the usual monopole ansatz

$$A_0^a = 0, \quad A_i^a = \varepsilon_{aik} \frac{n^k}{r}(1 - w(r)), \tag{2.7}$$

where $n^k = x^k/r$, $r = (x^2 + y^2 + z^2)^{1/2}$, and $w(r)$ is the real-valued function. After the integration over the sphere in (2.5) one obtains a two-dimensional action from which β can be eliminated by the coordinate rescaling $\sqrt{\beta}t \to t$, $\sqrt{\beta}r \to r$. The following static action results then

$$S = \int L dr, \quad L = r^2(1 - \mathcal{R}), \quad \text{with } \mathcal{R} = \sqrt{1 + 2\frac{w'^2}{r^2} + \frac{(1 - w^2)^2}{r^4}}, \tag{2.8}$$

where prime denotes the derivative with respect to r. The corresponding equation of motion reads

$$\left(\frac{w'}{\mathcal{R}}\right)' = \frac{w(w^2 - 1)}{r^2 \mathcal{R}}. \tag{2.9}$$

A trivial solution $w \equiv 0$ corresponds to the pointlike magnetic BI-monopole with the unit magnetic charge (embedded $U(1)$ solution). In the Born–Infeld theory it has a finite self-energy. For time-independent configurations the energy density is equal to minus the Lagrangian, so the total energy (mass) is

$$M = \int_0^\infty (\mathcal{R} - 1) r^2 dr. \tag{2.10}$$

For $w \equiv 0$ one finds

$$M = \int \left(\sqrt{r^4 + 1} - r^2\right) dr = \frac{\pi^{3/2}}{3\Gamma(3/4)^2} \approx 1.23604978. \tag{2.11}$$

Looking now for the essentially non–Abelian solutions of finite mass, we observe that in order to assure the convergence of the integral (2.10) the quantity $\mathcal{R} - 1$ must fall down faster than r^{-3} as $r \to \infty$. Thus, far from the core the BI corrections have to vanish and the Eq.(2.9) should reduce to the ordinary Yang-Mills equation, equivalent to the following two-dimensional autonomous system

$$\dot{w} = u, \quad \dot{u} = u + (w^2 - 1)w, \tag{2.12}$$

where a dot denotes the derivative with respect to $\tau = \ln r$. This dynamical system has three non-degenerate stationary points $(u = 0, w = 0, 1)$, from which $u = w = 0$ is a focus, while two others $u = 0, w = 1$ are saddle points with eigenvalues $\lambda = -1$ and $\lambda = 2$. The separatices along the directions $\lambda = -1$ start at infinity and after passing through the saddle points go to the focus with the eigenvalues $\lambda = (1i\sqrt{3})/2$. It has been proved in [14] that *the only finite-energy configurations with non-vanishing magnetic charge are the embedded U(1) BI-monopoles*. Indeed, such solutions should have asymptotically $w = 0$, which does not correspond to bounded solutions unless $w \equiv 0$. The remaining possibility is $w = 1, \dot{w} = 0$ asymptotically, which corresponds to zero magnetic charge. Coming back to r-variable one finds from (2.9)

$$w = 1 + \frac{c}{r} + O(r^{-2}), \tag{2.13}$$

where c is a free parameter. This gives a convergent integral (2.10) as $r \to \infty$. The two values $w = 1$ correspond to two neighboring topologically distinct Yang-Mills vacua. Now consider local solutions near the origin $r = 0$. For convergence of the total energy (2.10), w should tend to a finite limit as $r \to 0$. Then using the Eq.(2.9) one finds that the only allowed limiting values are $w = 1$ again. In view of the symmetry of (2.12) under reflection $w \to w$, one can take without loss of generality $w(0) = 1$. The following Taylor expansion satisfies the Eq.(2.12),

$$w = 1 - br^2 + \frac{b^2(44b^2+3)}{10(4b^2+1)} r^4 + O(r^6), \tag{2.14}$$

with b being the unique free parameter. As $r \to 0$, the function $\mathcal{R}$ tends to

$$\mathcal{R} = \mathcal{R}_0 + O(r^2), \quad \mathcal{R}_0 = 1 + 12b^2. \tag{2.15}$$

therefore it is not a solution of the initial system (2.10). What remains to be done is to find appropriate values of constant b leading to smooth finite-energy solutions by gluing together the two asymptotic solutions between 0 and ∞.

It has been proved in [14] that *any regular solution of the* Eq.(2.9) *belongs to the one-parameter family of local solutions* (2.14) *near the origin*. It follows that the global finite energy solution starting with (2.14) should meet some solution from the family (2.13) at infinity. Since both these local solutions are non-generic, one can at best match them for some discrete values of parameters. This technique has been used first in [15]

For some precisely tuned value of b the solution will remain a monotonous function of τ reaching the value -1 at infinity. This happens for $b_1 = 12.7463$. By a similar reasoning one shows that for another fine-tuned value $b_2 > b_1$ the integral curve $w(\tau)$ which has a minimum in the lower part of the strip will be stabilized by the friction term in the upper half of the strip $[-1, 1]$ and tend to $w = 1$. This solution will have two nodes. Continuing this process we obtain the increasing sequence of parameter values b_n for which the solutions remain entirely within the strip $[-1, 1]$ tending asymptotically to $(-1)^n$. The lower values b_n converge very rapidly to the limit value given by (2.11).

Some analogous solutions have been found in the symmetrized trace prescription in [16, 17].

In [18] we have introduced a new non-abelian generalization of the Born-Infeld lagrangian, and found a family of non-singular soliton-like solutions, using 't Hooft's ansatz for the $SU(2)$ gauge potential. As in the case discussed in [14], and in contrast with the usual Yang-Mills case, soliton and magnetic monopole solutions were possible without the presence of Higgs field or other scalar multiplets.

Our starting point is the gauge field tensor associated with a compact and semi-simple gauge group G, defined as a connection 1-form in the principal fibre bundle over Minkowskian spacetime, with its values in $\mathcal{A}_G$, the Lie algebra of G. We chose the representation of the connection in the tensorial product of a matrix representation of the Lie algebra $\mathcal{A}_G$ and the Grassman algebra of forms over M_4.

$$A = A^a_\mu dx^\mu \otimes T_a \,, \tag{2.16}$$

where $T_a, \;\; a, b, ... = 1, 2, ...N = dim(G)$ is the anti-hermitian basis of the particular representation R of dimension d_R of $\mathcal{A}_{\mathcal{G}}$.

By analogy with the abelian case, we want the lagrangian to satisfy the following properties:

- One should find the usual Yang-Mills theory in the limit $\beta \to \infty$
- The (non-abelian) analogue of the electric field strength should be bounded from above when the magnetic components vanish. (To satisfy this particular constraint, we must ensure that the polynomial expression under the root sign should start with terms $1 - \beta^{-2}(E^a)^2 + \ldots$ when $B^a = 0$)
- The action should be invariant under diffeomorphisms of R^4.

The idea is to compute a determinant in the tensor product of endomorphisms of $\mathbb{R}^{\not\supseteq}$, $End(\mathbb{R}^{\not\supseteq})$ and $R(\mathcal{A}_{\mathcal{G}})$. This enables us to introduce the following generalization of the Born-Infeld Lagrangian density for a non-abelian gauge field:

$$\mathcal{L} = \sqrt{|g|} - \left| \left(\nVdash_2 \otimes g_{\mu\nu} \otimes \nVdash_{d_R} + \beta^{-1} \otimes F^a_{\mu\nu} \otimes T_a \right) \right|^{\frac{1}{4d_R}} . \tag{2.17}$$

In the expression above, J denotes a $SL(2,\mathbb{C})$ matrix satisfying $J^2 = -\nVdash_2$, thus introducing a quasi-complex structure. This extra doubling of tensor space is necessary in order to ensure that the resulting Lagrangian is real. In the $SU(2)$ case, it is possible to compute the lagrangian and we obtain

$$L = 1 - \sqrt[4]{(1+2P-Q^2)^2 + (2K_3)^2} \; , \tag{2.18}$$

where

$$\begin{cases} 2P &= \frac{1}{2} F^a_{\mu\nu} F^{\mu\nu}_a \\ Q^2 &= \frac{1}{16} F^a_{\mu\nu} \star F^{b\mu\nu} F^c_{\alpha\beta} \star F^{d\alpha\beta} K_{acbd} \\ K_3 &= \frac{1}{6} \varepsilon_{abc} F^{a\mu}_{\nu} F^{b\nu}_{\alpha} F^{c\alpha}_{\mu} \end{cases} . \tag{2.19}$$

with $K_{abcd} = \delta_{ab}\delta_{bc} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$. We have then studied spherically symmetric static configurations by considering the well known 't Hooft ansatz

$$A = (1-k(r))\,(T_\theta \sin\theta d\varphi - T_\varphi d\theta) . \tag{2.20}$$

Then the action become

$$S = \int \left(1 - \sqrt[4]{\left(1 + \left(\frac{1-k^2}{r^2}\right)^2\right)\left(\left(1 + \frac{2\dot{k}^2}{r^4}\right)^2 + \left(\frac{1-k^2}{r^2}\right)^2\right)} \right) e^{3\tau} d\tau \; , \tag{2.21}$$

with $\tau = \log(r)$.

The equations of motion can be written in the standard form:

$$\begin{cases} \dot{k} &= u \\ \dot{u} &= \gamma(k,u,\tau)u + k(k^2-1) \end{cases} , \tag{2.22}$$

with

$$\begin{aligned} \gamma(k,u,\tau) = {} & 1 - 2\, \frac{u^2 + 2uk(1-k^2) + (1-k^2)^2}{r^4 + (1-k^2)^2} \\ & + \frac{6u(1-k^2)\left[ku^2 + 2u(1-k^2) + k(1-k^2)^2\right]\left[r^4 + 2u^2 + (1-k^2)^2\right]}{[r^4 + (1-k^2)^2]\,[(r^4+2u^2)^2 + (1-k^2)^2(r^4+6u^2)]} . \end{aligned} \tag{2.23}$$

Although the equations display asymptotic expansions analogous to those found in [20, 21, 14], careful analysis shows that solutions of the Bartnik-McKinnon type [22] are excluded here.

Near the origin there are two types of asymptotic development which satisfy the equations of motion:

$$k = k_0 + ar - k_0 \left(\frac{5a^2}{6g} + \frac{g}{12a^2} \right) r^2 + \frac{a^8(52-70g) - 9a^4g^3 + (g-1)g^4}{108a^5g^2} r^3 + O(r^4) \;, \tag{2.24}$$

where $g = 1 - k_0^2$, $a \neq 0$ and $g \neq 0$ and depending of two free parameters k_0 and a.

A second development depends on only one free parameter b, and starts as follows

$$k = \left(1 - br^2 + \frac{3b^2 + 92b^4 + 608b^6}{10 + 200b^2 + 1600b^4} r^4 + O(r^6) \right) \tag{2.25}$$

which correspond to solutions along the separtice with $\lambda = 2$ discussed in the previous section.

At infinity, the Taylor expansion can be made with respect to r^{-1}. It depends on one free parameter, denoted by c

$$k = \left(1 - \frac{c}{r} + \frac{3c^2}{4r^2} + O(\frac{1}{r^3}) \right) \;. \tag{2.26}$$

which correspond to solutions along the separatix with $\lambda = -1$ discussed in the previous section.

Taking these expansions as the first approximation either at $r = 0$ or at $r = \infty$, we then use standard numerical techniques in order to generate solutions valid everywhere. It was interesting to note that, when we started from infinity, no fine-tuning was necessary, and an arbitrarily fixed constant c would lead to a solution which, when extrapolated to $r = 0$, would define a particular pair of values of constants k_0 and a. On the contrary, starting from $r = 0$, arbitrarily chosen values of k_0 and a would not necessarily lead to good extrapolation at $r = \infty$. Therefore the three parameters occurring in the asymptotic expansions must be interrelated by two constraint equalities. Then the solutions can be labelled by only one real parameter, and then the two parameters k_0 and a of (2.24) are functions of the parameter τ_c.

We have evaluated the energy E of the solutions founded and the values of the parameter k_0 for τ_c varying from -10 to 20. The energy E is represented as a function of the parameter τ_c in Fig. 2. The energies converge to the limit $E_{\tau_c=\infty} = E_{n=\infty} = 1.23605...$ which coincides with the energy of the Born-Infeld monopole.

Our solutions do not interpolate between the two singular points at $k = 1$ and $k = -1$, but between the singular point at $k = 1$ for $r = \infty$ and a certain value k_0 (related to τ_c) which is always lower than 1 and bigger than -1 (as a matter of fact $k_0 = 0$ is a solution, which correspond to monopole solution). This is radically different from the sphaleron like solutions or solutions of Bartnik-McKinnon type found in [22, 14].

As in the Bartnik-McKinnon case, we can assign to each solution an integer n, with $n-1$ denoting the number of zeros of the function u or the winding number of the corresponding trajectory in the phase plane (k,u), as seen in Fig. 1, where a few solutions are plotted.

When the parameter τ_c goes from $-\infty$ to $+\infty$, we observe that this integer n grows from 1 to ∞. At certain special values of the parameter τ_c, this integer increases by 1. Here are the first critical values of τ_c:

τ_c	1.658	4.781	7.510	10.092	13.218	16.530	19.813

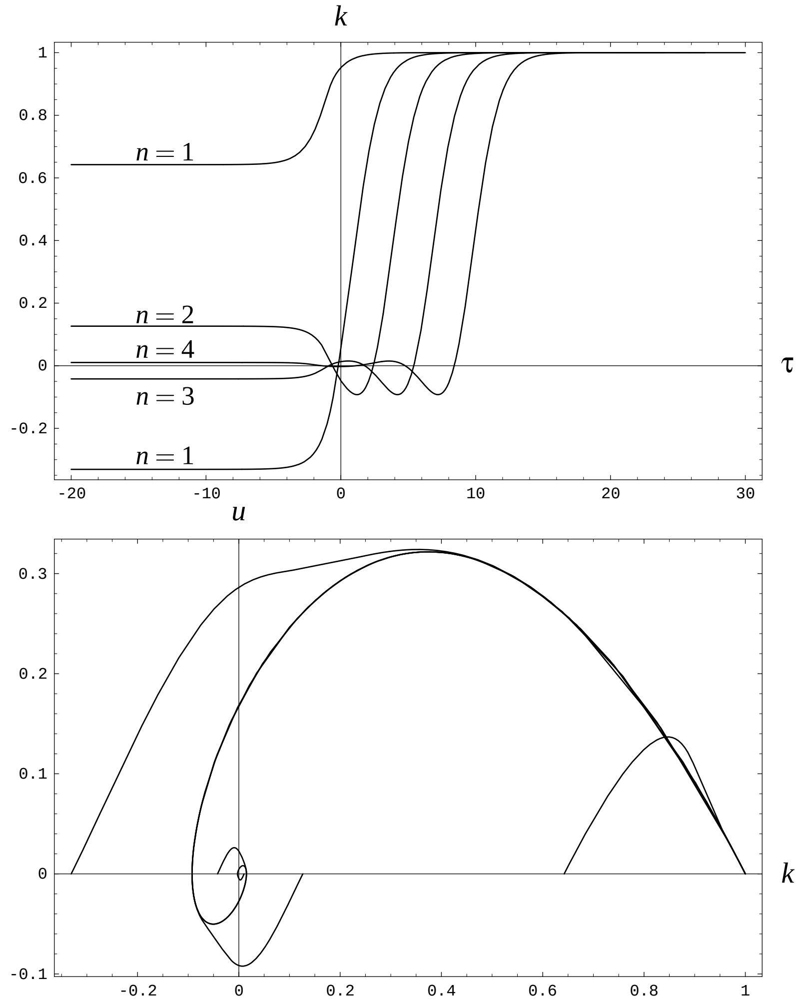

Figure 1: Plots of solutions for the values of $\tau_c = -3, 1.2, 4, 7, 10$.

3. Born-Infeld type lagrangian for Higgs fields from non-commutative geometry

In this section we study the Higgs-like fields which naturally appear in the version of standard model based on the non-commutative geometry [23]. We show that soliton-like solutions with finite energy can not be obtained with pure Higgs fields obeying this version of generalized Born-Infeld dynamics, in the case when the Higgs multiplet reduces to a single scalar. This situation corresponds to the particular choice of matrix-valued generalized Higgs field, when the corresponding matrix is proportional to the identity. This does not exclude the possibility of soliton-like solutions in more complicated cases, with many-component Higgs field.

We shall generalize now the non-commutative Maxwell theory developed in [23] in order to obtain a Born-Infeld like theory. Let us resume the notations and language of the theory. We consider the algebra $\mathcal{A} = C^\infty(V) \otimes M_n(\mathbb{C})$ with the vector fields spanned by the derivations of $C^\infty(V)$ and inner derivations of $M_n(\mathbb{C})$. The differential algebra is generated by the basis of linear 1-forms acting on the derivations. We can consider $\mathcal{A}$ as a bimodule over itself. Then one defines a gauge by the choice of an unitary element e of $\mathcal{A}$, satisfying $h(e,e) = 1$, with h an hermitian

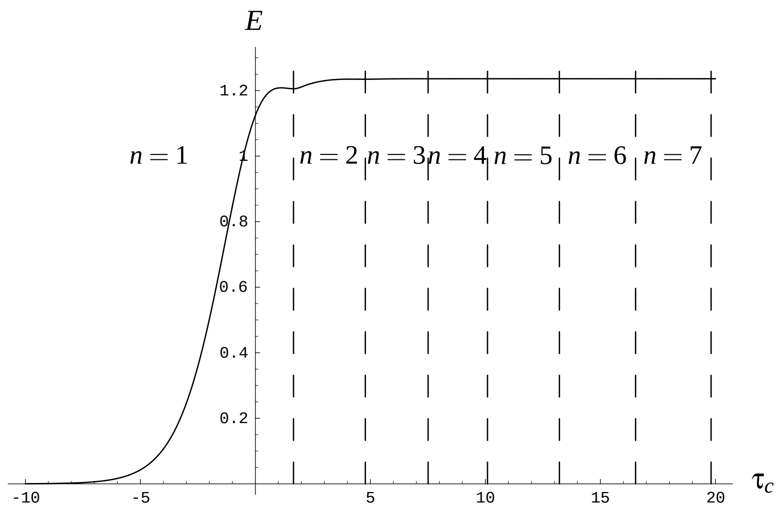

Figure 2: Energy as function of the parameter τ_c.

structure on $\mathcal{A}$. Then any element of $\mathcal{A}$ can be written in the form em with $m \in \mathcal{A}$ and a connection on $\mathcal{A}$ is a map:

$$\nabla : \mathcal{A} \to \Omega^1(\mathcal{A}),\ \ e\, m \to (\nabla e)\, m + e\, dm \tag{3.1}$$

In the gauge e, the connection can be completely characterized by an element ω of $\Omega^1(\mathcal{A})$:

$$\nabla e = e\, \omega\ .$$

One can also decompose ω in vertical and horizontal parts:

$$\omega = \omega_h + \omega_v\,,$$

with

$$\omega_h = A, \quad \omega_v = \theta + \phi\,. \tag{3.2}$$

Here A is like the Yang-Mills connection, whereas θ is the canonical 1-form of the matrix algebra, and plays the role of a preferred origin in the affine space of vertical connections. It satisfies the equation:

$$d\theta + \theta^2 = 0$$

Then ϕ is a tensorial form and can be identified with scalar field multiplet.

Choosing a local basis of derivations of $\mathcal{A}$: $\{e_\mu, e_a\}$, where for convenience e_μ are outer derivations of $C^\infty(V)$, and $e_a = ad(\lambda_a)$, with $\{\lambda_a\}$ a basis of anti-hermitian matrices of $M_n(\mathbb{C})$, are inner derivations.
The dual basis will be denoted by $\{\theta^\mu, \theta^a\}$. In this particular basis, we have:

$$A = A_\mu \theta^\mu\ ,\ \theta = -\lambda_a \theta^a\ ,\ \phi = \phi_a \theta^a\,.$$

If we choose the connection to be anti-hermitian, we can write $\phi = \phi_a^b \lambda_b \theta^a$. The curvature tensor associated with ω is

$$\Omega = d\omega + \omega^2\,,$$

we can also define the field strength

$$F = dA + A^2 \,.$$

Then by "dimensional reduction" one can identify

$$\Omega_{\mu\nu} = F_{\mu\nu} \qquad \Omega_{\mu a} = D_\mu \phi_a$$
$$\Omega_{a\mu} = -D_\mu \phi_a \qquad \Omega_{ab} = [\phi_a, \phi_b] - C^c_{ab}\phi_c \,,$$

C^c_{ab} are the constant structure in the $\{\lambda_a\}$ basis.

A gauge transformation is performed by the choice of a unitary element U of $M_n(\mathbb{C})$, satisfying $h(eU, eU) = 1$. Then in the gauge $e' = eU$

$$\omega' = U^{-1}\omega U + U^{-1} dU \,,$$

θ is invariant under gauge transformations, then

$$A' = U^{-1} A U + U^{-1} dU, \; \phi' = U^{-1}\phi U \,.$$

The generalization discussed in the previous section can be adapted to the non-commutative gauge theory. The lagrangian which we consider is

$$\sqrt{\det|g|} - \{|\det(\mathbb{1} \otimes g + J \otimes \hat{\Omega}|\}^{1/4n} \,,$$

and $\hat{\Omega} = \Omega_{\alpha\beta} L^{\hat{\alpha\beta}}$ with $L^{\hat{\alpha\beta}}$ the generators of the fundamental representation of $SO(4+n^2-1)$. $\Omega_{\alpha\beta}$ are the components of the curvature defined in previous section, and then are anti-hermitian elements of $M(n, \mathbb{C})$. J is an element of $SL(2, \mathbb{C})$ of square $-\mathbb{1}$.

The above lagrangian contains the contribution of two types of fields: the classical Yang-Mills potential, $A = A_\mu \theta^\mu$, corresponding to the usual space-time components of the connection one-form, and the scalar multiplet coming from its matrix components $\phi = \phi_a \theta^a = \phi^b_a \lambda_b \theta^a$. In the case when $\phi = 0$, this lagrangian coincide with the one studied in [18], and exposed in the previous section. The complete analysis of general solutions seems too tedious for the time being. This is why we shall restrict ourselves to a qualitative analysis of the case when the space time components of Ω do vanish $F_{\mu\nu} = 0$, leaving only the contribution of scalar multiplet degrees of freedom.

Let us recall the notations which will be used in the subsequent calculations. The basis of matrix representation of the $SU(2)$-algebra is chosen as follows

$$\lambda_a = -i\sigma_a \quad \lambda_a\lambda_b = -\delta_{ab} + \sum_c \varepsilon_{abc}\lambda_c \quad [\lambda_a, \lambda_b] = C^c_{ab} = 2\varepsilon_{abc}\lambda_c \,. \tag{3.3}$$

Now we have to evaluate the determinant of the following matrix

$$\begin{vmatrix} 1 & iD\hat{\phi} \\ -iD\hat{\phi} & 1 + i\hat{H} \,, \end{vmatrix} \tag{3.4}$$

where

$$\hat{H} = \{\Omega_{ab}\}_{a,b=1,2,3} \;,\; D\hat{\phi} = \left\{D_\mu \hat{\phi}_a\right\}_{a=1,2,3 \mu=0,1,2,3} \,. \tag{3.5}$$

From now on, we choose the simplest ansatz with one scalar field only

$$\phi = \varphi\, \theta\,.$$

After some algebra, we get the following result

$$L = 1 - \left\{1 + 6(D\varphi)^2 + 9(D\varphi)^4 + 16\varphi^2(\varphi-1)^2\right\}^{\frac{1}{4}} \sqrt{1 + 4\varphi^2(\varphi-1)^2}\,.$$

Let us show now that there no non-trivial static configurations can be found in this particular system. To this end, we shall generalize Derrick's theorem [24] to our case.

The idea of the proof is to use spatial dilatations of the field $\varphi(r) \to \varphi_\lambda(r) = \varphi(\lambda r)$ to generate a one parameter curve in the space of fields around such a solution.Thus the variational principle along this curve gives $\partial S[\varphi_\lambda]/\partial\lambda = 0$ at $\lambda = 1$, i.e.

$$\int 4\pi r^2 dr \left(\frac{\partial \mathcal{L}}{\partial \varphi'}\varphi' - 3\mathcal{L}\right) = 0\,. \tag{3.6}$$

We can show, by algebraic manipulations, that the quantity under the integral is always non-negative, and satisfies (3.6) if and only if it is zero. The solutions are just the trivial ones $\varphi' = 0$ and $\varphi = 0$ or 1 which exclude other non-trivial solutions.

We have performed a numerical analysis of the time dependent configurations of the scalar field resulting from the simplest ansatz $\varphi = \varphi(t)$. It gives an interesting phase space portrait and confirms the idea that BI-like theories set an upper bound on velocities (i.e. time derivatives of φ), and on the field strength as well. Such an ansatz could be of use in cosmology, when coupled with the scale factor $a(t)$ of Robertson-Friedmann metric. The investigation will be the subject of our forthcoming paper [25].

The equations of motion in this case take on the following form

$$\begin{aligned}
&\dot\varphi = u\,,\\
&(1+4X)g(X,Y)\dot u + 4ss'h(X,Y) = 0\,,
\end{aligned}$$

where

$$\begin{aligned}
&s = \varphi(\varphi-1)\ ,\ s' = 2\varphi - 1\\
&X = s^2\ ,\ Y = u^2\\
&g(X,Y) = 16X(1-9Y) + (1-3Y)^2\\
&h(X,Y) = ((1-3Y)^2 + 16X)(1 - Y + 8X) - 6(1+4X)(1-3Y)Y\,.
\end{aligned}$$

At some points of the phase space $\dot u$ is not well defined. These are the points at which the polynomial g vanishes (red curves in the figure). Nevertheless, in most of the cases singular behavior is only apparent, because the undetermined ratios $0/0$ prove to have a finite limit. The total number of singular points in the phase space is 16, but only 2 of them display a genuine singularity. In the 14 remaining cases the function $4ss'h(X,Y)$ vanishes at the same time as the function $g(X,Y)$, but they ratio remains finite. In the figure below one can observe the 16 aforementioned points. The only two points with genuine singularity are the ones without any vector attached to them, found on the central vertical line $\varphi = 0.5$ on both sides of the horizontal line and close to it.

The phase space portrait is symmetric by reflection around the vertical line $\varphi = 0.5$. Cyclic trajectories are contained inside the two pentagon-like areas circumscribed by separatrices. These areas are disposed symmetrically with respect to the vertical line $\varphi = 0$. One of these areas is represented in more detailed manner in fig. 5 below.

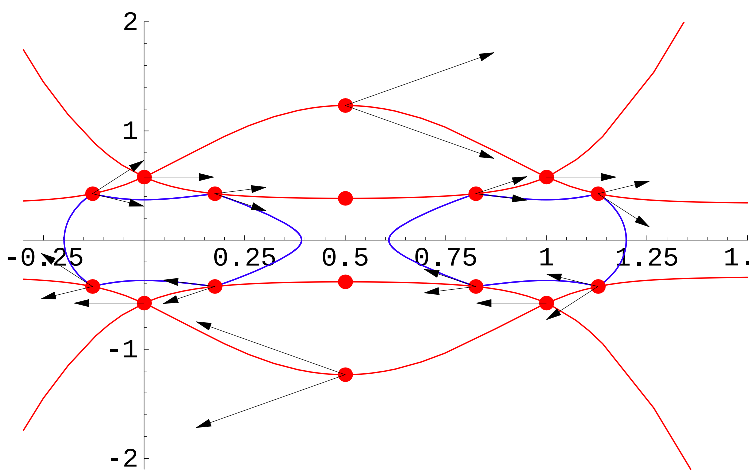

Figure 3: Characteristic curves and points in the phase space

One can note that in a certain region of the phase space the trajectories are periodic and defined for all values of time t. If one chooses the initial conditions outside this region, the integration ends up after some finite time. This means that the solutions $\varphi(t)$ obtained with these initial conditions have their second derivative divergent after finite time when they hit one of the curves on which $g = 0$.

Nevertheless some of these curves, with find-tuned initial conditions, can go beyond the singular curve $g = 0$ at points at which the indefinite expressions become finite again. This particular trajectories form a special set; they can be extended beyond the limits of the region shown in fig. 4. and be defined for all values of time $t \in \mathbb{R}$.

4. Conclusion and perspectives

Certain generalizations of the Born-Infeld type lagrangian for scalar fields have been proposed by several authors [26, 27]. However, in these papers only formal analogy was used; usually by inserting a classical scalar field lagrangian under the square root sign.

The highly non-linear behavior of the field Φ in this model suggests that when coupled to gravitation in a standard way, i.e. via minimal coupling resulting from the replacement of ordinary derivatives by their covariant counterparts, and adding he Einstein-Hilbert lagrangian for gravitational field it may lead to unusual behavior of cosmological models. The investigation of cosmological models using this scalar field will be the subject of our forthcoming paper [25].

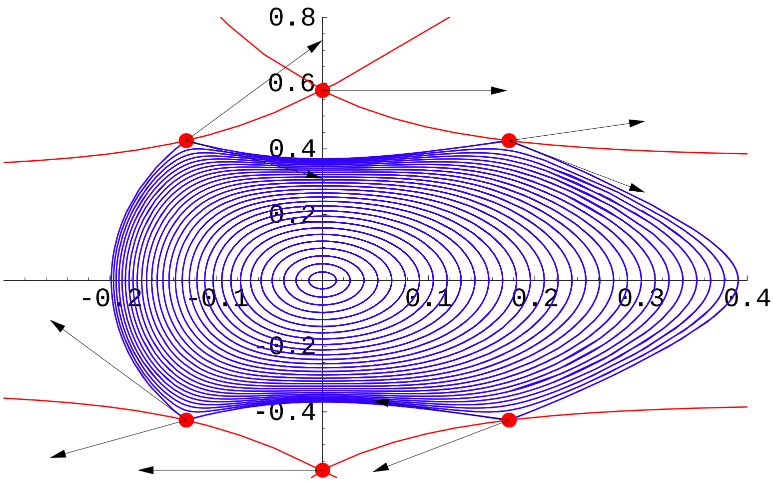

Figure 4: Trajectories in the confined region of the phase space

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