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Twistors, Generalizations and Exceptional Structures

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This paper is intended to describe twistors via the paravector model of Clifford algebras and to relate such description to conformal maps in the Clifford algebra over $\mathbb{R}^{4,1}$, besides pointing out some applications of the pure spinor formalism. We construct twistors in Minkowski spacetime as algebraic spinors associated with the Dirac-Clifford algebra $\mathbb{C}\otimes C\ell_{1,3}$ using one lower spacetime dimension than standard Clifford algebra formulations, since for this purpose the Clifford algebra over $\mathbb{R}^{4,1}$ is also used to describe conformal maps, instead of $\mathbb{R}^{2,4}$. It is possible to identify, via the pure spinor formalism, the twistor fiber in four, six and eight dimensions, respectively, with the coset spaces $\mathrm{SO}(4)/(\mathrm{SU}(2)\times\mathrm{U}(1)/\mathbb{Z}_2)\simeq\mathbb{CP}^1$, $\mathrm{SO}(6)/(\mathrm{SU}(3)\times\mathrm{U}(1)/\mathbb{Z}_2)\simeq\mathbb{CP}^3$ and $\mathrm{SO}(8)/(\mathrm{Spin}(6)\times\mathrm{Spin}(2)/\mathbb{Z}_2)$. The last homogeneous space is closely related to the SO(8) spinor decomposition preserving SO(8) symmetry in type IIB superstring theory. Indeed, aside the IIB superstring theory, there is no SO(8) spinor decomposition preserving SO(8) symmetry and, in this case, one can introduce distinct coordinates and conjugate momenta only if the Spin(8) symmetry is broken by a Spin(6)× Spin(2) subgroup of Spin(8). Also, it is shown how to generalize the Penrose flagpole, illustrating the use of the pure spinor formalism to construct a flagpole that is more general than the Penrose one, which arises when a defined parameter goes to zero. We investigate the relation between this flagpole and the SO($2n$)/U(n) twistorial structure, which emerges when one considers the action of a suitable classical group on the set Ξ of all totally isotropic subspaces of $\mathbb{C}^{2n}$, and an isomorphism from the set of pure spinors to Ξ. Finally we point out some relation between twistors fibrations and the classification of compact homogeneous quaternionic-Kähler manifolds (the so-called Wolf spaces), and exceptional Lie structures.

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1. Introduction

Nowadays the search for any unified theory that describes the four fundamental interactions demands a deep mathematical background and an interface between physics and mathematics. The relation between superstring theory in twistor spaces [1, 2] and the pure spinor formalism [3, 4] has been increasingly and widely investigated [5, 6]. With the motivation concerning the SO(8) spinor decomposition that preserves SO(8) symmetry in type IIB superstring theory [7], among others, it can be shown via the pure spinor formalism the well-known result asserting that a twistor in eight dimensions is an element of the homogeneous space SO(8)/(Spin(6)$\times$ Spin(2)/$\mathbb{Z}_2$) $\simeq$ SO(8)/U(4), and, in n dimensions, an element of SO($2n$)/U(n).

The main aim of this paper, besides pointing out some relation between twistors and pure spinors, is to describe conformal maps in Minkowski spacetime as the twisted adjoint representation of $\$\mathrm{pin}_+(2,4)$ (to be precisely defined in Sec. 2) on paravectors[1] [8, 9] of $\mathcal{C}\ell_{4,1}$, and to characterize twistors as algebraic spinors[2] [4] in $\mathbb{R}^{4,1}$. Although some papers have already described twistors using the algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}\simeq\mathcal{C}\ell_{4,1}$ [10, 11, 12], the present formulation sheds some new light on the use of the paravector model. This paper is presented as follows: in Sec. 2 we describe conformal transformations using the twisted adjoint representation of the group SU(2,2) $\simeq$ $\$\mathrm{pin}_+(2,4)$ on paravectors of $\mathcal{C}\ell_{4,1}$. In Sec. 3 twistors, the incidence relation between twistors and the Robinson congruence, via multivectors and the paravector model of $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}\simeq\mathcal{C}\ell_{4,1}$, are introduced. We show explicitly how our results can be led to the well-established ones of Keller [12], and consequently to the classical formulation introduced by Penrose [13, 14]. It is also described how one can obtain twistors as elements of SO($2n$)/U(n) via pure spinors. Finally in Sec. 4 we link twistor theory to Lie exceptional structures.

2. Conformal compactification and the paravector model

Given a vector space, endowed with a metric g of signature $p-q$, and denoted by $\mathbb{R}^{p,q}$, consider the injective map [9] $\mathbb{R}^{p,q}\ni x\mapsto(x,g(x,x),1)=(x,\lambda,\mu)\in\mathbb{R}^{p+1,q+1}$. The image of $\mathbb{R}^{p,q}$ under this map is a subset of the Klein absolute $x\cdot x-\lambda\mu=0$. This map induces an injective map from the conformal compactification $(S^p\times S^q)/\mathbb{Z}_2$ of $\mathbb{R}^{p,q}$ to the projective space $\mathbb{RP}^{p+1,q+1}$.

The conformal group Conf(p,q) is isomorphic to the quotient group $\mathrm{O}(p+1,q+1)/\mathbb{Z}_2$ [9], and since the group $\mathrm{O}(p+1,q+1)$ has four components, then Conf(p,q) has two (if p or q are even) or four components (otherwise) [9, 15]. Taking the case when $p=1$ and $q=3$, the group Conf(1,3) has four components, and the component Conf$_+(1,3)$ connected to the identity is the Möbius group[3] of $\mathbb{R}^{1,3}$. Besides, the orthochronous connected component is denoted by SConf$_+$(1,3). Consider a basis $\{\varepsilon_{\check{A}}\}_{\check{A}=0}^{5}$ of $\mathbb{R}^{2,4}$ and a basis $\{E_A\}_{A=0}^{4}$ of $\mathbb{R}^{4,1}$. This last basis can be obtained from $\{\varepsilon_{\check{A}}\}$ if the isomorphism $E_A\mapsto\varepsilon_A\varepsilon_5$ is defined.

Given ϕ an element of the Clifford algebra $\mathcal{C}\ell_{p,q}$ over $\mathbb{R}^{p,q}$, the reversion of ϕ is defined and denoted by $\tilde{\phi}=(-1)^{[k/2]}\phi$ ($[k]$ expresses the integer part of k), while the graded involution acting

[1] A paravector of the Clifford algebra $\mathcal{C}\ell_{p,q}$ is an element of $\mathbb{R}\oplus\mathbb{R}^{p,q}$.

[2] Algebraic spinors are elements of a minimal lateral ideal of a Clifford algebra.

[3] All Möbius maps are composition of rotations, translations, dilations and inversions [16].

PoS(WC2004)022

on ϕ is defined by $\hat{\phi} = (-1)^k\phi$. The Clifford conjugation $\bar{\phi}$ of ϕ is given by the reversion composed with the main automorphism.

If we take a vector $\alpha = \alpha^{\check{A}}\varepsilon_{\check{A}} \in \mathbb{R}^{2,4}$, a paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{C}\ell_{4,1}$ can be obtained as $\mathfrak{b} = \alpha\varepsilon_5 = \alpha^A E_A + \alpha^5$. From the periodicity theorem[4] [17] we have the isomorphism $\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{3,0} \simeq M(2,\mathbb{C}) \otimes \mathcal{C}\ell_{3,0}$, where $M(2,\mathbb{C})$ denotes the group of 2×2 matrices with complex entries. For i = 1,2,3 the isomorphism from $\mathcal{C}\ell_{4,1}$ to $\mathcal{C}\ell_{3,0}$ is given explicitly by $E_i \mapsto E_iE_0E_4 := \mathbf{e}_i$, where $\{\mathbf{e}_i\}$ denotes a basis of $\mathbb{R}^3$. Defining $E_\pm := \frac{1}{2}(E_4 \pm E_0)$, we can write $\mathfrak{b} = \alpha^5 + (\alpha^0 + \alpha^4)E_+ + (\alpha^4 - \alpha^0)E_- + \alpha^i\mathbf{e}_iE_4E_0$, and then it is possible, if we represent $E_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $E_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, to write $\mathfrak{b} = \begin{pmatrix} \alpha^5 + \alpha^i\mathbf{e}_i & \alpha^4 - \alpha^0 \\ \alpha^0 + \alpha^4 & \alpha^5 - \alpha^i\mathbf{e}_i \end{pmatrix}$. The vector $\alpha \in \mathbb{R}^{2,4}$ is in the Klein absolute, and so $\alpha^2 = 0$. Besides, we assert that $\mathfrak{b}$ is in the Klein absolute if and only if α is. Indeed, denoting $\lambda = \alpha^4 - \alpha^0$ and $\mu = \alpha^4 + \alpha^0$, if $\mathfrak{b}\bar{\mathfrak{b}} = 0$, the matrix element $(\mathfrak{b}\bar{\mathfrak{b}})_{11}$ is given by

$$(\mathfrak{b}\bar{\mathfrak{b}})_{11} = x\bar{x} - \lambda\mu = 0, \tag{2.1}$$

where $x := (\alpha^5 + \alpha^i\mathbf{e}_i) \in \mathbb{R} \oplus \mathbb{R}^3 \hookrightarrow \mathcal{C}\ell_{3,0}$. Choosing $\mu = 1$ then $\lambda = x\bar{x}$, and this choice is responsible for a projective description. Also, the paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1}$ can be rewritten as $\mathfrak{b} = \begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix}$. From eq.(2.1) we obtain $(\alpha^5 + \alpha^i\mathbf{e}_i)(\alpha^5 - \alpha^i\mathbf{e}_i) = (\alpha^4 - \alpha^0)(\alpha^4 + \alpha^0)$ from where $(\alpha^5)^2 + (\alpha^0)^2 - (\alpha^1)^2 - (\alpha^2)^2 - (\alpha^3)^2 - (\alpha^4)^2 = 0$, showing that α is indeed in the Klein absolute.

Now consider an element $g \in \mathrm{SU}(2,2) \simeq \$\mathrm{pin}_+(2,4) := \{g \in \mathcal{C}\ell_{4,1} \mid g\bar{g} = 1\}$. From the periodicity theorem, it can be represented as $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where $a, b, c, d \in \mathcal{C}\ell_{3,0}$.

In order to perform a rotation of the paravector $\mathfrak{b}$, we can use the twisted adjoint representation $\hat{\sigma} : \$\mathrm{pin}_+(2,4) \to \mathrm{SO}_+(2,4)$, defined by its action on paravectors by $\hat{\sigma}(g)(\mathfrak{b}) = g\mathfrak{b}\hat{g}^{-1} = g\mathfrak{b}\tilde{g}$. In terms of matrix representations (with entries in $\mathcal{C}\ell_{3,0}$), the group $\$\mathrm{pin}_+(2,4)$ acts on paravectors $\mathfrak{b}$ as $g\mathfrak{b}\tilde{g} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix}$. Fixing $\mu = 1$, $\mathfrak{b}$ is mapped on $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} = \Delta\begin{pmatrix} x' & x'\bar{x}' \\ 1 & \bar{x}' \end{pmatrix}$, where $x' = (ax+c)(bx+d)^{-1} \in \mathbb{R} \oplus \mathbb{R}^3$ and $\Delta = (bx+d)(\overline{bx+d}) \in \mathbb{R}$. In this sense the spacetime conformal maps are rotations in $\mathbb{R} \oplus \mathbb{R}^{4,1}$, performed by the twisted adjoint representation, just given above. All the spacetime conformal maps are expressed respectively by the following matrices [9, 16, 18]:

Conformal Map	Explicit Map	Matrix of $\$\mathrm{pin}_+(2,4)$
Translation	$x \mapsto x + h,\ \ h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$
Dilation	$x \mapsto \rho x,\ \rho \in \mathbb{R}$	$\begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix}$
Rotation	$x \mapsto \mathfrak{g}x\hat{\mathfrak{g}}^{-1},\ \mathfrak{g} \in \$\mathrm{pin}_+(1,3)$	$\begin{pmatrix} \mathfrak{g} & 0 \\ 0 & \hat{\mathfrak{g}} \end{pmatrix}$
Inversion	$x \mapsto -\bar{x}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Transvection	$x \mapsto x + x(hx+1)^{-1},\ \ h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$

[4]The periodicity theorem of Clifford algebras asserts that $\mathcal{C}\ell_{p+1,q+1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q}$.

This index-free algebraic formulation allows to trivially generalize the conformal maps of $\mathbb{R}^{1,3}$ to the ones of $\mathbb{R}^{p,q}$, if the periodicity theorem of Clifford algebras is used. The homomorphisms $\$\mathrm{pin}_+(2,4) \simeq \mathrm{SU}(2,2) \xrightarrow{2-1} \mathrm{SO}_+(2,4) \xrightarrow{2-1} \mathrm{SConf}_+(1,3)$ are explicitly constructed in [19].

The generators of Conf(1,3) are expressed, using a basis $\{\gamma_\mu\} \in \mathcal{C}\ell_{1,3}$ and denoting the volume element of $\mathbb{R}^{1,3}$ by $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, as $P_\mu = \frac{1}{2}(\gamma_\mu + i\gamma_\mu\gamma_5)$, $K_\mu = -\frac{1}{2}(\gamma_\mu - i\gamma_\mu\gamma_5)$, $D = \frac{1}{2}i\gamma_5$, and $M_{\mu\nu} = \frac{1}{2}(\gamma_\nu \wedge \gamma_\mu)$. They satisfy the commuting relations

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \qquad [K_\mu, K_\nu] = 0, \qquad [M_{\mu\nu}, D] = 0,\\
[M_{\mu\nu}, P_\lambda] &= -(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu), \quad [M_{\mu\nu}, K_\lambda] = -(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu),\\
\left[M_{\mu\nu}, M_{\sigma\rho}\right] &= g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma},\\
[P_\mu, K_\nu] &= 2(g_{\mu\nu}D - M_{\mu\nu}), \quad [P_\mu, D] = P_\mu, \qquad [K_\mu, D] = -K_\mu,
\end{aligned} \tag{2.2}$$

which are invariant under $P_\mu \mapsto -K_\mu$, $K_\mu \mapsto -P_\mu$ and $D \mapsto -D$.

3. Twistors as geometric multivectorial elements

In this section we present and discuss the construction of twistors as algebraic spinors of $\mathcal{C}\ell_{4,1}$, using the paravector model, and as elements of SO($2n$)/U(n), via the pure spinor formalism.

3.1 Twistors as algebraic spinors using the paravector model

The *reference twistor* $\eta_\mathbf{x}$ is defined [12], given $\mathbf{x} \in \mathbb{R}^{1,3}$ and a dotted covariant Weyl spinor[5] (DCWS) $\Pi = \frac{1}{2}(1 - i\gamma_5)\psi = (0, \xi)^t$, as the multivector

$$\eta_\mathbf{x} = (1 + \gamma_5\mathbf{x})\Pi. \tag{3.1}$$

The above expression is an index-free geometric algebra version of Penrose twistor in $\mathbb{R}^{1,3}$, since if a suitable representation[6] of $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ is used, we have

$$\eta_\mathbf{x} = (1 + \gamma_5\mathbf{x})\Pi = \left[\begin{pmatrix} i_2 & 0 \\ 0 & i_2 \end{pmatrix} + \begin{pmatrix} -i_2 & 0 \\ 0 & i_2 \end{pmatrix}\begin{pmatrix} 0 & \vec{x} \\ \vec{x}^c & 0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} i\vec{x}\xi \\ \xi \end{pmatrix}, \tag{3.2}$$

where $\vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}$. The symbol $\vec{x}^c$ denotes the $\mathbb{H}$-conjugation of x and $i_2 := i\mathbf{1}_{2\times 2}$.

The adjoint Dirac spinor is defined as $\mathring{\psi} = \psi^\dagger\gamma_0 = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \bar{\psi}_4)$ and the transposed twistor as $\mathring{\eta}_\mathbf{x} = \mathring{\psi}\frac{1}{2}(1 + i\gamma_5)(1 + \gamma_5\bar{\mathbf{x}}) = \mathring{\Pi}(1 + \gamma_5\bar{\mathbf{x}})$. The scalar product $\mathring{\eta}_\mathbf{x}\eta_\mathbf{x}$ represents the expected value of $\gamma_5\mathbf{x}$ with respect to the spinor Π, since $\mathring{\eta}_\mathbf{x}\eta_\mathbf{x} = \mathring{\Pi}\Pi + 2\mathring{\Pi}\gamma_5\mathbf{x}\Pi + \mathbf{x}^2\mathring{\Pi}\Pi = 2\mathring{\Pi}\gamma_5\mathbf{x}\Pi$. The tensor product $\eta_\mathbf{x}\mathring{\Pi} = (1 + \gamma_5\mathbf{x})\Pi\mathring{\Pi} = (1 + \gamma_5\mathbf{x})q$, where $q = \Pi\mathring{\Pi}$ is the chiral positive projection of the timelike vector $Q = \psi\mathring{\psi}$, is also presented [12]. It allows to interpret the relation between a twistor, a timelike vector q and the flagpole $\gamma_5\mathbf{x}q$, given by the following multivector:

$$\zeta_\mathbf{x} := \eta_\mathbf{x}\mathring{\Pi} = (1 + \gamma_5\mathbf{x})q = q + \gamma_5\mathbf{x}q = (1 - i\mathbf{x})q \in \mathcal{C}\ell_{4,1}. \tag{3.3}$$

[5] A Weyl spinor can always be written as $\frac{1}{2}(1 \pm i\gamma_5)\psi$, where ψ is a Dirac spinor.

[6] As Keller [12], we choose to use a representation that differs from the Weyl representation by a sign on the matrices representing γ_1, γ_2 and γ_3.

The incidence relation, that determines a point in spacetime from the intersection of two twistors is defined, leading to the Penrose description [13, 14], as

$$J_{\mathbf{xx}} := \overline{\eta}_{\mathbf{x}}\eta_{\mathbf{x}} = \mathring{\Pi}\gamma_5(\mathbf{x}-\mathbf{x})\Pi = 0. \tag{3.4}$$

The product $J_{\mathbf{xx}}$ is invariant if $\eta_{\mathbf{x}}$ is multiplied by a complex number. Then eight dimensions are reduced to six, which leads to the classical interpretation of a twistor related to the space $\mathbb{CP}^3 \simeq$ SO(6)/(SU(3)× U(1)/$\mathbb{Z}_2$) [6, 13, 14, 20].

Keller presents another inner product [12], corresponding to the same twistor, but relating distinct points in spacetime, as $J_{\mathbf{xx'}} = \overline{\eta}_{\mathbf{x}}\eta_{\mathbf{x'}} = \mathring{\Pi}\gamma_5(\mathbf{x}-\mathbf{x}')\Pi$. This product is null if and only if $\mathbf{x}=\mathbf{x}'$. The Robinson congruence [13] is defined if we fix $\mathbf{x}$ and let $\mathbf{x}'$ vary.

Let f be a primitive idempotent (PI) of $\mathbb{C}\otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}$ and $f_\pm := \frac{1}{2}(1+\mathbf{e}_3)$ be PIs of $\mathcal{C}\ell_{3,0}$. Since the Dirac spinor ψ is an element of the ideal $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f \simeq \mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{3,0}f_+ \oplus \mathcal{C}\ell_{3,0}f_-$, ψ indeed consists, as well-known, of the direct sum of two Weyl spinors[7].

Given a paravector $x = x^0 + x^A E_A \in \mathbb{R}\oplus\mathbb{R}^{4,1} \hookrightarrow \mathcal{C}\ell_{4,1}$ define $\chi = xE_4 \in \bigoplus_{k=0}^2 \Lambda^k(\mathbb{R}^{4,1})$.

Now we define the twistor as an algebraic spinor $\chi\frac{1}{2}(1-i\gamma_5)Uf \in (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f \simeq \mathcal{C}\ell_{3,0}$, where U is a Clifford multivector and so Uf is a Dirac spinor. The term[8] $\Pi := \frac{1}{2}(1-i\gamma_5)Uf = \binom{0}{\xi} \in \frac{1}{2}(1-i\gamma_5)(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})$ is a DCWS. If we take again a basis $\{E_A\}$ of $\mathcal{C}\ell_{4,1}$ and a basis $\{\gamma_\mu\}$ of $\mathcal{C}\ell_{1,3}$, the isomorphism $\mathcal{C}\ell_{4,1} \simeq \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ explicitly given by $E_0 = i\gamma_0$, $E_1 = \gamma_{10}$, $E_2 = \gamma_{20}$, $E_3 = \gamma_{30}$ and $E_4 = \gamma_5\gamma_0 = -\gamma_{123}$ is useful to prove the correspondence of this alternative formulation with eq.(3.2), and so, with a geometric algebra index-free version of the Penrose classical twistor formalism, by eq.(3.2). Indeed,

$$\begin{aligned}\chi\Pi &= (x^0E_4 + \alpha^0E_0E_4 + x^1E_1E_4 + x^2E_2E_4 + x^3E_3E_4 + \alpha^4)\Pi \\ &= x^0(-i\gamma_0\Pi) + x^k(\gamma_k\gamma_0)(-i\gamma_0\Pi) + \alpha^0(i\gamma_0)(-i\gamma_0\Pi) + \alpha^4\Pi \\ &= (1+\gamma_5\mathbf{x})\Pi = \begin{pmatrix} i\vec{x}\xi \\ \xi \end{pmatrix}.\end{aligned} \tag{3.5}$$

The incidence relation determines a spacetime manifold point if we take $J_{\bar{\chi}\chi} := \overline{xE_4U}xE_4U = -\overline{U}E_4\bar{x}xE_4U = 0$, since the paravector $x\in\mathbb{R}\oplus\mathbb{R}^{4,1}$ is in the Klein absolute ($x\bar{x}=0$).

3.2 Flagpoles and twistors from pure pinors and spinors

A generalized flagpole is given by the 2-form $G = \frac{1}{2}(i\mathfrak{u}\tilde{\mathfrak{u}} - i\mathfrak{u}_C\tilde{\mathfrak{u}}_C)$ [24], where $\mathfrak{u}_C$ is the charge conjugation of the pure spinor $\mathfrak{u}$. Given a real vector $p = < i\mathfrak{u}\mathfrak{u}_C >_1$, corresponding (modulo a real scalar) to a family of coplanar vectors determining the generalized flagpole, let ω be an element of a maximal totally isotropic subspace of V such that $\omega\mathfrak{u}^C = \mathfrak{u}$, $\omega\mathfrak{u} = 0$ and $\{\omega,\omega^*\} = 0$. It can be shown that $G = \exp(i\theta)p\omega + \exp(-i\theta)p\omega^*$ and $F := G\big|_{\theta=0} = p(\omega+\omega^*) = \mathrm{Re}\,(i\mathfrak{u}\tilde{\mathfrak{u}})$ is the Penrose flagpole [14, 24].

[7]The four types (dotted covariant, undotted covariant, dotted contravariant and undotted contravariant) of algebraic Weyl spinors are indeed elements of the respective minimal lateral ideals $\mathcal{C}\ell_{3,0}f_-$, $f_+\mathcal{C}\ell_{3,0}$, $f_-\mathcal{C}\ell_{3,0}$ and $\mathcal{C}\ell_{3,0}f_+$ of the Pauli algebra $\mathcal{C}\ell_{3,0}$ [21, 22, 23].

[8]In order to get a clear correspondence between our formalism and the Keller index-free formulation of twistors, by abuse of notation we adopt the same symbols to describe the DCWS.

Now, from the well-known correspondence between pure **pinors** and the group O(2n)/U(n) [20], it is possible to adapt the proof of this correspondence, in order to establish the natural correspondence between pure **spinors**, twistors and the group SO(2n)/U(n).

By definition, a spinor $\mathfrak{u}$ is said to be *pure* [3, 4] if the set $\Xi_{\mathfrak{u}} := \{\alpha \in \mathbb{C}^{2n} : \alpha(\mathfrak{u}) = 0\}$ has complex dimension n. Besides, the natural map from a pure spinor $\mathfrak{u}$ to $\Xi_{\mathfrak{u}}$ induces an equivariant isomorphism from the algebra of pure spinors (mod $\mathbb{C}^*$) to the set $\Xi_{\mathbb{C}}$ of all n dimensional totally null subspaces of $\mathbb{C}^{2n}$. Now the well-known result proved in [20], asserting that $\Xi_{\mathbb{C}} \simeq \mathrm{O}(2n)/\mathrm{U}(n)$, permits to link the pure spinors formulation to twistors. Indeed, the product of pure spinors is directly related to n-dimensional complex planes [6], which are invariant (mod U(1)) under U(n) actions. Thus it is possible, at least in even dimensions, to identify (via projective pure spinors) a twistor with an element of the group SO(2n)/U(n). In particular, twistors in four and six dimensions are respectively elements of $\mathrm{SO}(4)/\mathrm{U}(2) \simeq \mathbb{CP}^1$ and $\mathrm{SO}(6)/\mathrm{U}(3) \simeq \mathbb{CP}^3$. The investigation about an analogous mathematical structure and the physical implications of identifying twistors with elements of SO(2n)/U(n) is presented in [6].

4. Twistors and exceptional structures

It is well-known that it is possible, at least in three, four, six and ten dimensions, to construct a null vector from spinors. In string twistor formulations some manifolds can be identified with the set of all spinors corresponding to the same null vector, where in a particular case the homogeneous space SO(9)/G_2 arises [25]. Twistors are also an useful tool for the investigation of harmonic maps, as from the Calabi-Penrose twistor fibration $\mathbb{CP}^3 \to S^4$ [26]. The deep relation between twistors and exceptional structures is illustrated in the classification of compact homogeneous quaternionic-Kähler manifolds, the so-called *Wolf spaces* [27, 28]. The Wolf spaces associated with exceptional Lie algebras are E_6/SU(6)×Sp(1), E_7/Spin(12)×Sp(1), E_8/E_7×Sp(1), F_4/Sp(3)×Sp(1) and G_2/SO(4). More comments concerning such structures are beyond the scope of the present paper (see [27, 28, 29]).

5. Concluding remarks

We presented twistors in Minkowski spacetime as algebraic spinors associated to $\mathbb{C}\otimes C\ell_{1,3}$, using the paravector model, which was also used to describe all the conformal maps as the action of twisted adjoint representations on paravectors of the Clifford algebra over $\mathbb{R}^{4,1}$. The identification of the twistor identified with SO(2n)/U(n) is obtained, from the complex structure based on pure spinors formalism. As particular cases, twistors in four dimensions are elements of SO(4) modulo the double covering of electroweak group SU(2)×U(1), and in six dimensions twistors are elements of SO(6) modulo the double covering of the group SU(3)×U(1).

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