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# Fixing Ambiguities in $QED_2$ Considered in Non-Trivial Topology Sectors

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We study bosonization ambiguities in two-dimensional quantum electrodynamics in the presence of topologically charged gauge fields. The computation of fermionic correlation functions suggests that ambiguities may be absent in non-trivial topologies, provided that we do not allow changes of sector as we evaluate functional integrals. This removes an infinite arbitrariness from the theory.

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## 1. Introduction

Two-dimensional field theories are being considered, in many different aspects, since the pioneering works of Thirring [1], Schwinger [2] and others [3][4][5]. They appeared as rich laboratories for testing mechanisms not yet well understood in gauge field theories, such as screening and confinement. Recently, in two dimensional Quantum Electrodynamics ($QED_2$), it has been possible to study the perturbative divergence structure [6][7] and to perform exact renormalization [8], thus revealing distinct regimes, based on different ranges of values of the Jackiw-Johnson-Rajaraman parameter $a$ [9][10]. This parameter appears as a consequence of bosonization ambiguities, and apparently controls the quantum behavior of the model.

All these investigations have been performed in the so called "trivial topology sector", where the gauge field obeys trivial boundary conditions. A previous work [11], involving chiral fermions, showed that, when considering non-trivial topology sectors, an infinite set of parameters appears, seriously questioning the predictive power of the theory. In that case, the way of removing this arbitrariness from the theory (by requiring that transformations on the fields should not change the topological sector) determined that there were no contributions of non-trivial topology sectors to the correlation functions. This confirmed what was already known by other techniques [12]. However, another question remained: what happens in the case where these sectors effectively contribute to correlation functions, what is known to be the case when we consider Dirac fermions? The purpose of this paper is to investigate this question.

This paper is organized as follows: in section 2 we briefly review $QED_2$ in the general case where the gauge field can be given a topological charge; in section 3 we compute the contributions of these non-trivial sectors to correlation functions with general $a$ and give an argument to fix its value in all these sectors; and, in section 4, we present our final remarks.

## 2. The model

We will study quantum electrodynamics in two-dimensional euclidean space described by the action functional

$$S = \int d^2x \mathcal{L}(A_\mu, \overline{\psi}, \psi) = \int d^2x \left[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi} D(A_\mu)\psi\right], \tag{2.1}$$

where $D(A_\mu)$ is the Dirac operator

$$D(A_\mu) \equiv \gamma^\mu (i\partial_\mu + eA_\mu). \tag{2.2}$$

The generating functional of correlation functions is given by

$$Z[J^\mu, \overline{\eta}, \eta] = \int [dA_\mu][d\overline{\psi}][d\psi] \exp\left(-S + \langle J^\mu A_\mu\rangle + \langle\overline{\eta}\psi\rangle + \langle\overline{\psi}\eta\rangle\right), \tag{2.3}$$

where $J^\mu, \overline{\eta}$ and $\eta$ are the external sources associated with the fields $A_\mu, \psi$ and $\overline{\psi}$, respectively. In order to define the functional measure in (2.3), we write the fermionic fields as linear combinations

$$\psi(x) = \sum_n a_n \varphi_n(x) + \sum_i a_{0i}\varphi_{0i}(x), \tag{2.4}$$

$$\overline{\psi}(x) = \sum_n \overline{a}_n \varphi_n^\dagger(x) + \sum_i \overline{a}_{0i}\varphi_{0i}^\dagger(x), \tag{2.5}$$

of the eigenfunctions of $D$

$$D(A_\mu)\varphi_n(x) = \lambda_n\varphi_n(x), \tag{2.6}$$
$$D(A_\mu)\varphi_{0i}(x) = 0, \tag{2.7}$$

with $a_n$, $\overline{a}_n$ and $a_{0i}$, $\overline{a}_{0i}$ being grassmanian coeficients. Now, the fermionic functional measure is simply

$$[d\overline{\psi}][d\psi] = \prod_n d\overline{a}_n da_n \prod_i d\overline{a}_{0i} da_{0i},$$

such that, after an integration over Fermi fields, the fermionic part of the generating functional can be written as

$$Z_F[\overline{\eta},\eta] \propto \det{}' D.$$

In the above expression, $\det' D$ stands for the product of all nonvanishing eigenvalues of $D$.

As it is well known [12], the appearance of $N$ zero eigenvalues associated to the Dirac operator is closely related to the existence of classical configurations, in the gauge field sector, which can be written as [13][14]

$$eA_\mu^{(N)} = -\tilde{\partial}_\mu f,$$

where $\tilde{\partial}_\mu \equiv \varepsilon_{\mu\nu}\partial_\nu$ and the function $f(x)$ behaves, at infinity, as

$$\lim_{|x|\to\infty} f(x) \simeq -N\ln|x|.$$

These configurations carry topological charge $Q = N$, where $Q$ is given by

$$Q = \frac{1}{4\pi}\int d^2x\varepsilon_{\mu\nu}F_{\mu\nu}.$$

In the presence of such configurations, we can split the full quantum gauge field into a fixed background plus fluctuations:

$$A_\mu = A_\mu^{(N)} + a_\mu,$$

where $a_\mu$ is associated to $Q = 0$. In two dimensions we can always write any configuration of charge $N$ linearly in terms of a fixed one, due to the linear character of $Q$. The field $a_\mu$ can always be written as

$$ea_\mu = \partial_\mu\rho - \tilde{\partial}_\mu\phi. \tag{2.8}$$

Now we can bosonize the theory in this sector, performing the change of variables

$$\psi \to \exp(-i\rho + \phi\gamma_5)\psi,$$
$$\overline{\psi} \to \overline{\psi}\exp(i\rho + \phi\gamma_5),$$

where $\rho$ and $\phi$ were already defined in (2.8). After a long and detailed calculation, taking into account the Fujikawa jacobian, the computation of several functional determinants and an orthonormalization procedure for the zero modes of the Dirac operator associated to the fixed configuration $A_\mu^{(N)}$, [16][11] we end up with

$$Z[J^\mu,\overline{\eta},\eta] = \sum_N \int [da_\mu]\exp\left(-\overline{S} + \left\langle J^\mu A_\mu\right\rangle + \left\langle \overline{\eta}' G^{(N)}\eta'\right\rangle\right)\prod_{i=1}^{|N|}\left\langle \overline{\eta}'\Phi_{0i}^{(N)}\right\rangle\left\langle \Phi_{0i}^{(N)^\dagger}\eta'\right\rangle, \tag{2.9}$$

where

$$\overline{S} = \left\langle \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\rangle + \Gamma[a_\mu] + \overline{\Gamma}\left[a_\mu, A_\mu^{(N)}\right] + \Gamma'\left[A_\mu^{(N)}\right],$$

with

$$\begin{aligned} \Gamma[a_\mu] &= \frac{e^2}{4\pi}\int d^2x a_\mu \left(a(N)\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) a_\nu, \\ \overline{\Gamma}\left[A_\mu^{(N)}, a_\mu\right] &= \frac{e^2}{2\pi}\int d^2x a_\mu \left(a(N)\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) A_\nu^{(N)}, \\ \Gamma'\left[A_\mu^{(N)}\right] &= \frac{e^2 a(N)}{4\pi}\int d^2x f\Box f. \end{aligned} \tag{2.10}$$

In the above formulas, we notice the appearance of a parameter $a(N)$, as a consequence of bosonization ambiguities, that can be chosen independently in each topological sector. Other ingredients that also appear in (2.9) are

$$\begin{aligned} \eta' &= \exp(i\rho + \phi\gamma_5)\eta, \\ \overline{\eta}' &= \overline{\eta}\exp(-i\rho + \phi\gamma_5), \end{aligned} \tag{2.11}$$

and

$$G^{(N)}(x,y) = \{\exp(f(x) - f(y))\mathbf{P}_+ + \exp(-(f(x) - f(y)))\mathbf{P}_-\} G_F(x,y),$$

$G_F(x,y)$ being the free Dirac fermion propagator. Finally, there is the non-orthonormal set of zero modes of $D\left(A_\mu^{(N)}\right)$, which we called $\Phi_{0i}^{(N)}$, written in terms of the light-cone variables $z = x^0 + ix^1$ and $\overline{z} = x^0 - ix^1$, and given by [13][14]

$$\Phi_{0i}^{(N)} = \begin{cases} z^{i-1}\exp f \binom{1}{0}, & i = 1, \ldots, N, N > 0 \\ \overline{z}^{i-1}\exp(-f)\binom{0}{1}, & i = 1, \ldots, -N, N < 0. \end{cases} \tag{2.12}$$

## 3. Non-trivial contributions to correlation functions

Being directly proportional to fermionic sources, it is not difficult to see that there are no contributions to bosonic correlation functions and that bosonic-fermionic ones do not give different information (concerning the ambiguities) than that given by the fermionic functions alone. We have non vanishing contributions from non-trivial topologies to fermionic correlation functions of the following kind

$$\left\langle \prod_{i=1}^{k} \psi_{\alpha_i}(x_i) \prod_{j=1}^{k} \overline{\psi}_{\beta_j}(y_j) \right\rangle = \frac{1}{Z[0]} \frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)} \cdots \frac{\delta}{\delta\eta_{\beta_k}(y_k)} Z[0,\overline{\eta},\eta]\big|_{\overline{\eta}=\eta=0}.$$

It can be easily shown, by induction, that

$$\left\langle \prod_{i=1}^{k} \psi_{\alpha_i}(x_i) \prod_{j=1}^{k} \overline{\psi}_{\beta_j}(y_j) \right\rangle = \sum_{k=-N}^{N} \int [da_\mu] \exp\left(-\overline{S}\right) \det \begin{vmatrix} \Phi'^{(N)^\dagger} & \emptyset \\ \mathbf{G}'^{(N)} & \Phi'^{(N)} \end{vmatrix}, \tag{3.1}$$

where $\Phi'^{(N)^\dagger}$ is a $N\times k$ matrix given by

$$\Phi'^{(N)^\dagger} = \begin{pmatrix} \Phi'^{(N)^\dagger}_{01}(y_1)\cdots\Phi'^{(N)^\dagger}_{01}(y_k) \\ \vdots \\ \Phi'^{(N)^\dagger}_{0N}(y_1)\cdots\Phi'^{(N)^\dagger}_{0N}(y_k) \end{pmatrix},$$

$\Phi'^{(N)}$ is $k\times N$,

$$\Phi'^{(N)} = \begin{pmatrix} \Phi'^{(N)}_{01}(x_1)\cdots\Phi'^{(N)}_{0N}(x_1) \\ \vdots \\ \Phi'^{(N)}_{01}(x_k)\cdots\Phi'^{(N)}_{0N}(x_k) \end{pmatrix},$$

and

$$\mathbf{G}'^{(N)} = \begin{pmatrix} G'^{(N)}(x_1,y_1)\cdots G'^{(N)}(x_1,y_k) \\ \vdots \\ G'^{(N)}(x_k,y_1)\cdots G'^{(N)}(x_k,y_k) \end{pmatrix}$$

and $\emptyset$ (the null matrix) are square matrices $k\times k$ and $N\times N$ respectively, and

$$\begin{aligned} G'^{(N)}(x_i,y_j) &= \exp(-i\rho+\phi\gamma_5)\,G^{(N)}(x_i,y_j)\,(i\rho+\phi\gamma_5), \\ \Phi'^{(N)}_{0i}(x_j) &= \exp(-i\rho+\phi\gamma_5)\,\Phi^{(N)}_{0i}(x_j), \\ \Phi'^{(N)^\dagger}_{0i}(y_j) &= \Phi^{(N)^\dagger}_{0i}(y_j)\exp(i\rho+\phi\gamma_5). \end{aligned}$$

If we define

$$\chi_{ij} = \begin{cases} (z_j)^{i-1}\binom{1}{0}, & N>0 \\ (\bar{z}_j)^{i-1}\binom{0}{1}, & N<0 \end{cases}$$

and use the expressions for the zero modes (2.12) we can show that

$$\begin{aligned} \det\left(\Phi'^{(N)}\right) &= \exp\left(\sum_{i=1}^{|N|} i\rho(y_i)\right)\exp\left(\pm\sum_{i=1}^{|N|} f(y_i)+\phi(y_i)\right)\det(\chi), \\ \det\left(\Phi'^{(N)^\dagger}\right) &= \exp\left(-\sum_{i=1}^{|N|} i\rho(x_i)\right)\exp\left(\pm\sum_{i=1}^{|N|} f(x_i)+\phi(x_i)\right)\det\left(\chi^\dagger\right) \end{aligned}$$

and

$$\begin{aligned} \det\left(\mathbf{G}'^{(\mathbf{N})}\right) &= \exp\left(\sum_{i=|N|+1}^{k} i\left(\rho(y_i)-\rho(x_i)\right)\right)\times \\ &\quad \exp\left(\pm\sum_{i=|N|+1}^{k} f(y_i)-f(x_i)+\phi(y_i)-\phi(x_i)\right)\det(\mathbf{G_F}), \end{aligned}$$

where, according to the positiveness or not of $N$,

$$\begin{aligned} \det(\chi) &= \prod_{\substack{i,j=1 \\ i>j}}^{|N|}\left|z_i-z_j\right|\otimes\begin{pmatrix}1\\0\end{pmatrix}, \qquad N>0 \\ \det\left(\chi^\dagger\right) &= \prod_{\substack{i,j=1 \\ i>j}}^{|N|}\left|\bar{z}_i-\bar{z}_j\right|\otimes\begin{pmatrix}1 & 0\end{pmatrix}, \qquad N>0 \end{aligned}$$

or

$$\det(\chi) = \prod_{\substack{i,j=1\\i>j}}^{|N|} |\bar{z}_i - \bar{z}_j| \otimes \begin{pmatrix}0\\1\end{pmatrix}, \qquad N<0$$
$$\det(\chi^\dagger) = \prod_{\substack{i,j=1\\i>j}}^{|N|} |z_i - z_j| \otimes \begin{pmatrix}0 & 1\end{pmatrix}, \qquad N<0.$$

Collecting these results, we arrive at

$$\left\langle \prod_{i=1}^{k} \psi_{\alpha_i}(x_i) \prod_{j=1}^{k} \overline{\psi}_{\beta_j}(y_j) \right\rangle = \sum_{k=-N}^{N} \int [da_\mu] \exp\left(-\overline{S}_{\text{sources}}\right) \det \begin{vmatrix} \chi^\dagger & \emptyset \\ \mathbf{G}_F & \chi \end{vmatrix}, \tag{3.2}$$

where

$$\overline{S}_{\text{sources}} = \overline{S} - \left\langle i\left(j_\rho + j'_\rho\right)\rho\right\rangle \mp \left\langle (j+j')(f+\phi)\right\rangle,$$

the signs $\mp$ refer to $N>0$ and $N<0$ respectively, and $j_\rho$, $j'_\rho$, $j$ and $j'$ are defined by

$$j_\rho = \sum_{i=1}^{|N|} \delta(y_i - z) - \delta(x_i - z),$$
$$j'_\rho = \sum_{i=|N|+1}^{k} \delta(y_i - z) - \delta(x_i - z),$$
$$j = \sum_{i=1}^{|N|} \delta(y_i - z) + \delta(x_i - z),$$
$$j' = \sum_{i=|N|+1}^{k} \delta(y_i - z) - \delta(x_i - z),$$

with $\langle\ \rangle$ representing integration over $z$.

There is still a last integration over the scalar fields $\rho$ and $\phi$ in terms of which the gauge field is written. So we write the effective action $\overline{S}$ in terms of these fields

$$\overline{S} = \frac{1}{2e^2}\left\langle (\phi+f)\Box\left(\Box - \frac{e^2 a(N)}{\pi}\right)(\phi+f)\right\rangle - \frac{(a(N)-1)}{2\pi}\langle \rho\Box\rho\rangle,$$

and do the following change of variables using the sources $j_\rho$, $j'_\rho$, $j$ and $j'$:

$$\sigma = \rho + \frac{i}{\lambda}\left\langle \Delta_F\left(j_\rho + j'_\rho\right)\right\rangle \tag{3.3}$$

and

$$\varphi = \phi + f \mp e^2\left\langle \Delta(m; x-y)(j+j')\right\rangle \tag{3.4}$$

where

$$\Delta_F(x-y) \equiv \Box^{-1}(x-y) = \frac{1}{2\pi}\ln m|x-y|$$

and

$$\Delta(m; x-y) \equiv \left[\Box\left(\Box - m^2\right)\right]^{-1}(x-y) = -\frac{1}{2\pi m^2}\left\{K_0[m|x-y|] + \ln m|x-y|\right\}$$

($K_0$ is the zeroth-order modified Bessel function) and we have defined $\lambda \equiv (a(N)-1)/\pi$ and $m^2 = \left(e^2 a(N)\right)/\pi$. Now we have for $\overline{S}$ plus the sources the expression

$$\begin{aligned}\overline{S}_{\text{sources}} &= \frac{1}{2e^2}\left\langle \varphi\Box\left(\Box - m^2\right)\varphi\right\rangle - \frac{e^2}{2}\left\langle \left(j+j'\right)\Delta(m)\left(j+j'\right)\right\rangle - \\ &\quad \frac{\lambda}{2}\left\langle \sigma\Box\sigma\right\rangle - \frac{1}{2\lambda}\left\langle \left(j_\rho + j'_\rho\right)\Delta_F\left(j_\rho + j'_\rho\right)\right\rangle .\end{aligned}$$

As we have already said, the scalar fields $\rho$ and $\phi$ are such that $a_\mu$ does not carry a topological charge in the limit $|x| \to \infty$. So it is desirable that the new fields $\sigma$ and $\varphi$ behave like the old ones, going to zero at infinity. If this would not be the case, it would be equivalent to perform transformations that change the topological sector, which would lead us to compute jacobians over noncompact spaces, what is very difficult to obtain [19][20]. So, altough keeping in mind the general case, we will restrict ourselves to transformations which do not change the topological sector.

In the case of the $\sigma$ field we have

$$\begin{aligned}\lim_{|x|\to\infty}\sigma(x) &= \lim_{|x|\to\infty}\rho(x) + \frac{i}{\lambda}\lim_{|x|\to\infty}\left\langle \Delta_F(x-z)\left(j_\rho + j'_\rho\right)\right\rangle \\ &= \frac{i}{2\pi\lambda}\lim_{|x|\to\infty}\left\{\sum_{i=1}^{k}\left(\ln m|x-x_i| - \ln m|x-y_i|\right)\right\} \\ &= 0,\end{aligned}$$

given that $\lim_{|x|\to\infty}\rho(x) = 0$, in agreement with the conditions imposed.

For the field $\varphi$, we have

$$\begin{aligned}\lim_{|x|\to\infty}\varphi(x) &= \lim_{|x|\to\infty} f(x) + \lim_{|x|\to\infty}\phi(x) \mp \lim_{|x|\to\infty} e^2\left\langle \Delta(m;x-z)\left(j+j'\right)\right\rangle \\ &= -N\ln|x| \pm \frac{e^2}{2\pi m^2}\lim_{|x|\to\infty}\left\langle \left(K_0\left[m|x-z|\right] + \ln m|x-z|\right)\left(j+j'\right)\right\rangle \\ &= -N\ln|x| \pm \frac{1}{2a(N)} 2|N|\ln|x| \\ &= -\left(N \mp \frac{|N|}{a(N)}\right)\ln|x|,\end{aligned}$$

once $K_0$ is well behaved in the limit considered and $\lim_{|x|\to\infty}\phi(x) = 0$. Here, the $\mp$ sign corresponds to sectors with topological charge $N$ and $-N$, respectivelly. The assymptotic behavior of $\varphi$ is then

$$\lim_{|x|\to\infty}\varphi(x) = \begin{cases} -\left(N - \frac{N}{a(N)}\right)\ln|x|, & N>0, \\ -\left(N - \frac{N}{a(N)}\right)\ln|x|, & N<0. \end{cases}$$

which is singular unless we have

$$a(N) = 1, \qquad \forall \quad N \neq 0.$$

## 4. Final remarks

As we have seen, there is, in principle, an infinite amount of ambiguity in the theory, due to arbitrary choices of $a(N)$ for each $N$. A simple criterium to choose $a(N)$ seems to be that of not allowing changes of topological sector when we change integration variables. It gives a value for $a(N)$ which coincides with the one obtained through the requirement of gauge invariance. The connection between gauge invariance and preservation of topology is not completely clear, as it sugests the existence of some relation between short and long distance properties of the theory, and perhaps can only be clarified if one could compute the correlation functions without these criteria. It is our aim to explore also this direction in the near future.

## References

[1] W. Thirring, *A soluble relativistic field theory?, Ann. of Phys.* **3** (1958) 91.

[2] J. Schwinger, *Gauge invariance and mass, Phys. Rev.* **125** (1962) 397; J. Schwinger, *Gauge invariance and mass II, Phys. Rev.* **128** (1962) 2425.

[3] S. Coleman, *The quantum sine-Gordon equation as the massive Thirring model, Phys. Rev.* **D11** (1975) 2088.

[4] S. Mandelstam, *Soliton operators for the quantized sine-Gordon equation, Phys. Rev.* **D11** (1975) 3026.

[5] J. Lowenstein and J. Swieca, *Quantum electrodynamics in two-dimensions, Ann. Phys.* (N.Y.) **68** (1971) 172.

[6] R. Casana and S. A. Dias, *Renormalization of two-dimensional quantum electrodynamics, Int. J. Mod. Phys.* **A15** (2000) 4603.

[7] R. Casana and S. A. Dias, *On the origin of divergences in massless QED$_2$, J. Phys. G: Nucl. Part. Phys.* **27** (2001) 1501.

[8] R. Casana and S. A. Dias, *Exact renormalization of massless QED$_2$, Int. J. Mod. Phys.* **A17** (2002) 4601.

[9] R. Jackiw and K. Johnson, *Anomalies of the axial vector current, Phys. Rev.* **182** (1969) 1459.

[10] R. Jackiw and R. Rajaraman, *Vector meson mass generation through chiral anomalies, Phys. Rev. Lett.* **54** (1985) 1219; R. Jackiw and R. Rajaraman, *Erratum on "Vector Meson Mass Generation Through Chiral Anomalies", Phys. Rev. Lett.* **54** (1985) 2060;

[11] S. A. Dias and C. A. Linhares, *Non-trivial topology and the chiral Schwinger model, Phys. Rev.* **D47** (1993) 1672.

[12] E. Abdalla, M. C. B. Abdalla and K. D. Rothe, *Non-perturbative methods in two dimensional quantum field theory*, second edition, World Scientific, Singapore 2001.

[13] K. Bardakci and M. Crescimano, *Monopole backgrounds in the world sheet, Nucl. Phys.* **B313** (1989) 269.

[14] M. V. Manias, C. M. Naón and M. L. Trobo, *New topological aspects of fermionic correlation functions in the Schwinger model, Phys. Rev.* **D41** (1990) 3174.

[15] K. D. Rothe and B. Schroer, *Green functions, determinants and induced actions in gauge theories*, in *Field Theoretical Methods in Particle Physics*, ed. by W. Rühl, Plenum Press, New York 1980.

[16] S. A. Dias and M. T. Thomaz, *Zeta function regularization of chiral jacobians for singular Dirac operators: finite chiral rotations*, *Phys. Rev.* **D44** (1991) 1811.

[17] R. E. Gamboa Saravi, M. A. Muschietti and J. E. Solomin, *On the regularized determinant for noninvertible elliptic operators*, *Comm. Math. Phys.* **93** (1984) 407.

[18] S. A. Dias and C. A. Linhares, *Fermion determinant for the generalized Schwinger model*, *Phys. Rev.* **D45** (1992) 2162.

[19] M. V. Manias, C. M. Naón and M. L. Trobo, *Path integral bosonization in a topological backgroumd: the Thirring model*, *Int. Jour. Mod. Phys.* **A5** (1990) 2853.

[20] O. Barraza, H. Falomir, R. E. Gamboa Saravi and E. M. Santangelo, *P determinants and boundary values*, *J. Math. Phys.* **33** (1992) 2046.