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# The Riemann Zeta Function and Vacuum Spectrum

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A variant for the Hilbert and Polya spectral interpretation of the Riemann zeta function is proposed. Instead of looking for a self-adjoint linear operator H, whose spectrum coincides with the Riemann zeta zeros, we look for the complex poles of the S matrix that are mapped into the critical line in coincidence with the nontrivial Riemann zeroes. The associated quantum system, an infinity of “virtual resonances" described by the corresponding S matrix poles, can be interpreted as the quantum vacuum. The distribution of energy levels differences associated to these resonances shows the same characteristic features of random matrix theory.

*Fourth International Winter Conference on Mathematical Methods in Physics*
*09 - 13 August 2004*
*Centro Brasileiro de Pesquisas Fisicas (CBPF/MCT), Rio de Janeiro, Brazil*

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Published by SISSA http://pos.sissa.it/

PoS(WC2004)026

## 1. Introduction

There is a conjecture, quoted to be made by Hilbert and Polya, that the zeros of the Riemann zeta function $\zeta(z)$ on the critical line are eigenvalues of a “mysterious" complex Hermitian operator H. We report in this presentation a variant of the above conjecture where we associate the unknown quantum system to the “vacuum". This vacuum is interpreted as an infinity of “virtual resonances", described by complex poles of the scattering S matrix.

Actually the Riemann zeta zeros fluctuations are interesting by their universality, being observed in quantal spectra of different physical systems (see Refs.[1]), and by the connection they have with chaotic dynamics, (for a review see Bohigas [2]). Ever since Montgomery's [3] conjectured these zeros behave like the eigenvalues of a random hermitian matrix, attempts have been made in order to find a quantum system with the Hamiltonian represented by such an operator (see Berry and Keating [4], and references therein). Our goal here is to present a new conjecture in order to relate the origin of this universality to the statistical fluctuation of the vacuum that provides the same bakground for the different physics systems in interaction with it.

We will begin by a short outline of the concepts of the Riemann zeta function, prime numbers and the Jost function. After, motivated by recent work [5] in which was shown numerically a one to one connection between the large zeros of Jost function in the complex momenta plane with large prime numbers and large complex Riemann zeta zeros, we propose a distribution for these Jost zeros representing the quantum vacuum. Finally, rather than look for a potential that put these Jost zeros on the desired place, we show that the corresponding resonances energies has the same statistical fluctuation given by the random matrix theory (RMT). We conclude suggesting an approximate model for the above quantum vacuum.

## 2. Prime numbers, Riemann's zeta function and Jost function

A comparison between prime numbers, Riemann's zeta function and Jost function has been given in a recent paper [5]. In this section we summarize it in order to introduce the proposed map (2.8) between complex zeros of Jost function with the complex zeros of the zeta function. For a recent review concerning the relevance of prime numbers to Physics see Rosu [6].

The approximate number of primes $\pi(x)$ less than a given $x$, also called the prime counting function, is given by the prime number theorem $\pi(x) \sim x/\log x$ (see Titchmarsh [7], Chapter- III). This relation gives the asymptotic approximation for $n$th prime $p_n$,

$$p_n = n \log\, n \quad , \qquad \text{as } n \to \infty \, . \tag{2.1}$$

The connection between the distribution of prime numbers $\pi(x)$ and the complex zeros of the zeta function started with Riemann's 1859 paper (see Edwards [8], p. 299). Riemann's zeta function is defined (Ref.[7], p.1), either by the Dirichlet series or by the Euler product

$$\zeta(z) = \sum_n n^{-z} = \prod_p (1 - p^{-z})^{-1} \, , \qquad Re\, z > 1 \, , \tag{2.2}$$

where n runs through all integers and p runs over all primes. This function can be analytically continued to the whole complex plane, except at $z = 1$ where it has a simple pole with residue 1.

It satisfies the functional equation $\zeta(z) = 2^z\pi^{z-1}\sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$ , called the reflection formula, where $\Gamma(z)$ is the Euler gamma function. It is known that $\zeta(z)$ has simple zeros at points $z = -2n,\ n = 1,2,\ldots$, which are called trivial zeros, and an infinity of complex zeros lying in the strip $0 < Re\, z < 1$. Due to the reflection formula they are symmetrically situated with respect to the axis $Re\, z = 1/2$, and since $\zeta(z^*) = \zeta^*(z)$, they are also symmetric about the real axis. Consequently it suffices to consider the zeros in the upper half of strip $1/2 \leq Re\, z < 1$. It is possible to enumerate these complex zeros as $z_n = b_n + i\, t_n$ , with $t_1 \leq t_2 \leq t_3 \leq \ldots$, and the following result can be proven (see Titchmarsh [7] , p.214)

$$|z_n| \sim t_n \sim \frac{2\pi n}{\log n}, \qquad \text{as } n \to \infty. \tag{2.3}$$

The Riemann Hypothesis (RH) is the conjecture, not yet proven, that all complex zeros of $\zeta(z)$ lie on the axis $Re\, z = 1/2$, called the "critical line". This is usually considered as the most important unsolved problem in Mathematics. Based on this conjecture, Riemann derived an exact formula for $\pi(x)$ taking into account local prime fluctuations in terms of nontrivial zeta zeros (Ref.[8], p.299).

On the other hand, the Jost function has played a central role in the development of the analytic properties of the scattering amplitudes. In order to recall its properties, let us consider the scattering of a non-relativistic particle, without spin, of mass $m$ by a spherically symmetric local potential, $V(r)$, everywhere finite, behaving at infinity as $V(r) = O(r^{-1-\varepsilon}),\quad \varepsilon > 0$. The Jost functions $f_\pm(k)$ are defined (see Newton [9], p.341) as the Wronskian W, $f_\pm(k) = W[f_\pm(k,r),\ \varphi(k,r)]$, where $\varphi(k,r)$ is the regular solution of the radial Schrödinger equation

$$[\frac{d^2}{dr^2} + k^2 - V(r)]\ \varphi(k,r) = 0\ , \tag{2.4}$$

(in units for which $\hbar = 2m = 1$) $k$ being the wave number and the Jost solutions, $f_\pm(k,r)$, are two linearly independent solutions of eq.(2.4). They satisfy the boundary conditions, $\lim_{r\to\infty}[e^{\mp ikr} f_\pm(k,r)] = 1$, corresponding to incoming and outgoing waves of unit amplitude.

The properties of the solution of the differential equation (2.4) define the domain of analyticity of the Jost functions $f_\pm(k)$ on the complex $k$-plane as well as its symmetry properties, such as $f_+^*(k^*) = f_-(k)$. The phase of the Jost function is just minus the scattering phase shift $\delta(k)$, that is $f_\pm(k) = |f_\pm(k)|\ \ e^{\mp i\delta(k)}$ , so that the usual S matrix is given by

$$S(k) \equiv e^{2i\delta(k)} = \frac{f_-(k)}{f_+(k)}\ . \tag{2.5}$$

The complex poles ($Re\ k \neq 0$) of $S(k)$, or zeros of $f_+(k)$, correspond to the solutions of the Schrödinger equation with purely outgoing, or incoming, wave boundary conditions. Resonances show up as complex poles with negative imaginary parts, their complex energies being

$$k_n^2 = \Xi_n - i\ \frac{\Gamma_n}{2}, \tag{2.6}$$

where $\Xi_n$ and $\Gamma_n$ represents the energy and the width, respectively, associated with $n$th resonance state in order of distance from origin. For small $\Gamma_n$, resonances appear as long-lived quasistationary states populated in the scattering process. If the width is sufficiently broad no resonance effect will be observed and we will call this kind of S matrix pole as "virtual resonances" throughout.

Resonances, in fact, are represented by pairs of symmetrical S matrix poles in the complex k-plane, a capture state pole in the third quadrant and the decaying state in the fourth quadrant, since they give to the asymptotic solution an incoming growing wave and an outgoing decaying wave, respectively, exponential in time [10]. This could be described by Gamow vectors treated by Bohm and Gadella [11] as pairs of S matrix poles corresponding to decay and growth states, where the traditional Hilbert space description is replaced by the generalized Rigged Hilbert Space in order to account for time asymmetry of a resonant process. The transient states which last only a very short time corresponding to the above "virtual resonances", to which our instruments are insensitive, will be called here as "vacuum". Examples of broad resonances are the well known large poles, related basically to the cutoff in the potential without any physical interpretation (see Nussensveig [12], p.178).

In truncating the potential, i.e., the potential is set equal to zero for $r \geq R > 0$, which is the cutoff of the potential at arbitrarily large distances $R$, it is possible to obtain explicit asymptotic formulas for the Jost zeros. With this restriction it can be shown that equation $f_+(k) = 0$ is entire of order $1/2$ and according to Piccard's theorem has infinitely many roots for arbitrary values of the potential. The asymptotic expansion of $k_n$, for large n, after introducing dimensionless parameter $\beta = kR$, is given by (Ref.[9], p.362)

$$\beta_n = n\pi - i\frac{(\sigma+2)\log|n|}{2} + 0(1) \tag{2.7}$$

where $n = \pm 1,\ \pm 2,\ \pm 3\cdots$ and $\sigma$ is defined by the first term of the potential asymptotic expansion, near $r = R$, through $V(r) = C(R-r)^{\sigma} + \cdots$, $\sigma \geq 0$ and $r \leq R$.

The connection between the complex zeros of the Jost function and those one of the Riemann zeta function is provided by the transformation,

$$z = -i\frac{\beta^2}{2\, Im\, \beta^2}\ , \tag{2.8}$$

by which the lower half of complex $\beta$-plane ($Re\ \beta \neq 0$) is mapped onto the critical axis, $Re\, z = 1/2$, of complex $z$-plane.

Now we show that for cutoff potentials the transformation (2.8) gives rise to complex Jost zeros with the same asymptotic behavior as the complex Riemann zeta zeros, being all in the critical line. After introducing dimensionless quantities, energy $E_n = R^2\Xi_n$ and widths $G_n = R^2\Gamma_n$ , the equation (2.6) is written as $\beta_n^2 = E_n - iG_n/2$, then by (2.8) we get

$$z_n = \frac{1}{2} + i\frac{E_n}{G_n}\ , \tag{2.9}$$

from (2.7), with $\sigma = 0$, we see that $\{Im\ z_n\}$ has the same asymptotic expansion as $\{t_n\}$, given by (2.3), i.e.,

$$\frac{E_n}{G_n} = \frac{t_n}{8} \quad \text{as} \quad n \to \infty\ , \tag{2.10}$$

which means that for each resonance, the ratio between the energy and the width is given by the height of the zeta zero on the critical line. On the other hand, dimentionless widths $\{G_n\}$, defined in (2.9) as

$$G_n = 4\ Re\ \beta_n\ \ Im\ \beta_n\ , \tag{2.11}$$

after taking into account (2.7), when $\sigma = 0$, shows the same asymptotic expansion for large primes (2.1), given by the prime number theorem,

$$G_n = 4\pi p_n \quad \text{as} \quad n \to \infty. \tag{2.12}$$

Then $n$th large complex Jost zeros are also related to $n$th large primes, showing an asymptotic connection between primes and complex Riemann zeta zeros, in a one-to-one correspondence.

## 3. Hilbert-Polya conjecture

The Hilbert-Polya conjecture is the spectral interpretation of the complex zeros of the Riemann zeta function as eigenvalues of a self-adjoint linear operator H in some Hilbert space. Such H could prove the RH. We suggest a variant for the above conjecture: Instead of looking for H, whose spectrum have to coincide with the Riemann zeta zeros, we are looking for a potential that gives a Jost function with all zeros on the lower half of complex $\beta$-plane ($Re\ \beta \neq 0$) that coincide with the complex zeros of the Riemannn zeta function after the transformation (2.8). In this way the Riemann hypothesis follows. For real potentials, these complex $\beta$ zeros are located symmetrically about the imaginary axis, then by (2.8) they will be mapped symmetrically about the real axis into the critical line. The associated quantum system could be identified with the quantum vacuum, interpreted as an infinity of “virtual resonances", described by corresponding S matrix poles.

The firsts to associate the Riemann Hypothesis with transient states were Pavlov and Faddeev [13], by relating the nontrivial zeros of the zeta function to the complex poles of the scattering matrix of a particle on a surface of negative curvature. Khuri [14] has recently proposed a modification to the inverse scattering problem in order to obtain the potential whose coupling constant spectrum coincides with the Riemann zeta zeros.

Assuming that the RH is true, we conjecture a vacuum spectrum with widths given by $G_n = p_n$, according to (2.12), and the corresponding energies given by $E_n = G_n t_n = p_n t_n$, according to (2.10). Before trying to find a potential that will give these S matrix poles we will study numerically the statistics fluctuation of these “virtual levels".

## 4. RMT and energy distribution of virtual resonances

The random matrix theory (RMT) describes fluctuation properties of quantal spectra. In lack of analytical or numerical methods to obtain the spectra of complicated Hamiltonians, Wigner and Dyson (see Refs. [1]) analyzed ensembles of random matrices, in which the matrix elements are considered to be independent random variables, that have in common only symmetry properties like hermiticity and time-reversal invariance. The RMT predicts, for the nearest-neighbour spacings of the energy levels distribution $P(s)$, the form which is described by the Wigner surmise: $P(s) = A\ s^D\ exp(-Bs^2)$, where $s_n = (E_{n+1} - E_n)$ are spaces between adjacent levels of the unfolded spectrum with mean distance $< s_n >= 1$. It is obtained by accumulating the number of spacings that lying between $(s, s + \triangle s)$ and then normalizing $P(s)$ to unity. $A$ and $B$ are normalizing constants and $D$ is a parameter which depends on the symmetry of the system which characterizes the repulsion between neighbor levels. For hermitian time-reversal Hamiltonians

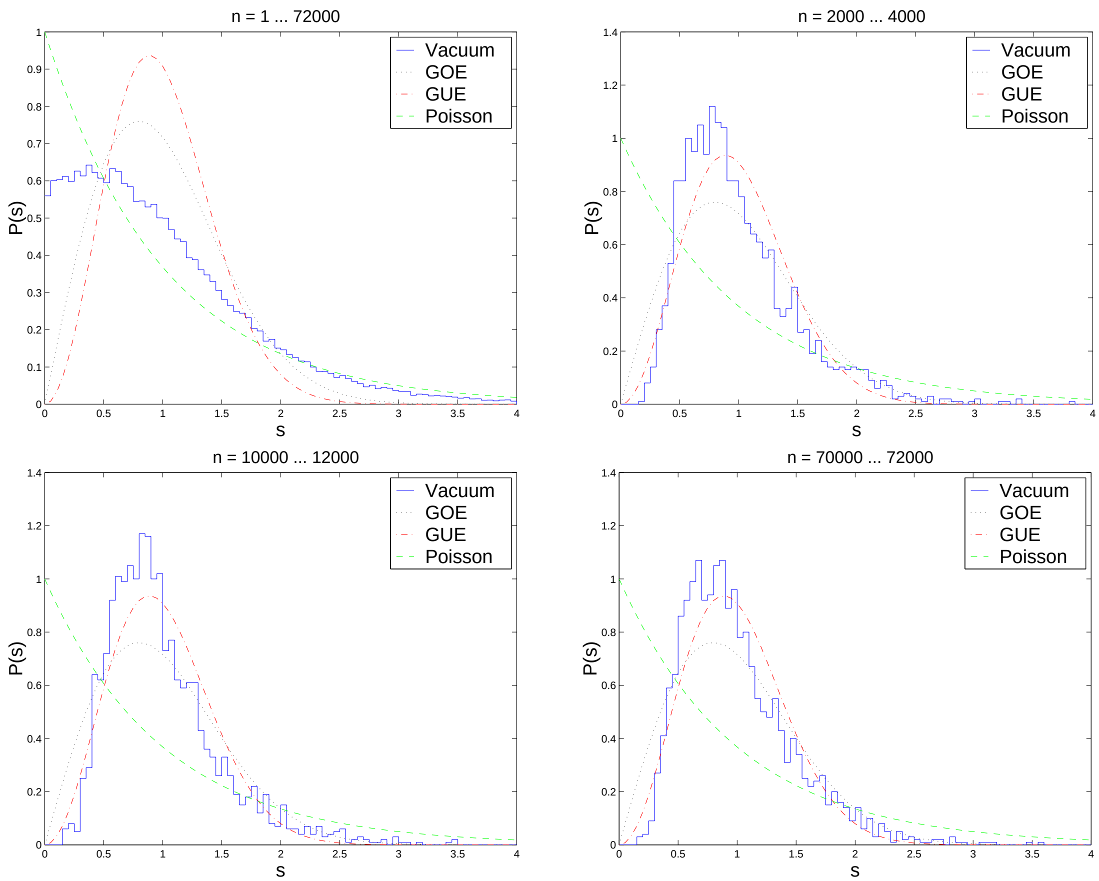


**Figure 1:** The statistical of level-spacings in vacuum spectra $\{p_n t_n\}$. The histogram at the top left gives a sequence comprising the first 72000 vacuum levels. The histogram at the top right comprises 2000 consecutive levels in the analysis corresponding from the $2000th$ to the $4000th$ level. The histograms at the bottom give 2000 consecutive levels centered at different $E$ values.

(Gaussian Orthogonal Ensemble) $D = 1$ and $P_{GOE}(s) = (\pi/2)\ s\ exp(-\pi s^2/4)$. For complex hermitian matrices, when time-reversal invariance does not hold (Gaussian Unitary Ensemble) $D = 2$ and $P_{GUE}(s) = (32/\pi^2)\ s^2\ exp(-4s^2/\pi)$. If the eigenvalues of a system are completely uncorrelated (Poisson Ensemble ) we have $P_{PE}(s) = exp(-s)$.

The nearest-neighbor spacing distribution $P(s)$ corresponding to the vacuum spectrum energy, i.e. resonances energies given by $\{p_n t_n\}$, where prime sequence $\{p_n\}$ are taken from the table [16] and $\{t_n\}$ computed by Odlyzko [17] are shown in Figure 1. The curves corresponding to Poisson distribution (dashed line), GOE distribution (dotted line) and GUE distribution (dotted-dashed line) are drawn for comparison. For a sequence comprising the first 72 000 vacuum levels we obtain, see the top left of Figure 1, a mixing between Poisson and GOE distribution. In order to compare with spectra of atomic nuclei at higher energies, in regions of high density, we need an interval of energy $\triangle E$ centered at $E$ like $E >> \triangle E >> 1$. As can be seen from the numerical results, see the top right and the bottom of Figure 1, when $E >> \triangle E$, the nearest-neighbor spacing distribution of the vacuum spectrum are consistent with a kind of mixing of GOE distribution and GUE, with the same characteristic features of random matrix, i.e. level repulsion at short distances and suppression of fluctuations at large distances. The histograms at the bottom of the Figure 1 gives 2000 consecutive levels centered at differents $E$ values in order to show the spectral rigidity, i.e. a very small fluctuation around its average.

## 5. A model for the quantum vacuum

Following [5] we present a model that approximates the above quantum vacuum. It could be represented by the non-relativistic s-wave scattering by a spherically symmetric potential barrier, $V(r) = V_0$ for $r < R$ and $V(r) = 0$ for $r \geq R$. From the stationary scattering solution one obtains the Jost function $f_+(k)$ and after introducing dimensionless parameters, $\beta = kR$ and $v = V_0 R^2$, its zeros are given by the solutions of the complex transcendental equation $\sqrt{\beta^2 - v} \cot \sqrt{\beta^2 - v} = i\beta$, for each value of potential strength $v$. The displacements of the roots, $\beta_n$, in the $\beta$-plane with the variation of the potential strength was shown many years ago by Nussensveig [15]. All these zeros are in the lower half of complex $\beta$-plane and located symmetrically about the imaginary axis. The energy/width ratios $\{4\pi E_n/G_n\}$ where calculated (see [5], Fig.1a ) from the Jost zeros $\beta_n$, for $v = 2$, in order of distance from the origin and show an approximate agreement from the beginning with $\{t_n\}$, for n up to $6 \times 10^4$. The same Jost zeros $\beta_n$, for $v = 2$, after transformation (2.11), gives the dimensionless widths $\{G_n/4\pi\}$ in agreement with the global behavior of the sequence of primes, from the beginning in the same range (see [5], Fig. 1b ). The local behavior, defined as deviations from the average density of the zeta zeros, are not obtained by this potential [5].

The qualitative agreement of the above model suggests the investigation of the many-body problem described by one particle being scattered by an effective cutoff potential where, by analogy to the mean field, a kind of residual interaction is introduced phenomenologically, in order to describe the “vacuum" fluctuation with the same statistical distribution of the vacuum energy levels $\{p_n t_n\}$. In this case, the universality of level fluctuation law of the spectra of different quantum systems (nuclei, atoms and molecules) [1, 2] could be understood by the “vacuum" role as a dissipative system [18].

## 6. Conclusion

In summary, the spectral interpretation of the Riemann zeta function is associated to the vacuum spectrum by means of a variant of the Hilbert-Polya conjecture. The distribution of the virtual resonances would reflect a chaotic nature of the quantum vacuum. The energy/width ratios of the large virtual resonances are associated with the nontrivial zeta zeros while the corresponding widths are related to the prime sequence. Finally, a weak repulsive cutoff potential is proposed as an approximate model for the quantum vacuum.

## Acknowledgements

It is a pleasure to thank C. A. Garcia Canal for useful conversations.

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