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Exponential stability of one-dimensional hyperbolic systems

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This note is devoted to spectral theory and asymptotic behavior of one-dimensional hyperbolic systems. We give in particular a practical way to compute the essential type of the associated semigroup. As is by now well-known, the sign of the essential type gives the asymptotic behavior of the semigroup up to a finite-dimensional space of initial data.

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1. Introduction

This paper is devoted to spectral theory of one-dimensional hyperbolic systems in *normal* form

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} U \\ V \end{pmatrix} = M(x)\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + A(x)\begin{pmatrix} U \\ V \end{pmatrix} & \mathbb{R}^+ \times (0,1) \\ U(t,0) = BV(t,0)\ , \quad V(t,1) = CU(t,1) & t > 0 \\ \begin{pmatrix} U \\ V \end{pmatrix}(0,x) = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}(x)\ , & x \in (0,1). \end{cases} \tag{1.1}$$

where M and A are $n \times n$ matrices $(n \geq 2)$ such that

$$\begin{aligned} M &= diag\,(\lambda_1,...,\lambda_p,\mu_1,...,\mu_l)\,,\ \ p+l = n,\, 1 \leq p \leq n-1 \\ A &= \left(a_{jk}\right)_{1\leq j,k\leq n} \end{aligned}$$

and $U \in \mathbb{R}^p, V \in \mathbb{R}^l$. We denote by B, C constant matrices with appropriate size. (Similar hyperbolic systems with *dynamic boundary conditions* are also dealt with by the authors in [3]). Some physical examples are also given in the last section. We will present in the last section how the stabilization of the Timoshenko beam system fails within our formalism; a more elementary application to discrete kinetic models is also given.

The following "unperturbed" system proves useful:

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} U \\ V \end{pmatrix} = M(x)\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + \widehat{D}(x)\begin{pmatrix} U \\ V \end{pmatrix} & \mathbb{R}_+ \times (0,1) \\ U(t,0) = BV(t,0)\ , \quad V(t,1) = CU(t,1) & t > 0 \\ \begin{pmatrix} U \\ V \end{pmatrix}(0,x) = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}(x)\ , & x \in (0,1). \end{cases} \tag{1.2}$$

where

$$\widehat{D} = diag\,(a_{11},\cdots,a_{nn}) \tag{1.3}$$

is the diagonal matrix composed by the diagonal entries of A.

Under the following assumptions

$(H1)$ $\lambda_k, \mu_j \in C^1([0,1])$ and $\lambda_k > 0 > \mu_j$ in $[0,1]$ for $k = 1,...,p$ and $j = 1,...,l$;

$(H2)$ $A \in C([0,1], M_n(\mathbb{R}))$;

$(H3)$ If for $k \neq m$ there exists some $x_1 \in [0,1]$ such that $\lambda_k(x_1) = \lambda_m(x_1)$ (or $\mu_k(x_1) = \mu_m(x_1)$) then $a_{km} \equiv 0$ in $[0,1]$,

F. Neves, S. Ribeiro and O. Lopes [5] showed that in L^q spaces $(1 \leq q < \infty)$ the difference between the semigroups generated by the systems (2.5) and (1.2) is a compact operator from which it follows, by standard arguments, that the two semigroups have the same essential spectrum and consequently the same essential type (see for instance [8] for definitions of these notions). Owing to the fact that the essential type for (1.2) is easily computable, such a compactness result provides some information on the time asymptotic behavior ($t \to \infty$) of solutions of both systems. (Actually, more generally hyperbolic systems with *dynamic boundary conditions* are also considered in [5]).

In this paper, we weaken Assumption $(H3)$. Indeed, if we replace $(H1)-(H3)$ by

$(H4)$ $\lambda_k, \mu_j \in C^\infty([0,1])$ and $\lambda_k > 0 > \mu_j$ on $[0,1]$ for $k = 1,...,p$ and $j = 1,...,l$;

$(H5)$ $A \in C^\infty([0,1], M_n(\mathbb{R}))$;

$(H6)$ For $k \neq m$, $\lambda_k - \lambda_m$ (or $\mu_k - \mu_m$) has at most finitely many roots with finite order,

We get the following result:

Theorem 1. *Let $(H4)-(H5)-(H6)$ be satisfied. Then:*

(i) The semigroups generated by the systems (2.5) and (1.2) have the same essential type.

(ii) Assume that, if for some $k \neq m$, $\lambda_k - \lambda_m$ (or $\mu_k - \mu_m$) has roots $x_1, \cdots, x_N$ of order $l_1, \cdots, l_N$, then a_{km} vanishes at $x_1, \cdots, x_N$ at orders $s_i \geq l_i$. Then the difference between the semigroups generated by the systems (2.5) and (1.2) is compact.

Remark 2. *The smoothness assumptions in $(H4)-(H5)$ are unnecessary when the eigenvalue curves λ_k (or μ_k) do not interesect; we need L^∞ regularity only. Moreover, when $\lambda_k - \lambda_m$ (or $\mu_k - \mu_m$) has a root at some order, then a stationnary phase argument used in our proof imposes some C^m smoothness of λ_k, μ_j and A in the neighborhood of such roots where the integer m depends on the orders of the roots. However, for the simplicity of the statement, we assume C^∞ regularity.*

If instead of $(H6)$ we assume:

$(H7)$ There exist $k \neq m$ such that $\lambda_k - \lambda_m$ (or $\mu_k - \mu_m$) vanishes on $[a,b] \subset [0,1]$,

then Theorem 1 is no longer true and the above decomposition of the semigroup generated by (2.5) is not relevant. More precisely, under $(H7)$, instead of $\widehat{D}$, we introduce the matrix $\widetilde{D} = (d_{qr})_{1 \leq q,r \leq n}$ where

$$d_{qr}(x) = \begin{cases} a_{qr}(x) & q = r \\ a_{km}(x) & q = k, r = m. \\ a_{mk}(x) & q = m, r = k, \\ 0 & otherwise \end{cases} \tag{1.4}$$

and deal with a new ”unperturbed” system:

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} U \\ V \end{pmatrix} = M(x)\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + \widetilde{D}(x)\begin{pmatrix} U \\ V \end{pmatrix} \; in \mathbb{R}_+ \times (0,1) \\ U(t,0) = BV(t,0) \;, \quad V(t,1) = CU(t,1) \qquad t > 0 \\ \begin{pmatrix} U \\ V \end{pmatrix}(0,x) = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}(x) \;, \qquad x \in (0,1). \end{cases} \tag{1.5}$$

We prove the following:

Theorem 3. *Under* $(H4)-(H5)-(H7)$, *the semigroups associated with (2.5) et (1.2) have different essential types.*

Under $(H4)-(H5)-(H7)$, *the difference between the semigroups generated by (2.5) and (1.5) is compact.*

Remark 4. *Theorem 3 has been obtained recently (cf [1]) under the assumption that* $[a,b] = [0,1]$ *by using tools similar to those used in [5].*

Our proofs of Theorem 1 and Theorem 3 are new and rely on recent functional analytic tools, in particular those given by [4], [9], [10] and [7]. Roughly speaking, instead of the *direct* analysis of the difference of the semigroups given by Neves, Ribeiro and Lopes [5], we provide a *resolvent approach* consisting in analysing the behavior of the difference of the resolvents of their generators for *large imaginary part* of the spectral parameter. The mathematical analysis is performed in L^2 setting and the obtained results extend to L^q spaces $(1 < q < \infty)$ by interpolation arguments. This point of view provides us with a *systematic* analysis of the delicate issue of intersecting curves eigenvalues.

2. Some applications

We give some applications of the previous theory to some examples of physical interest.

2.1 Boundary stabilization of the Timoshenko beam system.

The equations of motion of a Timoshenko beam are

$$\begin{cases} \alpha w_{tt} = (\beta(\varphi + w_x))_x \\ \gamma\varphi_{tt} = (\delta\varphi_x)_x - \beta(\varphi + w_x) \end{cases} \quad on \; (0,1)\times\mathbb{R}^+ \tag{2.1}$$

Here, t is the time variable and x the space coordinate along the beam. The function w is the transverse displacement of the beam and φ is the rotation angle of a filament of the beam. The coefficients $\alpha, \beta, \gamma$and δ are the mass per unit length, the polar moment of inertia of a cross section, Young’s modulus of elasticity, the moment of inertia of a cross section and the shear modulus repectively. We assume that:

$$\alpha, \beta, \gamma, \delta \in \mathbf{C}^1([0,1], (0,+\infty)). \tag{2.2}$$

and the question is to know if the natural energy of this the beam which is

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left\{ \alpha \mid w_t \mid^2 + \gamma \mid \varphi_t \mid^2 + \beta \mid \varphi + w_x \mid^2 + \delta \mid \varphi_x \mid^2 \right\} dx \tag{2.3}$$

decays exponentially whenever we deal with the following boundary conditions:

$$\begin{cases} w(0,t) = w(1,t) = \varphi(0,t) = 0 \\ \sqrt{\delta(1)}\varphi_x(1,t) = -d\ \sqrt{\gamma(1)}\varphi_t(1,t) \end{cases}. \tag{2.4}$$

where $d > 0$ is a real number.

Note that this example has also been considered in [2] and in [6] but the original choice here is in the fact that dissipation for the system comes only from the last condition in (2.4). We will see in the following developpements that exponential decay of the energy $\mathcal{E}$ can be expected only if the wave speeds are equal at least on a subintervall.

Introducing the Riemann invariants:

$$\begin{aligned} u_1 &= \frac{1}{2}\left(\sqrt{\alpha}w_t + \sqrt{\beta}(w_x + \varphi)\right), \\ u_2 &= \frac{1}{2}\left(\sqrt{\gamma}\varphi_t + \sqrt{\delta}\varphi_x\right), \\ v_1 &= \frac{1}{2}\left(\sqrt{\alpha}w_t - \sqrt{\beta}(w_x + \varphi)\right), \\ v_2 &= \frac{1}{2}\left(\sqrt{\gamma}\varphi_t - \sqrt{\delta}\varphi_x\right), \end{aligned}$$

transforms system (2.1) into:

$$Y_t = MY_x + AY \quad in\ (0,1) \times \mathbb{R}^+ \tag{2.5}$$

where M is the diagonal 4×4 matrix given by

$$M = diag\left(\sqrt{\frac{\beta}{\alpha}}, \sqrt{\frac{\delta}{\gamma}}, -\sqrt{\frac{\beta}{\alpha}}, -\sqrt{\frac{\delta}{\gamma}}\right) \tag{2.6}$$

and

$$A = \begin{pmatrix} \left(\sqrt{\frac{\beta}{\alpha}}\right)' & -\frac{1}{2}\sqrt{\frac{\beta}{\gamma}} & \frac{\alpha}{2}\left(\sqrt{\frac{\beta}{\alpha}}\right)' & \frac{1}{2}\sqrt{\frac{\beta}{\gamma}} \\ \frac{1}{2}\sqrt{\frac{\beta}{\gamma}}, & \left(\sqrt{\frac{\delta}{\gamma}}\right)' & -\frac{1}{2}\sqrt{\frac{\beta}{\gamma}}, & \frac{\gamma}{2}\left(\sqrt{\frac{\delta}{\gamma}}\right)' \\ -\frac{\alpha}{2}\left(\sqrt{\frac{\beta}{\alpha}}\right)' & \frac{1}{2}\sqrt{\frac{\beta}{\gamma}} & -\left(\sqrt{\frac{\beta}{\alpha}}\right)' & \frac{1}{2}\sqrt{\frac{\beta}{\gamma}} \\ -\frac{1}{2}\sqrt{\frac{\beta}{\gamma}} & -\frac{\gamma}{2}\left(\sqrt{\frac{\delta}{\gamma}}\right)' & -\frac{1}{2}\sqrt{\frac{\beta}{\gamma}} & -\left(\sqrt{\frac{\delta}{\gamma}}\right)' \end{pmatrix} \tag{2.7}$$

The boundary conditions (2.4) become:

$$u_1 + v_1 = 0, x = 0; 1,\ t > 0, \tag{2.8}$$

$$u_2 + v_2 = 0, x = 0,\ t > 0, \tag{2.9}$$

$$(1+d)\, u_2 + (1-d)\, v_2 = 0, x = 1,\ t > 0. \tag{2.10}$$

We set

$$\lambda_1 = \sqrt{\frac{\beta}{\alpha}}, \lambda_2 = \sqrt{\frac{\delta}{\gamma}}, \mu_1 = -\sqrt{\frac{\beta}{\alpha}}, \mu_2 = -\sqrt{\frac{\delta}{\gamma}}. \tag{2.11}$$

If we assume $(H6)$, the reduced system is

$$Y_t = MY_x + DY \quad in\ (0,1) \times \mathbb{R}^+ \tag{2.12}$$

with $D = diag(A) = diag\left(\lambda_1', \lambda_2', \mu_1', \mu_2'\right)$ and a short computation leads to the two families of eigenvalues of this system associated with the boundary conditions (2.10):

$$\rho_k^1 = i\frac{k\pi}{\int_0^1 \frac{dx}{\lambda_1(x)}}, \rho_k^2 = \frac{\ln\left|\frac{1-d}{1+d}\right|}{2\int_0^1 \frac{dx}{\lambda_2(x)}} + i\frac{k\pi}{\int_0^1 \frac{dx}{\lambda_2(x)}}, k \in \mathbb{Z}. \tag{2.13}$$

Thus the essential type is $\omega_e = 0$ and the natural energy cannot decay exponentially.

If we assume $(H7)$ with $(a,b) = (0,1)$, this time, in the reduced system we have:

$$D = \begin{pmatrix} \lambda' & -\tau & 0 & 0 \\ \tau & \lambda' & 0 & 0 \\ 0 & 0 & -\lambda' & \tau \\ 0 & 0 & -\tau & -\lambda' \end{pmatrix} \tag{2.14}$$

with $\lambda = \lambda_i (= -\mu_i)$, $i = 1,2$ and $\tau = \frac{1}{2}\sqrt{\frac{\beta}{\gamma}}$. The differential system:

$$\rho Y = MY_x + DY \tag{2.15}$$

can be written:

$$\begin{aligned} \rho U &= \lambda\frac{dU}{dx} + \lambda' U + \tau\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U \\ \rho V &= -\lambda\frac{dV}{dx} - \lambda' V - \tau\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V \end{aligned}$$

The change of variables:

$$\begin{aligned} U &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\widetilde{U} \\ V &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\widetilde{V} \end{aligned}$$

transforms the previous equations into:

$$\begin{aligned} \rho\widetilde{U} &= \lambda\frac{d\widetilde{U}}{dx} + \lambda'\widetilde{U} + \tau\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\widetilde{U} \\ \rho\widetilde{V} &= -\lambda\frac{d\widetilde{V}}{dx} - \lambda'\widetilde{V} - \tau\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\widetilde{V} \end{aligned}$$

and the boundary conditions (2.10) become:

$$\begin{aligned}\widetilde{U}(0) &= -\widetilde{V}(0)\\ \widetilde{V}(1) &= -\begin{pmatrix} 1+\gamma & -1+\gamma \\ -1+\gamma & 1+\gamma \end{pmatrix}\widetilde{U}(1)\end{aligned}$$

with $\gamma = \frac{1+d}{1-d}$ (we assume $d \neq 1$ but the case $d = 1$ could be treated in the same way). Some computations lead to the eigenvalue equation:

$$4\gamma e^{4\lambda_0\rho} - 2(1+\gamma)e^{2\lambda_0\rho} + 1 = 0, \tag{2.16}$$

where $\lambda_0 = \int_0^1 \frac{dx}{\lambda(x)}$, which gives the two families of eigenvalues:

$$\begin{aligned}\rho_k^1 &= -\frac{\ln|\gamma|}{2\lambda_0} + i\frac{k\pi}{\lambda_0}, k \in \mathbb{Z}\\ \rho_k^2 &= -\frac{\ln 2}{2\lambda_0} + i\frac{k\pi}{\lambda_0}, k \in \mathbb{Z}.\end{aligned}$$

Since actually system (2.12) is, up to a change of variable, a diagonal system, it follows that (with the same argument as in [5]):

$$\omega_e = \frac{\max(-\ln|\gamma|, -\ln 2)}{2\lambda_0} \tag{2.17}$$

Thus:

$$\omega_e < 0 \Longleftrightarrow |\gamma| > 1 \tag{2.18}$$

and this condition is always satisfied if $d > 0$. So, again, the exponential stability occurs up to a finite dimension space of initial data.

Note that this example has also been considered in [2] and in [6] but with more conditions in (2.4).

2.2 Discrete kinetic models.

Consider a monokinetic equation in a slab with thickness 1

$$\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \sigma(x,\mu)f(x,\mu,t) = \int_{-1}^{1} k(x,\mu,\mu')f(x,\mu',t)d\mu', \tag{2.19}$$

for any $(x,\mu,t) \in (0,1)\times(-1,1)\times(0,\infty)$, where $\mu \in (-1,1)$ is the cosine of the angle between the velocity of particle and the axis of reference (orthogonal to the slab). This equation is supplemented by a initial condition $f(x,\mu,0)$ and a boundary condition

$$f_-(0,.) = B_0(f_+(0,.)), \quad f_+(1,.) = B_1(f_-(1,.)) \tag{2.20}$$

where

$$f_-(0,.) : \mu \in (-1,0) \to f(0,\mu) \tag{2.21}$$

$$f_+(0,.) : \mu \in (0,1) \to f(0,\mu) \tag{2.22}$$

$$f_-(1,.) : \mu \in (-1,0) \to f(1,\mu) \tag{2.23}$$

$$f_+(1,.): \mu \in (0,1) \to f(1,\mu) \tag{2.24}$$

are related by boundary operators B_0 and B_1. In neutron transport theory $B_k = 0$ $(k = 0,1)$, while nonzero (Maxwell) boundary operators appear in the kinetic theory of gases. We are concerned here with the *discrete* (with respect to the μ-variable) version of this model:

$$\frac{\partial f_i}{\partial t} + \mu_i \frac{\partial f_i}{\partial x} + \sigma_i(x) f_i(x,t) = \sum_{j=-N}^{+N} k_{i,j}(x) f_j(x,t), \quad i = -N, ..., -1, +1, ...N \tag{2.25}$$

where $0 < \mu_1 < \mu_2 < \cdots < \mu_N$ denote the positive angles and $\mu_{-N} < \cdots < \mu_{-1} < 0$ denote the negative angles; there are as much negative angles as positive angles The boundary condition (2.20) is

$$U(0,t) = BV(0,t), \ V(t,1) = CU(1,t) \tag{2.26}$$

where

$$U(x,t) := \begin{pmatrix} f_1(x,t)) \\ . \\ . \\ f_N(x,t) \end{pmatrix}, V(x,t) := \begin{pmatrix} f_{-1}(x,t)) \\ . \\ . \\ f_{-N}(x,t) \end{pmatrix}. \tag{2.27}$$

and the matrix B (resp. C) is a discrete version of the operator B_0 (resp. B_1). The hyperbolic system we obtain is much simpler than that considered in this paper since the "velocities" μ_i $(i = -N, ..., -1, +1, ...N)$ are constant with respect to the space variable and then the phenomenon of crossing curve eigenvalues does not occur. So, either all the μ_i are distinct, either some of them are identical. In one or the other case, we can apply Theorem 1 or Theorem 3. For instance, in the first situation (all the eigenvalues are distinct), the semigroup governing (2.25)-(2.20) has the same essential spectrum as the semigroup governing (2.28)and (2.20) where

$$\frac{\partial f_i}{\partial t} + \mu_i \frac{\partial f_i}{\partial x} + (\sigma_i(x) - k_{i,i}(x)) f_i(x,t) = 0, \ i = -N, ..., -1, +1, ...N. \tag{2.28}$$

This is a new spectral result in transport theory. ■

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