

Control problems for Euler-Bernoulli thermoelastic plates without memory and thermoelastic systems with memory

Maria Grazia Naso*

Università degli Studi di Brescia

E-mail: naso@ing.unibs.it

In this note we analyze some problems related to the controllability of thermoelastic systems. In particular, we consider two classes of problems: thermoelastic systems with thermal memory and Euler-Bernoulli thermoelastic plates without memory. For every model we discuss about some needed assumptions in order to obtain the control of the state.

Control Systems: Theory, Numerics and Applications

30 March - 1 April 2005

Rome

*Speaker.

1. Introduction

We address some results related to the controllability of thermoelastic systems. What means to study a controllability problem? Consider an evolution system described in terms of Partial Differential Equations. We are allowed to act on the state of the system by means of a suitable choice of controls (source terms of the system or boundary conditions): given a time interval $t \in (0, T)$, and initial and final states in a Hilbert space, we look for a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

In this note we are interested to present two classes of problems: thermoelastic systems of memory-type dealing with ‘hyperbolic-like’ dynamics, and Euler-Bernoulli thermoelastic plates without memory, where the described model is non-hyperbolic and associated with analyticity of the underlying generator. For these systems we resume our results obtained in the controllability context [8, 11, 22, 23]. We start to recall some definitions about this subject. Assume that H and U are Hilbert spaces and $A : H \rightarrow H$ and $B : U \rightarrow H$ are linear operators. Consider the linear differential system

$$\frac{d}{dt}z(t) = Az(t) + Bf(t), \quad z(0) = z^0 \in H, \quad t \geq 0. \quad (1.1)$$

Let us introduce the set of reachable final states as

$$R(T; z^0) = \{z(T) : f \in L^2(0, T; U)\},$$

where $T > 0$. We can list the following types of controllability:

- (a) *Approximate controllability*: system (1.1) is said to be approximately controllable at time T if the set of reachable states $R(T; z^0)$ is dense in H for every $z^0 \in H$.
- (b) *Exact controllability*: system (1.1) is said to be exactly controllable at time T if $R(T; z^0) = H$ for all $z^0 \in H$. That is, system (1.1) can be driven from any state to any state belonging to the same space of states where the system evolves.
- (c) *Null controllability*: system (1.1) is said to be null controllable at time T if $0 \in R(T; z^0)$ for all $z^0 \in H$. This means that an arbitrary state can be transferred to the null state at time T .

When system (1.1) is reversible in time, null and exact controllability are equivalent notions. Every exactly controllable system is null and approximately controllable too. In general, viceversa is not true (see, for instance, the heat equation with distributed control in the domain Ω [26]). Note that null controllability is a physically interesting notion since the null state is an equilibrium point for system (1.1), and for linear systems this is equivalent to require the controllability to trajectories. Recall that system (1.1) is said to be *controllable to trajectories* at time $T > 0$ if for any initial data z^0 in a suitable space H , there exists a control function $f \in L^2(0, T; U)$ such that the corresponding solution z of (1.1) is defined on $[0, T]$ and satisfies

$$z(T) = \widehat{z}(T),$$

where \widehat{z} is a solution of (1.1) defined on $[0, T]$ associated to given initial data \widehat{z}_0 in the same space H and a given function $\widehat{f} \in L^2(0, T; U)$.

Controllability of thermoelastic systems is a subject which has attracted considerable attention in the literature. Many efforts have been devoted to studying the controllability of thermoelastic systems (see for instance [2, 4, 5, 10, 20, 25, 27]), under varying boundary conditions, and with different choices of control on the boundary or in the control domain. Some results are concerned with models whose underlying dynamics are governed by analytic semigroups (see for instance [8, 11, 17] and references therein). For such systems, because of the smoothing effects associated with analyticity, the null controllability is a more natural question than for the hyperbolic problems. In fact for models which exhibit hyperbolic characteristics, the notion of exact controllability is a more natural property to study. In this case, the control time T has to be sufficiently large due to the finite speed of related propagation. For a more careful review in some known results on analogous problems, we refer to [22] and references therein. Here, we want recall some results of controllability for systems with memory. Leugering [21] proves the reachability for a plate equation with a memory. Lasiecka [16] establishes exact controllability with boundary control for Kirchhoff plates and viscoelastic Kirchhoff plates with a general memory kernel depending on time and space variables. Kim [13, 14] studies controllability problems for systems with large memory by a unique continuation property, which is proved by adapting an idea of Bardos, Lebeau, and Rauch [7]. Barbu and Iannelli [6] study the control for the heat equation with memory. In particular, they show the exact controllability of the one-dimensional linear equation for a sufficiently large interval of time. Pandolfi [24] consider Gurtin-Pipkin equation with control in the Dirichlet boundary condition and he proves exact controllability as a consequence of the known exact controllability of the wave equation.

We briefly sketch the plan of this note. In Section 2 boundary controllability for thermoelastic systems with thermal memory is recalled. Section 3 is devoted to the internal controllability of Euler-Bernoulli thermoelastic plate without memory. In every section we will introduce suitable functional setting and notations which will be used in such section. It has always to be understood that there is a dependence on x , even if, in order to simplify the notation, it will usually not be written.

2. Thermoelastic systems with memory

In this section we are interested in the study of the exact controllability of the thermoelastic system with thermal memory. As well known, in the classical linear theory of thermo-elasticity, the Fourier law is used to describe the heat conduction in the body (see, for instance [9]). The classical theory has two main shortcomings. First, it is unable to account for the thermal memory effect which may prevail in some materials, particularly at low temperatures. Subsequently, the heat equation of this system is of parabolic type and predicts an unrealistic result, namely that a thermal disturbance at one point of the body is instantly felt everywhere in the body. These observations yield that Fourier law is not a good model, and suggest to look for another more general constitutive assumption relating the heat flux to the thermal history of the material. Then, in the model under consideration the Fourier law for the heat flux is replaced by the Gurtin-Pipkin law [12], in order to consider the memory effect which may prevail in some materials, particularly at low temperatures. This produces the convolution term appearing in the second equation of (2.1) that, in particular, entails finite propagation speed of thermal disturbances, so that in this case the

thermoelastic system is fully hyperbolic, which, in particular, implies finite speed of propagation of thermal disturbances (which is physically more realistic). Let Ω be a bounded, open, connected subset of \mathbb{R}^n , $n \geq 1$, with smooth boundary $\Gamma = \partial\Omega$. Assume $T > 0$ and set $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \Gamma$. The state system is as follows

$$\begin{aligned} u_{tt}(t) - \mu \Delta u(t) - (\mu + \lambda) \nabla \operatorname{div} u(t) + \alpha \nabla \theta(t) &= 0 \text{ in } Q, \\ \theta_t(t) - (k * \Delta \theta)(t) + \beta \operatorname{div} u_t(t) &= 0 \text{ in } Q, \\ u(t) = f(t), \quad (k * \theta)(t) = g(t) &\text{ on } \Sigma, \\ u(0) = u_0, \quad u_t(0) = v_0, \quad \theta(0) = \theta_0 &\text{ in } \Omega, \end{aligned} \quad (2.1)$$

where u is the *displacement vector*, θ is the *relative temperature*, μ, λ are the *Lamé coefficients* satisfying $\mu > 0$ and $\lambda + \mu > 0$. The subscript \cdot_t denotes time derivative. The constants $\alpha, \beta > 0$ are coupling parameters depending on properties of the material. By $k * \theta$ we are denoting the convolution product, that is

$$(k * \theta)(t) = \int_0^t k(t - \tau) \theta(\tau) d\tau.$$

In the above system we are assuming that θ has vanishing memory. The case of non-vanishing memory is similar, since it can be considered as a further given function in the right hand side.

Let us now discuss the exact controllability problem. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit normal on Γ directed towards the exterior of Ω . Let $x^0 \in \mathbb{R}^n$ and

$$m(x) = x - x^0 = (x_i - x_i^0), \quad \Gamma_0 = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}, \quad \Sigma_0 = (0, T) \times \Gamma_0.$$

The following statement holds:

Theorem 1. *Let Ω be a open, bounded, connected subset in \mathbb{R}^n with boundary Γ of class C^2 .*

(a) *Suppose that $k(t) \in H^2(\mathbb{R}^+)$ is a positive function and β is a positive constant. Then, there exist $\varepsilon > 0$ and $T_0 > 0$ such that for any initial and final state $(u_0, v_0, \theta_0), (u_T, v_T, \theta_T) \in [L^2(\Omega)]^n \times [H^{-1}(\Omega)]^n \times H^{-1}(\Omega)$ there exists a boundary control $(f, g) \in [L^2(\Sigma_0)]^n \times L^2(\Sigma_0)$ such that the solution of system (2.1) satisfies*

$$u(T) = u_T, \quad u_t(T) = v_T, \quad \theta(T) = \theta_T, \quad (2.2)$$

for any $T > T_0$, provided that $\|k'\|_{H^1(\mathbb{R}^+)} + \beta < \varepsilon$.

(b) *Assume that memory kernel satisfies conditions*

$$k \in H^2(\mathbb{R}^+) \cap C^2(\mathbb{R}^+), \quad k_0 = k(0) > 0, \quad \frac{|k'(0)|}{k_0} < \frac{1}{1 + \sqrt{\alpha\beta}}.$$

The coupling parameters α, β are chosen such that condition

$$\alpha\beta < \frac{\mu + \lambda}{(n-1)^2} \quad (2.3)$$

is satisfied when $n \geq 2$. Then, there exists $T_0 > 0$ such that for any $T > T_0$ and for any initial and final state $(u_0, v_0, \theta_0), (u_T, v_T, \theta_T) \in [L^2(\Omega)]^n \times [H^{-1}(\Omega)]^n \times H^{-1}(\Omega)$ there exists a boundary control $(f, g) \in [L^2(\Sigma)]^n \times L^2(\Sigma)$ such that the solution of system (2.1) satisfies (2.2).

Remark 1.

- (a) In the case considered by Theorem 1-(a), we study the control problem for thermoelastic model with memory by boundary mechanical and thermal controls which can be applied also only in a part Γ_0 of the boundary Γ . The ‘strong’ assumption on the memory kernel is such that its size, in a suitable norm, has to be sufficiently small. The procedure follows a direct approach: first the adjoint system is evaluated; subsequently, the direct and inverse inequalities are shown. The hypothesis on the size of the relaxation function is required to obtain, by energy estimates and multipliers techniques, the inverse inequality, which gives the exact controllability result [22].
- (b) In the case proposed by Theorem 1-(b) we use all over boundary mechanical and thermal controls, but no restriction on the size of memory kernel is required. The inverse inequality is shown by contradiction by means of the introduction of a resolvent kernel and the application of a unique continuation property [23].

3. Euler-Bernoulli thermoelastic plates without memory

In this section we investigate the null controllability of Euler Bernoulli thermoelastic plates without memory when the control (heat source) acts in the thermal equation. The plate, we consider here, is derived in the light of [15]. Transverse shear effects are neglected (Euler-Bernoulli model), and the plate is hinged on its edge. In addition to internal and external heat source, the temperature dynamics are driven by internal frictional forces caused by the motion of the plate. The latter connection is expressed by the second law of thermodynamics for irreversible processes, which relates the entropy to the elastic strains. Accounting for thermal effects, we assume that the heat flux law involves only the temperature gradient by the Fourier law.

Let Ω be a bounded, open, connected subset of \mathbb{R}^2 , with a C^∞ boundary and ω any open subset of Ω . Let $T > 0$ and set $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \partial\Omega$. We consider a model which describes the small vibrations of a homogeneous, elastically and thermally isotropic Euler-Bernoulli plate, under the influence of a control function $f \in L^2((0, T) \times \omega)$. In absence of exterior forces, and with hinged mechanical and Dirichlet thermal boundary conditions, the system we are going to study is the following one

$$\left\{ \begin{array}{ll} u_{tt}(t) + \Delta^2 u(t) + \Delta \theta(t) = 0 & \text{in } Q, \\ \theta_t(t) - \Delta \theta(t) - \Delta u_t(t) = \chi_\omega f(t) & \text{in } Q, \\ u(t) = 0, \Delta u(t) = 0 & \text{on } \Sigma, \\ \theta(t) = 0 & \text{on } \Sigma, \\ u(0) = u_0, u_t(0) = u_1, \theta(0) = \theta_0 & \text{on } \Omega. \end{array} \right. \quad (3.1)$$

Here, u is the *vertical deflection* of the plate and θ is the *variation of temperature* of the plate with respect to its reference temperature. The subscript \cdot denotes time derivative, χ_ω is the characteristic function of ω , and u_0, u_1, θ_0 are initial data in a suitable space.

Two results are obtained (see [8]). Firstly, we study the case when $\omega \equiv \Omega$, and we show the null controllability, or equivalently, the controllability to trajectories at any time $T > 0$. Then, we prove the same result in the case $\omega \subset \Omega$. To do this, let $H = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$ be the Hilbert space equipped with the inner product

$$\langle z_1, z_2 \rangle_H = \int_{\Omega} (\Delta u_1 \Delta u_2 + v_1 v_2 + \theta_1 \theta_2) dx, \quad \text{where } z_i = [u_i, v_i, \theta_i]^\top, \quad i = 1, 2.$$

The induced norm is denoted by $\|\cdot\|_H$. Setting $v(t) = u_t(t)$ and $z(t) = [u(t), v(t), \theta(t)]^\top$, $z_0 = [u_0, v_0, \theta_0]^\top$, problem (3.1) can be rewritten as an abstract linear evolution equation in H of the form

$$\begin{cases} z_t = Az + Bf, \\ z(0) = z_0 \in H, \end{cases} \quad (3.2)$$

where we set the operator $A : D(A) \rightarrow H$ by

$$A = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{bmatrix} \quad (3.3)$$

with domain $D(A) = \{z \in H : \Delta u, v, \theta \in H^2(\Omega) \cap H_0^1(\Omega)\}$, and the control operator $B : L^2(\omega) \rightarrow H$ by $Bf = [0, 0, f]^\top$. Given $T > 0$, the problem of the null controllability of system (3.2) consists in to prove that, for any $z_0 \in H$, there exists a control $f \in L^2((0, T) \times \omega)$ such that the solution $z(t; z_0, f)$ of (3.2) satisfies $z(T; z_0, f) = 0$. This property is equivalent to (see for instance [26], Theorem 2.6, p. 213): there exists a positive constant C_T such that

$$\|e^{A^*T} y_0\|_H^2 \leq C_T \int_0^T \|B^* e^{A^*t} y_0\|_{L^2(\omega)}^2 dt, \quad \text{for any } y_0 \in H. \quad (3.4)$$

Note that the term in the right-hand side of (3.4) depends on the norm in $L^2(\omega)$.

We compute now

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & 0 & \Delta \\ 0 & -\Delta & \Delta \end{bmatrix},$$

with domain $D(A^*) = D(A)$, and $B^* = [0 \ 0 \ I]$. The adjoint system with respect to (3.1) is

$$\begin{cases} \varphi_{tt} + \Delta^2 \varphi + \Delta w = 0 & \text{in } Q, \\ w_t - \Delta w - \Delta \varphi_t = 0 & \text{in } Q, \\ \varphi = 0, \Delta \varphi = 0, w = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_0, \varphi_t(0) = \varphi_1, w(0) = w_0 & \text{on } \Omega. \end{cases} \quad (3.5)$$

Its solution can be written as

$$\begin{bmatrix} \varphi(t) \\ \varphi_t(t) \\ w(t) \end{bmatrix}^\top = e^{A^*t} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ w_0 \end{bmatrix}^\top, \quad (3.6)$$

and

$$B^* e^{A^*t} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ w_0 \end{bmatrix}^\top = w(t).$$

Condition (3.4) is equivalent to require that there exists a positive constant C_T such that

$$\|\Delta\varphi(T)\|_{L^2(\Omega)}^2 + \|\varphi_t(T)\|_{L^2(\Omega)}^2 + \|w(T)\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|w(t)\|_{L^2(\omega)}^2 dt, \quad (3.7)$$

for any solution (3.6) of system (3.5).

Then, our main result is

Theorem 2. *Problem (3.1) is controllable to trajectories at any time $T > 0$ on the space H within the class of $L^2((0, T) \times \omega)$ -controls, when*

(a) $\omega \equiv \Omega$;

(b) $\omega \subset \Omega$.

Remark 2.

- (a) In the case of Theorem 2-(a), an analogous result was obtained by Lasiecka and Triggiani in [17]. In [8] our technique is supported by introducing a quadratic function depending on the time. Multipliers method is applied to construct this function [1, 2, 3].
- (b) In the case of Theorem 2-(b), by applying an iterative method and the observability estimates on the eigenfunctions of the Laplacian operator due to Lebeau and Robbiano in [19] (see also [20]), we show [8] that system (3.1) is null controllable at any time $T > 0$. In our proof, the analyticity property of semigroup associated to the thermoelastic system (recall there is no rotational inertia term, see Lasiecka and Triggiani [18]), and the commutative property of the operators, which comes from the hinged boundary conditions, are crucial.
- (c) In [11] the analysis and construction of the minimization procedure related to the controllability to trajectories for problem (3.1) are considered by applying both penalty and duality arguments. Numerical approximation of the optimality system is carried out through the use of spectral element methods in space and finite difference schemes in time. Numerical results obtained on several test cases are shown.

References

- [1] F. Ammar Khodja and A. Benabdallah, *Sufficient conditions for uniform stabilization of second order equations by dynamical controllers*, Dynam. Contin. Discrete Impuls. Systems **7** (2000), no. 2, 207–222.
- [2] G. Avalos, *Exact controllability of a thermoelastic system with control in the thermal component only*, Differential Integral Equations **13** (2000), no. 4-6, 613–630.

- [3] G. Avalos and I. Lasiecka, *Boundary controllability of thermoelastic plates via the free boundary conditions*, SIAM J. Control Optim. **38** (2000), no. 2, 337–383.
- [4] ———, *Mechanical and thermal null controllability of thermoelastic plates and singularity of the associated minimal energy function*, Control Cybernet. (2003), no. 32, 473–490.
- [5] ———, *The null controllability of thermoelastic plates and singularity of the associated minimal energy function*, J. Math. Anal. Appl. (2004), no. 1, 34–61.
- [6] V. Barbu and M. Iannelli, *Controllability of the heat equation with memory*, Differential Integral Equations **13** (2000), no. 10-12, 1393–1412.
- [7] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.
- [8] A. Benabdallah and M. G. Naso, *Null controllability of a thermoelastic plate*, Abstr. Appl. Anal. **7** (2002), no. 11, 585–599.
- [9] D. E. Carlson, *Linear thermoelasticity*, Handbuch der physik (C. Truesdell, ed.), vol. VIa/2, Springer, New York, 1972, pp. 297–345.
- [10] L. de Teresa and E. Zuazua, *Controllability of the linear system of thermoelastic plates*, Adv. Differential Equations **1** (1996), no. 3, 369–402.
- [11] P. Gervasio and M.G. Naso, *Numerical approximation of controllability of trajectories for Euler-Bernoulli thermoelastic plates*, Math. Models Methods Appl. Sci. **14** (2004), no. 5, 701–733.
- [12] M.E. Gurtin and A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.
- [13] J.U. Kim, *Control of a plate equation with large memory*, Differential Integral Equations **5** (1992), no. 2, 261–279.
- [14] ———, *Control of a second-order integro-differential equation*, SIAM J. Control Optimiz. **31** (1993), no. 1, 101–110.
- [15] J. Lagnese and J.-L. Lions, *Modelling analysis and control of thin plates*, Masson, Paris, 1988.
- [16] I. Lasiecka, *Controllability of a viscoelastic Kirchhoff plate*, Internat. Ser. Numer. Math. **91** (1989), 237–247.
- [17] I. Lasiecka and R. Triggiani, *Exact null controllability of structurally damped and thermo-elastic parabolic models*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **9** (1998), no. 1, 43–69.
- [18] ———, *Two direct proofs on the analyticity of the s.c. semigroup arising in abstract thermo-elastic equations*, Adv. Differential Equations **3** (1998), no. 3, 387–416.
- [19] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), no. 1-2, 335–356.
- [20] G. Lebeau and E. Zuazua, *Null-controllability of a system of linear thermoelasticity*, Arch. Rational Mech. Anal. **141** (1998), no. 4, 297–329.
- [21] G. Leugering, *Exact boundary controllability of an integro-differential equation*, Appl. Math. Optim. **15** (1987), 223–250.
- [22] J.E. Muñoz Rivera and M.G. Naso, *Exact boundary controllability in thermoelasticity with memory*, Adv. Differential Equations **8** (2003), no. 4, 471–490.

- [23] J.E. Muñoz Rivera and M.G. Naso, *Exact controllability for hyperbolic thermoelastic systems with large memory*, Adv. Differential Equations **9** (2004), no. 11-12, 1369–1394.
- [24] L. Pandolfi, *Controllability of the Gurtin-Pipkin equation*, in proceedings of *Control Systems: Theory, Numerics and Applications*, PoS(CSTNA2005)015.
- [25] R. Triggiani, *Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems*, Adv. Differential Equations **8** (2003), no. 2, 189–229.
- [26] J. Zabczyk, *Mathematical control theory: an introduction*, Birkhäuser Boston Inc., Boston, MA, 1992.
- [27] E. Zuazua, *Controllability of the linear system of thermoelasticity*, J. Math. Pures Appl. (9) **74** (1995), no. 4, 291–315.

POS(CSTNA2005)014