PROCEEDINGS OF SCIENCE

Verification Theorems for HJB equations

Mauro Garavello*
Dipartimento di Matematica e Applicazioni,
Università di Milano Bicocca
Via R. Cozzi 53 - Edificio U5
20125 Milano (Italy)
E-mail: mauro.garavello@unimib.it

We study an optimal control problem in Bolza form and we consider the value function associated to this problem. We prove two verification theorems which ensure that, if a function W satisfies some suitable weak continuity assumptions and a Hamilton-Jacobi-Bellman inequality outside a rectifiable set of codimension one, then it coincides with the value function.

Control Systems: Theory, Numerics and Applications
30 March - 1 April 2005
Rome

*Speaker.

Published by SISSA http://pos.sissa.it/

1. Introduction

In this paper we consider a control system of the type:

$$\dot{x} = f(t,x,u), \qquad u \in U \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state, $U \subset \mathbb{R}^q$ is the control space and f is the controlled dynamic. Given a target $S \subset \mathbb{R}^n$, a running cost $L(t,x,u)$, a final cost $\psi(t,x)$ and an initial condition (t_0,x_0), we consider the optimal control problem in Bolza form. We define in the usual way the value function $V(t_0,x_0)$ to be the infimum of the problem with initial condition (t_0,x_0). It is well known that, under suitable conditions, V satisfies the Hamilton-Jacobi-Bellman equation in viscosity sense [1] and it is possible to isolate V as the unique solution. The proof is based on the dynamic programming principle.

Therefore given a function W, it is possible to determine if W coincide with the value function, checking if it is a viscosity solution to the HJB equation. This type of theorems, called verification theorems, are useful, for example, when a candidate value function is produced by means of the construction of a synthesis [13]. It is then natural to ask for minimal conditions under which a function W coincides with the value function. If we know that W was obtained via a synthesis then the inequality $W \geq V$ is granted by construction, thus we take this assumption. Then, for W to coincide with the value function, we prove it is sufficient that, outside a rectifiable set of codimension one, both W is differentiable and it satisfies a Hamilton-Jacobi-Bellman inequality. Moreover, we make use of only some weak continuity assumptions, already used in [13] to prove optimality of a regular extremal synthesis, see Theorem 1 and Theorem 2 for details. A first result in this direction can be found in [2], where the HJB inequality is asked outside a countable collection of Lipschitz continuous manifolds of positive codimension. Notice that, for an optimal control problem, the value function is indeed differentiable outside a closed rectifiable set of codimension one, see [4].

We start considering the main assumptions for the problem and presenting two technical lemmas, one of which dealing with the cardinality of the intersections between admissible trajectories and a rectifiable set.

The first case we treat is the problem of finite time. We define a value function as the infimum, over all admissible trajectories reaching the target in finite time. The main result of this part is Theorem 1 which permits to verify if the function W coincides with the value function. In particular we need the differentiability of W outside a rectifiable set and the fact that W must satisfy a HJB differential inequality in the same set.

Next, we consider the infinite time problem. In this case the value function (5.1) is defined as the infimum of the cost functional over all admissible trajectories reaching the target in infinite time. The main result of this section is Theorem 2 which gives sufficient conditions on the function W to ensure the equality between W and the value function. In this case, we consider a suitable neighborhood S_1 of the target S and we suppose that the final cost ψ is defined on S_1 in order to give sense to the limit in the definition of the value function (5.1). As a corollary of Theorem 1 and Theorem 2 we can treat a mixed case (see also [12]), considering at the same time the trajectories reaching the target both in finite time and in infinite time.

A key ingredient for Theorem 1 and Theorem 2 is the positiveness of the Lagrangian L, in order to prevent some bad phenomena such as the permanence of the system for an arbitrary interval of times in a region where L is negative making the value function equal to $-\infty$.

Acknowledgements

The author wishes to thank Prof. B. Piccoli for having proposed him the study of this problem and for his useful advice.

2. Preliminaries

We consider a control system:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (t,x) \in \Omega, \quad u(t) \in U \tag{2.1}$$

where

(A-1) Ω is an open and connected subset of $\mathbb{R} \times \mathbb{R}^n$.

(A-2) U is a non-empty subset of $\mathbb{R}^q$, for some $q \geq 1$, $q \in \mathbb{N}$.

(A-3) $\mathcal{U} = L^p(\mathbb{R}; U)$ with $1 \leq p < +\infty$ is the set of admissible controls.

(A-4) $f : \Omega \times U \to \mathbb{R}^n$ is measurable in t, continuous in (x,u), differentiable in x and, for each $u \in U$, $D_x f(\cdot,\cdot,u)$ is bounded on compact sets. Moreover there exists $\varphi_1 : \mathbb{R} \to \mathbb{R}^+$ integrable and for every K, compact subset of Ω, there exist a modulus of continuity ω_K and a constant $L_K > 0$ such that, if $(t,x) \in K$ and $(t,y) \in K$, then

$$\begin{cases} |f(t,x,u) - f(t,y,u)| \leq \omega_K(|x-y|) \\ (f(t,x,u) - f(t,y,u)) \cdot (x-y) \leq L_K |x-y|^2 \\ |f(t,x,u)| \leq L_K(\varphi_1(t) + |u|^p). \end{cases} \tag{2.2}$$

We consider a function $L : \Omega \times U \to \mathbb{R}$ and assume:

(A-5) L is measurable in t and continuous in (x,u). Moreover, there exist $\varphi_2 : \mathbb{R} \to \mathbb{R}^+$ integrable and, for every $R \geq 0$, $C_R \geq 0$ such that

$$|L(t,x,u)| \leq C_R(\varphi_2(t) + |u|^p), \quad |(t,x)| \leq R \tag{2.3}$$

In this paper we indicate with $x(\cdot; u, t_0, x_0)$ the solution to (2.1) such that $x(t_0; u, t_0, x_0) = x_0$. Define the value function:

$$V(t_0,x_0) := \inf_{\substack{u \in \mathcal{U} \\ (T, x(T;u,t_0,x_0)) \in S}} \left\{ \int_{t_0}^{T} L(s, x(s;u,t_0,x_0), u(s))ds + \psi(T, x(T;u,t_0,x_0)) \right\} \tag{2.4}$$

where S - the target - is a closed subset of $\mathbb{R} \times \mathbb{R}^n$ contained in Ω, $\psi : S \to \mathbb{R}$ is the final cost. Now we need the following definitions (introduced in [13]):

Definition 1. *We call a function $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ weakly upper semicontinuous (w.u.s.c.) at a point x if $\liminf_{y\to x} \limsup_{z\to y} W(z) \le W(x)$. Moreover W is w.u.s.c. if it is w.u.s.c. at every point of $\mathbb{R}^n$.*

Remark 1. *A function $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is w.u.s.c. at x if and only if, for any sequence $\delta_j > 0$, $\delta_j \to 0$ there exist a sequence $x_j \to x$ and a sequence $\varepsilon_j > 0$ such that*

$$\left|y - x_j\right| \le \varepsilon_j \quad \Rightarrow \quad W(y) \le W(x) + \delta_j. \tag{2.5}$$

Note that we can choose $0 < \varepsilon_j \le \delta_j$.

Definition 2. *Suppose that we have a time-varying Lipschitz-continuous vector field X on $\mathbb{R}^n$ and $W : \Omega \to \mathbb{R} \cup \{+\infty\}$. We say that W has the no downward jumps property (NDJ) along X if for any $[a,b] \ni t \mapsto \gamma(t)$, solution to $\dot\gamma(t) = X(t,\gamma(t))$ such that $(t,\gamma(t)) \in \Omega$ $\forall t \in [a,b]$, we have $\liminf_{h\downarrow 0} W(t-h,\gamma(t-h)) \le W(t,\gamma(t))$, whenever $t \in]a,b]$.*

Definition 3. *A subset A of Ω is an n-dimensional rectifiable set if there exist A_1 and A_2 such that $A = A_1 \cup A_2$, A_1 is a finite or countable union of connected Lipschitz submanifolds of positive codimension, and $\mathcal{H}^n(A_2) = 0$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.*

3. Two lemmas

In this section we present two technical lemmas.

Lemma 1. *Fix an element $\omega \in U$, $t' < t''$ and $x \in \mathbb{R}^n$ with $(t'',x) \in \Omega$. Assume that there exists $\mathcal{W}$, an open neighborhood of x in $\mathbb{R}^n$, such that $\zeta^y(\cdot)$, the solution to $\dot\zeta^y(t) = f(t,\zeta^y(t),\omega)$ with $\zeta^y(t'') = y$, is defined on $[t',t'']$ for any $y \in \mathcal{W}$ and $(t,\zeta^y(t)) \in \Omega$ $\forall t \in [t',t'']$. Let A be an n-dimensional rectifiable set.*

Then for a.e. $y \in \mathcal{W}$ the set $B^y := \{t \in [t',t''] : (t,\zeta^y(t)) \in A\}$ is finite or countable.

Proof. We can write $A = A_1 \cup A_2$, where $A_1 = \cup_j M_j$ and $\{M_j\}_{j\in J}$ is a finite or countable family of connected submanifolds of $\mathbb{R}^{n+1}$ of codimension $d_j > 0$, and $\mathcal{H}^n(A_2) = 0$. After replacing each M_j by a finite or countable family of open submanifolds of M_j, we may assume that the M_j are embedded. Define $\widetilde{\mathcal{W}} :=]t',t''[\times \mathcal{W}$ and let Φ be the map $\widetilde{\mathcal{W}} \ni (t,y) \mapsto (t,\zeta^y(t)) \in \Omega$. The Jacobian of Φ is

$$\mathbf{J}\Phi = \left(\begin{array}{c|c} 1 & 0 \cdots\cdots 0 \\ \hline \mathbf{b} & \mathbf{V}^\zeta(t;t',\mathbf{Id}) \end{array}\right) \tag{3.1}$$

where $\mathbf{b}$ is the column vector $f(t,\zeta^y(t),\omega)$ and $\mathbf{V}^\zeta(t;t',\mathbf{Id})$ is the fundamental matrix solution to the linear system

$$\dot v(t) = -D_x f(t,\zeta^y(-t+t'+t''),w)\cdot v(t) \tag{3.2}$$

such that $\mathbf{V}^\zeta(t';t',\mathbf{Id}) = \mathbf{Id}$. So the determinant of $\mathbf{J}\Phi$ is equal to the determinant of $\mathbf{V}^\zeta(t;t',\mathbf{Id})$, which is equal to $\exp\int_{t'}^t \mathrm{tr}(-D_x f(s,\zeta^y(-s+t'+t''),\omega))ds$, by Liouville's theorem (see [9]). In particular $det(\mathbf{J}\Phi)$ is strictly positive for any $t \in [t', t'']$. Moreover, by (A-4) $tr(-D_x f)$ is bounded on compact sets and then there exist $c > 0, C > 0$ such that $0 < c \leq det(\mathbf{J}\Phi) \leq C$.

So Φ is a Lipschitz diffeomorphism. In particular we have $\mathcal{H}^n(\Phi^{-1}(A_2)) = 0$. Now, for each j, let us consider $\widetilde{M}_j := \Phi^{-1}(M_j)$. It is a locally Lipschitz embedded submanifold of codimension $d_j > 0$. So we may suppose that $\widetilde{M}_j$ is locally described as the image of a bi–Lipschitz continuous function $\varphi_j : U \to \mathbb{R}^{n+1}$, where U is an open and bounded subset of $\mathbb{R}^m$, with $1 \leq m \leq n$.

First we suppose that $m < n$. We obviously have that $\mathcal{H}^m(\varphi_j(U)) \leq L\mathcal{H}^m(U) < +\infty$ where L is the lipschitz constant for φ_j and so (see [5]) $\mathcal{H}^n(\varphi_j(U)) = 0$. This implies that $\mathcal{H}^n(\Pi(\widetilde{M}_j)) = 0$. Since $\mathcal{L}^n$ coincides with $\mathcal{H}^n$ in $\mathbb{R}^n$ (see [5]), we conclude that $\mathcal{L}^n(\Pi(\widetilde{M}_j)) = 0$.

So we may suppose that $m = n$. We define the set

$$Z_j := \left\{(t,y) \in \widetilde{M}_j : \varphi_j \text{ not differentiable at } x = \varphi_j^{-1}(t,y)\right\}.$$

By Rademaker theorem, we obtain that $\mathcal{H}^n(Z_j) = 0$ and so $\mathcal{L}^n(\Pi(Z_j)) = 0$. We now consider the function

$$\Pi \circ \varphi_j : U \to \mathbb{R}^n,$$

the set

$$S_j := \left\{u \in U : \Pi \circ \varphi_j \text{ differentiable at } u \text{ and } D'(\Pi \circ \varphi_j)(u) \text{ not surjective}\right\}$$

and the set

$$Z_j^{(1)} := \varphi_j(S_j).$$

By Sard's lemma (see [7]), $\mathcal{L}^n(\Pi(Z_j^{(1)})) = 0$. So the set

$$\mathcal{B} := \Pi(\Phi^{-1}(A_2)) \cup (\bigcup_j \Pi(Z_j)) \cup (\bigcup_j \Pi(Z_j^{(1)}))$$

has Lebesgue measure 0 in $\mathbb{R}^n$.

Let $y \in \mathcal{W} \setminus \mathcal{B}$. Then $(t, \zeta^y(t)) \notin A_2$ if $t' < t < t''$. To obtain the thesis, it is sufficient to show that, for each j, the set $E_j = \left\{t \in]t', t''[: (t, \zeta^y(t)) \in M_j\right\}$ is at most countable. Fix j and suppose $\bar{t} \in E_j$. Then $(\bar{t}, \zeta^y(\bar{t})) \in M_j$ and $(\bar{t}, y) \in \widetilde{M}_j$. By the fact that $y \notin \mathcal{B}$, we have $\frac{\partial}{\partial t} \notin T_{\widetilde{M}_j}(\bar{t}, y)$. This fact permits to conclude that $(t, y) \notin \widetilde{M}_j$ if $0 < |t - \bar{t}| < \varepsilon$ for ε sufficiently small. Therefore t is an isolated point of E_j and the lemma is proved. □

Lemma 2. *Let g be a real-valued function on a compact interval $[a,b]$. Assume that there exists a finite or countable subset E of $[a,b]$ with the following properties:*

(A-1) $\liminf_{h\downarrow 0} \frac{g(x+h)-g(x)}{h} \geq 0$ *for all* $x \in [a,b[\setminus E$,

(A-2) $\liminf_{h\downarrow 0} g(x+h) \geq g(x)$ *for all* $x \in [a,b[$,

(A-3) $\liminf_{h\downarrow 0} g(x-h) \leq g(x)$ *for all* $x \in]a,b]$.

Then $g(b) \geq g(a)$.

For a proof of this lemma see [13, Lemma B.1].

4. Problem with finite time

We indicate with ∂Q the topological boundary of an arbitrary $Q \subseteq \mathbb{R} \times \mathbb{R}^n$.

Theorem 1. *Suppose (A-1)-(A-5) hold. Let $Q \subseteq \Omega$ be an open subset containing S. Let $W : \overline{Q} \to \mathbb{R}$ be a lower semicontinuous function verifying the NDJ property along every time-varying vector field of the type $f(t,x,u)$ with $u \in U$ fixed. Moreover we assume that, for each t, $W(t,\cdot)$ is w.u.s.c. and that:*

i) $W \geq V$.

ii) $W \leq \psi$ *on* S.

iii) *At every point* $(t,x) \in \partial Q$ *one has*

$$W(t,x) = \sup_{(s,y) \in Q} W(s,y).$$

iv) *There exists an n-dimensional rectifiable set set $A \subseteq \Omega$ such that W is differentiable on $Q \setminus A$ and satisfies*

$$W_s(s,y) + \inf_{\omega \in U} \{W_y(s,y) \cdot f(s,y,\omega) + L(s,y,\omega)\} \geq 0 \quad \text{on } Q \setminus A.$$

v) $L \geq 0$.

Then $W = V$ on Q. If $Q = \Omega$ we can drop hypotheses iii) and v).

Proof. We have to prove that $W \leq V$. Suppose by contradiction that there exists $(t_0, x_0) \in Q$ such that $W(t_0,x_0) > V(t_0,x_0)$. Then there exists $M \in \mathbb{R}$ such that $W(t_0,x_0) > M > V(t_0,x_0)$. So we can find $\varepsilon > 0$, $\delta > 0$ such that

$$V(t_0,x_0) + \varepsilon \leq M - \varepsilon \tag{4.1}$$

and, by the lower semicontinuity of W,

$$|x - x_0| < \delta \quad \Rightarrow \quad W(t_0,x) > V(t_0,x_0) + \varepsilon. \tag{4.2}$$

We can find $u^* \in \mathcal{U}$ such that $x^*(\cdot) := x(\cdot; u^*, t_0, x_0)$ satisfies $(T, x^*(T)) \in S$ and

$$\int_{t_0}^{T} L(s, x^*(s), u^*(s))ds + \psi(T, x^*(T)) \leq V(t_0,x_0) + \frac{\varepsilon}{2}\,. \tag{4.3}$$

Moreover, by [3, Théorèm IV.9], there exist $h \in L^p([t_0,T])$ and, for every $\eta > 0$, $u^\sharp = u^\sharp(\eta) \in \mathcal{U}$ such that $\|u^\sharp - u^*\|_{L^p([t_0,T])} \leq \eta$, $u^\sharp$ piecewise constant, left continuous and $\left|u^\sharp\right| \leq h$ a.e. Hence, if we denote by $x^\sharp(\cdot)$ the trajectory $x(\cdot; u^\sharp, T, x^*(T))$, for η sufficiently small, we have

$$\left| \int_{t_0}^{T} [L(s, x^\sharp(s), u^\sharp(s)) - L(s, x^*(s), u^*(s))]ds \right| \leq \frac{\varepsilon}{2} \tag{4.4}$$

and

$$\left|x^\sharp(t) - x^*(t)\right| < \frac{\delta}{2} \qquad \forall t \in [t_0, T]. \tag{4.5}$$

Fix an interval $]t', t'']$ such that $u^\sharp(t) \equiv \omega$ in $]t', t'']$. Suppose that $(t, x^\sharp(t)) \in Q$ $\forall t \in [t', t'']$. Let $\zeta^y(t)$ be the trajectory associated to the constant control ω and such that $\zeta^y(t'') = y$. By the fact that $d(\partial Q, \{(t, x^\sharp(t)) : t \in [t', t'']\}) > 0$, we can find an open neighborhood $\mathcal{W}$ of $x^\sharp(t'')$ in $\mathbb{R}^n$ such that $(t'', y) \in Q$ $\forall y \in \mathcal{W}$ and $\{(t, \zeta^y(t)) : t \in [t', t'']\} \subseteq Q$ $\forall y \in \mathcal{W}$. By Lemma 1, we have that for a.e. $y \in \mathcal{W}$ the set $B^y := \{t \in [t', t''] : (t, \zeta^y(t)) \in A\}$ is at most countable.

Since W is w.u.s.c. for every fixed t, then for every $\delta_j \to 0$, $\delta_j > 0$ there exist $x_j \to x^\sharp(t'')$, and $0 < \varepsilon_j \le \delta_j$ such that

$$\left|x_j - y\right| \le \varepsilon_j \quad \Rightarrow \quad W(t'', y) \le W(t'', x^\sharp(t'')) + \delta_j. \tag{4.6}$$

For j sufficiently big, we can find y_j, $\left|y_j - x_j\right| \le \varepsilon_j$, such that B^{y_j} is at most countable. Consider the following function defined on $[t', t'']$:

$$\varphi_j(t) := W(t, \zeta^{y_j}(t)) + \int_{t'}^{t} L(s, \zeta^{y_j}(s), \omega) ds.$$

By the choice of y_j and the hypotheses *iv*), φ_j is differentiable a.e. with a nonnegative derivative. By the lower semicontinuity of W and the NDJ condition, it follows that φ_j verifies the hypotheses of Lemma 2 and so $\varphi_j(t') \le \varphi_j(t'')$. Thus

$$W(t', \zeta^{y_j}(t')) \le W(t'', \zeta^{y_j}(t'')) + \int_{t'}^{t''} L(s, \zeta^{y_j}(s), \omega) ds. \tag{4.7}$$

Now, using (4.6) and the fact that $\zeta^{y_j}(t'') = y_j$ we obtain

$$W(t', \zeta^{y_j}(t')) \le W(t'', x^\sharp(t'')) + \delta_j + \int_{t'}^{t''} L(s, \zeta^{y_j}(s), \omega) ds. \tag{4.8}$$

Pass to the limit as $j \to +\infty$:

$$W(t', x^\sharp(t')) \le W(t'', x^\sharp(t'')) + \int_{t'}^{t''} L(s, x^\sharp(s), \omega) ds. \tag{4.9}$$

First consider the case $\{(t, x^\sharp(t)) : t \in [t_0, T]\} \subseteq Q$. Summing (4.9) over each interval on which $u^\sharp$ is constant we have

$$W(t_0, x^\sharp(t_0)) \le W(T, x^\sharp(T)) + \int_{t_0}^{T} L(s, x^\sharp(s), u^\sharp(s)) ds. \tag{4.10}$$

Now, $x^\sharp(T) = x^*(T)$ by definition and so, using (4.2)-(4.3)-(4.4)

$$\begin{aligned} W(t_0, x^\sharp(t_0)) &\le W(T, x^*(T)) + \int_{t_0}^{T} L(s, x^\sharp(s), u^\sharp(s)) ds \\ &\le \psi(T, x^*(T)) + \int_{t_0}^{T} L(s, x^\sharp(s), u^\sharp(s)) ds \\ &\le V(t_0, x_0) + \frac{\varepsilon}{2} - \int_{t_0}^{T} L(s, x^*(s), u^*(s)) ds \\ &\quad + \int_{t_0}^{T} L(s, x^\sharp(s), u^\sharp(s)) ds \\ &\le V(t_0, x_0) + \varepsilon < W(t_0, x^\sharp(t_0)). \end{aligned}$$

This is a contradiction.

Suppose now $\left\{(t,x^\sharp(t)) : t \in [t_0,T]\right\} \not\subseteq Q$. Define

$$\hat{\tau} := \inf\left\{t \leq T : (s,x^\sharp(s)) \in Q \quad \forall s \in [t,T]\right\}. \tag{4.11}$$

In particular $(\hat{\tau},x^\sharp(\hat{\tau})) \in \partial Q$. Using the same argument to pass from (4.9) to (4.10), we obtain that for every $\tau > \hat{\tau}$

$$W(\tau,x^\sharp(\tau)) \leq W(T,x^*(T)) + \int_\tau^T L(s,x^\sharp(s),u^\sharp(s))ds \tag{4.12}$$

and so

$$\begin{aligned} W(\tau,x^\sharp(\tau)) &\leq \psi(T,x^*(T)) + \int_\tau^T L(s,x^\sharp(s),u^\sharp(s))ds \\ &\leq V(t_0,x_0) + \frac{\varepsilon}{2} - \int_{t_0}^T L(s,x^*(s),u^*(s))ds \\ &\quad + \int_\tau^T L(s,x^\sharp(s),u^\sharp(s))ds. \end{aligned} \tag{4.13}$$

Since $L \geq 0$, we obtain for all $\tau > \hat{\tau}$

$$\begin{aligned} W(\tau,x^\sharp(\tau)) &\leq V(t_0,x_0) + \varepsilon \\ &\leq M - \varepsilon \\ &< W(t_0,x_0) - \varepsilon. \end{aligned} \tag{4.14}$$

Passing to the liminf as $\tau \to \hat{\tau}$ and using the lower semicontinuity of W, we conclude

$$W(\hat{\tau},x^\sharp(\hat{\tau})) < W(t_0,x_0) - \varepsilon \tag{4.15}$$

and so by *iii*)

$$W(t_0,x_0) < W(t_0,x_0) - \varepsilon \tag{4.16}$$

which is a contradiction. This concludes the proof of the theorem. □

The hypotheses of the positiveness of L is quite optimal as the next example shows. However, it is sufficient that the system could not stay for too long in a region where the Lagrangian is negative, so one can relax the assumption v) in this way.

Example 1. *Consider the system $\dot{x} = u$, $U = [-1,1]$ and $\mathcal{U} = L^1(\mathbb{R};U)$, $\Omega = \mathbb{R}^2$, $S = \mathbb{R} \times \{0\}$, $Q = \mathbb{R}\times]-1,1[$ with the Lagrangian $L(t,x,u) = u^2 + x^4 - 6x^3 + 7x^2$ (see Figure 1) and $\psi \equiv 0$ on S. Since the Lagrangian is negative in a region where the system can stay for an arbitrary interval of times, clearly the value function for this problem is equal to $-\infty$. If $W \equiv C$ on $\overline{Q}$ with C negative constant, then W verifies all the hypotheses of the Theorem 1, but v). In fact i), ii), iii) are obvious, while iv) holds because L is positive on Q and W is differentiable on Q. So there exist infinitely many functions W defined on $\overline{Q}$ verifying the hypotheses of Theorem 1, but v), which are different from V.*

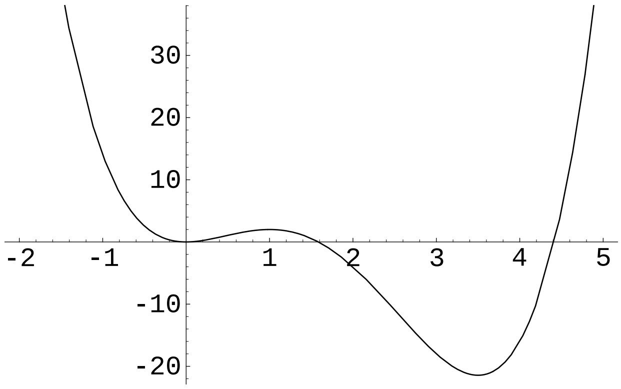

Figure 1: $L(t,x,0)$

5. Problem with infinite time

In this section we consider the control system (2.1) and assume that (A-1)-(A-5) hold with $0 \leq C_R \leq C$ for some $C > 0$ and every $R > 0$. Moreover we suppose that the target S is a closed subset of $\mathbb{R} \times \mathbb{R}^n$ which satisfies:

$(*)$ For any $T > 0$, there exists $(t,x) \in S$ with $t \geq T$.

Let S_1 be an open neighborhood of S contained in Ω. Assume that the final cost ψ is defined on S_1 and, if $d((t,x(t;u,t_0,x_0)),S) \to 0$ as $t \to +\infty$, then the trajectory $x(\cdot;u,t_0,x_0)$ is definitively in S_1, that is:

$(**)$ there exists $T > t$ such that $(s,x(s;u,t_0,x_0)) \in S_1$ for all $s \geq T$.

Define the value function $V(t_0,x_0)$ as

$$\inf_{\substack{u\in\mathcal{U} \\ d((t,x(t;u,t_0,x_0)),S)\to 0 \\ \text{as } t\to+\infty}} \left\{ \int_{t_0}^{+\infty} L(s,x(s;u,t_0,x_0),u(s))ds + \limsup_{t\to+\infty} \psi(t,x(t;u,t_0,x_0)) \right\} \tag{5.1}$$

In other words, we consider only the trajectories that approach the target S in infinite time. Notice that this condition does not imply that $(T,x(T)) \notin S$ for every $T \geq t_0$.

We now prove a verification theorem for a function W defined on $\overline{Q}$, where Q is an open subset of Ω containing the target.

Theorem 2. *Let $Q \subseteq \Omega$ be an open subset containing S. Let $W : \overline{Q} \to \mathbb{R}$ be a lower semicontinuous function verifying the NDJ property along every time-varying vector field X of the type $f(t,x,u)$ with $u \in U$ fixed. Moreover assume that, for each t, $W(t,\cdot)$ is w.u.s.c. and that:*

i) $W \geq V$.

ii) $W \leq \psi$ *on* S_1.

iii) *At every point* $(t,x) \in \partial Q$ *one has*

$$W(t,x) = \sup_{(s,y)\in Q} W(s,y).$$

iv) *There exists a thin set* $A \subseteq \Omega$ *such that* W *is differentiable in* $Q \setminus A$ *and satisfies*

$$W_s(s,y) + \inf_{\omega\in U} \{W_y(s,y)\cdot f(s,y,\omega) + L(s,y,\omega)\} \geq 0 \quad \text{in } Q\setminus A.$$

v) $L \geq 0$.

Then $W = V$ *on* Q*. If* $Q = \Omega$ *we can drop hypotheses iii) and v).*

The proof is eesentially similar to the proof of Theorem 1. For a complete proof of this theorem see also [8].

Remark 2. *If we assume that there exists* $\eta > 0$ *such that* $S + B(0,\eta) \subseteq S_1$*, where* $B(0,\eta)$ *is the ball in* $\mathbb{R}^{n+1}$ *centered in* 0 *with radius* η*, then hypothesis* $(**)$ *obviously holds. In fact suppose* $d((t,x(t;u,t_0,x_0)),S) \to 0$ *as* $t \to +\infty$*. Then there exists* $T > 0$ *such that* $d((s,x(s;u,t_0,x_0)),S) < \frac{\eta}{2}$ *for all* $s \geq T$*. So we can choose an element* $(t(s),y(s)) \in S$ *in order to have*

$$d((s,x(s;u,t_0,x_0)),(t(s),y(s))) < \frac{\eta}{2}$$

for all $s \geq T$*. So the points* $(s,x(s;u,t_0,x_0)) \in S + B(0,\eta) \subseteq S_1$ *for every* $s \geq T$*.*

Remark 3. *We obtain a generalization of Theorems 1 and 2 considering the same problem (2.1) with assumptions (A-1)-(A-4), but we accept at the same time all the trajectories that hit the target in finite time or that tend to the target in infinite time. Obviously an analogous theorem as 1 and 2 holds.*

References

[1] M. Bardi, I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser Boston Inc., Boston 1997.

[2] A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, Oxford University Press, Oxford 2000.

[3] H. Brezis, *Analyse fonctionnelle: Théorie et applications*, Masson, Paris 1983.

[4] P. Cannarsa, A. Mennucci, C. Sinestrari, *Regularity results for Solutions of a Class of Hamilton-Jacobi Equations*, Arch. Rational Mech. Anal., **140** (1997), pp. 197–223.

[5] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC press 1992.

[6] G. B. Folland, *Real Analysis: Modern Techniques and their Applications*, J. Wiley and sons, New York 1999.

[7] I. Fonseca, W. Gangbo, *Degree theory in analysis and applications*, Clarendon Press, Oxford, 1995.

[8] M. Garavello, *Verification theorems for Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optim. **42**, (2003), no. 5, pp. 1623–1642.

[9] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons Inc., New York 1964.

[10] M. Malisoff, *On the Bellman equation for control problems with exit times and unbounded cost functionals*, in proceedings of the 38^{th} *IEEE Conference on Decision and Control*, Phoenix, Arizona USA, 1999.

[11] M. Malisoff, H. J. Sussmann, *Further Results on the Bellman Equation for Optimal Control Problems with Exit Times and Nonnegative Instantaneous Costs*, in proceedings of *39th IEEE Conference on Decision and Control*, Sydney, Australia 2000.

[12] B. Piccoli, *Infinite time regular synthesis*, ESAIM, Control Optim. Calc. Var. **3**, (1998), pp. 381–405.

[13] B. Piccoli, H. J. Sussmann, *Regular Synthesis and Sufficient Conditions for Optimality*, SIAM J. Control Optim., **39** (2000), pp. 359–410.

[14] P. Soravia, $\mathcal{H}_\infty$ *Control of Nonlinear Systems: Differential Games and Viscosity Solutions*, SIAM J. Control Optim., **34** (1996), pp. 1071–1097.