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Progress in Kaon Phenomenology from Lattice QCD

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Kaon physics is an area in which lattice QCD can have an important impact in Phenomenology. I review the recent progress and current status of three particular topics: kaon beta-decay, the kaon B-parameter and the non-leptonic decay of the kaon into two pions.

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1. Introduction

In this talk I will cover three topics in kaon phenomenolgy: kaon beta decay, $K^0 - \overline{K^0}$ mixing and $K \to \pi\pi$ decays.

2. Kaon beta decay

The kaon beta decay reactions $K^+ \to \pi^0 l^+ \nu_l$ and $K^0 \to \pi^- l^+ \nu_l$, where l represents either the electron or the muon, have an important role in determining the value of the CKM matrix element $|V_{us}|$ [1]. While recently the extraction of $|V_{us}|$ from the ratio f_K/f_π [2] has, due to the existence of 2+1 flavour dynamical calculations with impressive control over systematic errors [3, 4], become competitive in this regard, I will not discuss this topic, as the extraction of the decay constants and their relevance to the CKM matrix are covered in other talks at this conference ([5] and [6] respectively).

The decay rate for this class of processes reads

$$\Gamma(K_{l3}) = \frac{G_\mu^2}{192\pi^2} m_K^5 |V_{us}|^2 C_K^2 |f_+(0)|^2 I(f_+, f_-) \tag{2.1}$$

where C_K is a Clebsh-Gordan coefficient (1 for neutral kaon decays; $1/\sqrt{2}$ for charged), and f_+ and f_- are the form factors:

$$\langle \pi(p_f) | \bar{s}\gamma_\mu u | K(p_i) \rangle = (p_i + p_f)_\mu f_+(q) + q_\mu f_-(q) \; ; \; q = p_i - p_f \, . \tag{2.2}$$

The role of lattice calculations is to determine $f^+(0)$, with a sub-percent accuracy needed to be of phenomenological interest. While this may naively seem an extremely challenging target, it is not as stringent a constraint as might be first thought for the same reasons that kaon beta decay has long been the favoured method of extraction of $|V_{us}|$: a combination of vector-current conservation and the Ademello-Gatto theorem. The former implies that $f^+(0)$ is unity in the $SU(3)$ limit, while the latter implies that the deviations from unity are quadratic in the difference in quark masses. One consequence of this is that if we expand $f_+(q)$ in chiral perturbation theory,

$$f_+(q^2) = 1 + f_2 + f_4 + O(p^6) \, , \tag{2.3}$$

where the terms denoted f_i are $O\left((p/\Lambda_\chi)^i\right)$, with Λ_χ being the chiral scale, f_2 does not depend on any new low energy constants, and so can be completely determined in terms of the known meson masses and decay constants. Using these physical inputs this leads to a value of $f_2 = -0.023$. At the next order, however, terms quadratic in the quark mass are allowed and so f_4 does depend on new low energy constants. A full, two-loop, calculation of f_4 exists which [7], in principle, allows the value of f_4 to be extracted by studying the slope with respect to transfered momenta of the scalar form factor (defined later). However, better experimental resolution is need before this technique can become practical.

In lieu of such an extraction various approximation schemes have been used to estimate the value of f_4. A review of these results is beyond the scope of this talk, but it is useful – for a point of reference – to mention the extraction of Leutwyler and Roos [8], who, using a general

parameterisation of the meson wavefunction, together with an estimate of the generic size of $SU(3)$ symmetry breaking effects, get an estimate of

$$f_4(0) = -0.016(8) \tag{2.4}$$
$$f_+(0) = +0.961(8)\,. \tag{2.5}$$

This is the value currently quoted in the particle data-book [9].

The calculation of $f^+(0)$ is clearly amenable to methods of lattice QCD. However, to be practical it is necessary that, as with the case with continuum QCD, that the systematic errors in the calculation are suppressed by a factor proportional to $(m_s - m_d)^2$. In this case, rather than view the target as a 1% accuracy on $f^+(0)$, it is better to think of it as a $\sim 25\%$ error on $1 - f^+(0)$. A method of calculation on the lattice which has this quality was outlined in [10]: the so-called double ratio method. The starting point of this approach is to calculate the ratio

$$\frac{\langle \pi | \bar{s}\gamma_0 u | K \rangle \langle K | \bar{u}\gamma_0 s | \pi \rangle}{\langle \pi | \bar{u}\gamma_0 u | \pi \rangle \langle K | \bar{s}\gamma_0 s | K \rangle} = \left[f_0\left(q^2_{max}\right) \right]^2 \frac{(M_K + M_\pi)^2}{4 M_K M_\pi}\ ;\ q^2_{max} = (M_K - M_\pi)^2 \tag{2.6}$$

where

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{M_K^2 - M_\pi^2} f_-(q^2) \tag{2.7}$$

is the scalar form factor previously mentioned. In the $SU(3)$ limit this ratio is exactly unity and – because

$$f_0(0, M_\pi, M_K) = f_+(0, M_\pi, M_K)\,, \tag{2.8}$$

and there is no transfered momenta when $M_K = M_\pi$ – exactly equal to $f^+(0)$. It should also be noted that the ratio is invariant under interchange of s- and d-quarks, disallowing corrections proportional to $(m_s - m_d)$ appearing out of this limit. Further, it has been shown using chiral perturbation theory that that the Ademello-Gatto theorem still applies in the quenched approximation, and in particular that f_2 remains fully determined in terms of the meson masses and decay constants [1].

Equation 2.6 determines $f^0(q^2, M_K, M_\pi)$, not $f^+(0)$. To extract $f^+(0)$ it is needed to first extrapolate in q^2 to the $q^2 = 0$ point at fixed (non-physical) meson masses, then extrapolate the result to the physical masses. For the former extrapolation it is usual to use an ansatz of the form

- $f_0(q^2) = f_0(0)/(1 - \lambda^{(pol)} q^2)$
- $f_0(q^2) = f_0(0)(1 + \lambda^{(1)} q^2)$
- $f_0(q^2) = f_0(0)(1 + \lambda^{(2)} q^2 + c q^4)$

where $\lambda^{(pol)}$,$\lambda^{(1)}$, $\lambda^{(2)}$ and c are free parameters. For this to be possible, the transfered momenta must be varied independently of the meson masses. While such information could be extracted directly from the matrix element of the vector current between Kaon and Pion states which carry

[1] Since this talk was given this results has been extended to partially quenched chiral perturbation theory for both 2 and $2+1$ dynamical flavours.

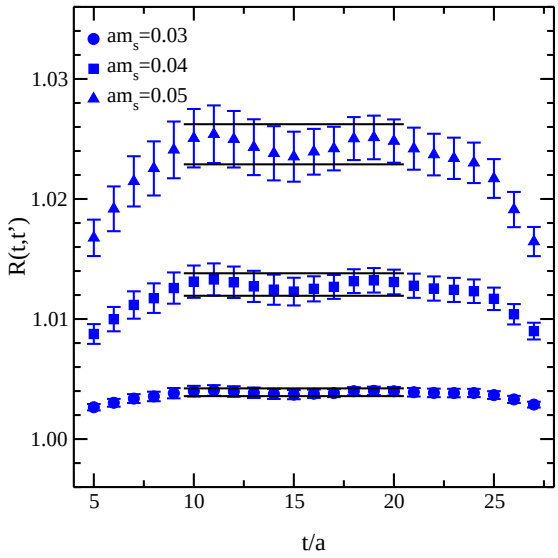

Figure 1: Example plateau the ratio in Equation 2.6. Taken from the RBC data.

momenta, as with the zero transfered momentum case, this approach would suffer from large systematic errors which do not vanish in the $SU(3)$ limit. However, as shown in [10], it is possible to construct double ratios of correlators that allow the extraction of

$$F(p,p') = \frac{f_+(q^2)}{f_0(q^2_{\rm max})}\left(1+\frac{E_K(p)-E_\pi(p')}{E_K(p)+E_\pi(p')}\,\xi(q^2)\right), \tag{2.9}$$

where p and p' refer to the kaon an pion momenta and $E_{\pi(K)}(p)$ is the energy of the pion (kaon) state with momenta p, and

$$\xi(q^2) = f_-(q^2)/f_+(q^2) \tag{2.10}$$

such that – in a similar fashion to the extraction of $f_O(q_{max})$ from Equation 2.6 – the systematic errors have a suppression factor of $(m_s-m_d)^2$. These may then be combined to extract $f_0(q^2)$.

In the past year the main improvements that have been made is the move away from the quenched approximation, and I will briefly review the two new results presented at this conference using two-flavour QCD [11, 12] and one which has appeared in the past year using 2+1 flavours of dynamical quarks [13]. First I will cover the work of the RBC collaboration [11]. This uses the $N_f = 2$, dynamical domain wall ensembles described in [14]. Briefly, these consist of a single lattice spacing of $a^{-1} \sim 1.7$GeV, a relatively small lattice volume ($16^3 \times 32$), three sea quark masses ranging from the strange quark mass to half of that, with 94 configurations for each sea-quark mass, although – at present – the analysis has only been completed for lightest of these. Figure 1 shows an example of the ratio in Equation 2.6 for several different values of the bare strange quark mass. As can be seen, as the ratio approached unity as $m_s \to 0.02$, which is the (bare) down quark mass used, with the deviations away from unity being statistically well resolved away from this limit.

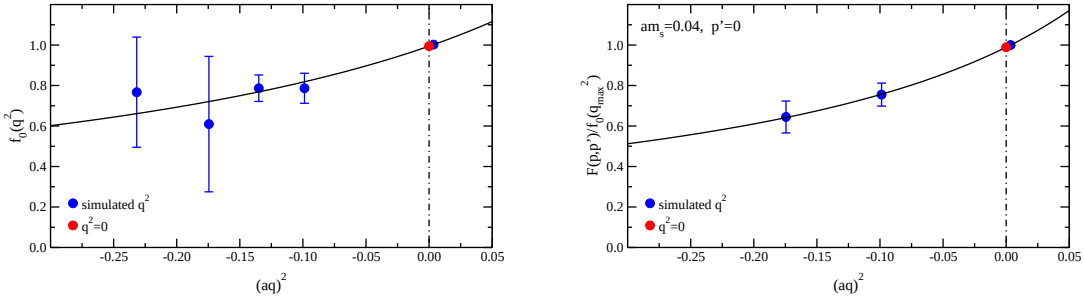

Figure 2: LEFT: The interpolation of $f_O(q^2)$ versus (Minkowski) q^2 to the physical point. RIGHT: The extrapolation of F to $q^2 = 0$

In the left-hand panel of Figure 2 I show the extrapolation of $f_0(q^2)$ versus (Minkowski) q^2. The point with $q^2 > 0$ is from the data with no momentum insertion in the meson sources. It is instructive to note both that, as would be expected, it has a much smaller statistical error-bar than the data where momentum is inserted, and it is the dominant point in the extrapolation to the $q^2 = 0$ limit. These nice properties (true for all the lattice calculations I will discuss) allow for a statistically accurate extraction of the $q^2 = 0$ point. It should be noted, however, that were lattice calculations to work with the physical values of the meson masses, a calculation with mesons at rest would be much further from the $q^2 = 0$ point. An alternate approach to the extrapolation – due to the CP-PACS collaboration [12] – which has been found to have a slightly smaller statistical error is to separately extrapolate F and ξ to $q^2 = 0$ and then combined these extrapolated values to give $f_+(0)$. An example of the extrapolation of F is shown in the right-hand panel of Figure 2.

Finally, $f^+(0)$ must be extrapolated to the physical masses. To take out the known quark mass dependence, it is usual to work with the quantity

$$R = \frac{\Delta f}{M_K^2 - M_\pi^2} \,. \tag{2.11}$$

where

$$\Delta f = f - 1 - f_2 \tag{2.12}$$

The mass dependence of this quantity will start at NNLO in chiral perturbation theory. One proviso to this point is that the form of f_2 is different for the quenched, two-flavour and full theories. The full theory result is long known, while the quenched form was calculated in [10]. However, the form for the remaining situation was not known until very recently [15], so the RBC work makes use of f_2 from full QCD, although it has since been checked that switching to the $N_f = 2$, partially quenched result only changes the answer within the statistical error. As discussed previously, any further mass dependence will show up at the two-loop level. While two-loop calculations for full QCD exist, this is not the case for the quenched, two-flavour or partially quenched theory and so an ansatz is used to extrapolate R to the physical point. The RBC work made use of two different fit forms:

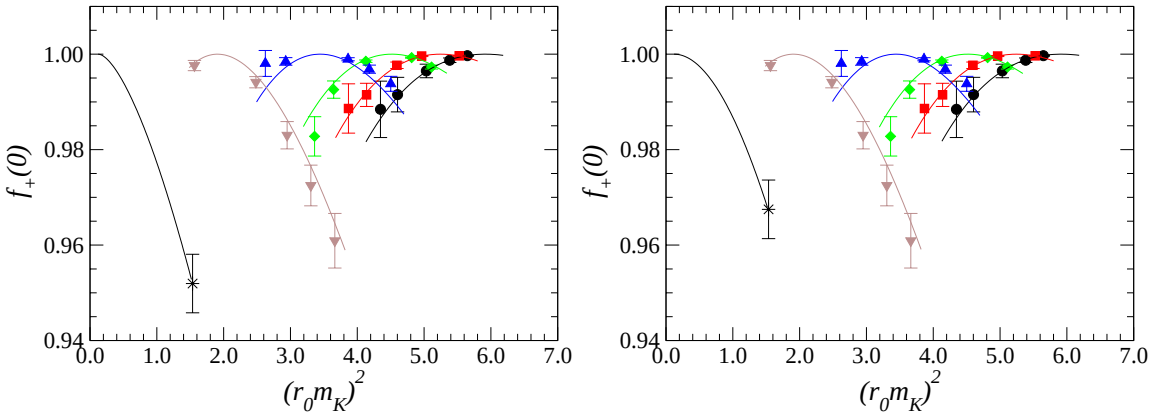

Figure 3: LEFT: Results of the fit to R including f_2 RIGHT: Results of the fit to R with $f_2 = 0$

1. $R = A + B(M_K^2 + M_\pi^2)$
2. $R = A + Bm_s$

with A and B as free parameters. These gives results of $f_+(0) = 0.955(12)$ and $f_+(0) = 0.966(6)$ respectively. It should be noted, however, that this is an extrapolation in the valence quark mass, and until the study is extended to more than one sea-quark mass any number should be counted as a ballpark estimate only.

The CP-PACS collaboration has also presented a result in two-flavour QCD at this conference [12]. Rather than Domain Wall Fermions, this calculation uses Wilson fermions. While numerically cheaper, one downside of this approach is that due to the lack of good chiral symmetry on the lattice, $O(a)$ corrections must be taken into account. While using an improved action will remove such terms from spectral quantities such as the meson masses, operators must also be improved. In particular the on-shell, $O(a)$ improved vector current is a sum of two operators, related by an improvement coefficient which must be determined. While the contribution of this second operator vanishes when the Kaon and pion are at rest, when they carry momenta this is not the case and a determination of this coefficient is required for this calculation.

The calculation uses the lattice described in [16] which were generated using the clover improved Wilson fermion action and the Wilson gauge action with $\beta = 5.2$. Five different sea-quark masses are used, giving rise to pseudo-scalar meson masses in the range 500 MeV to 1 GeV. In total 1200 configurations were used, with a separation of 10 HMC trajectories between each one. Aside from the introduction of the alternate method for the q^2 extrapolation described above, this work also compares two methods for the chiral extrapolation. These consist of either taking the value of f_2 from [15], or setting f_2 to zero and then fitting to

$$R = A + B(M_K^2 + M_\pi^2)\,. \tag{2.13}$$

The former of these is clearly the more rigorous approach, in that it deviates from the predicted chiral perturbation theory form at higher order in the pseudo-goldstone boson masses than the latter. However, as can be seen from Figure 3 (taken from [12]), both forms fit the data well, giving final results of $f_+(0) = 0.952(6)$ and $f_+(0) = 0.967(6)$ respectively. Since the masses used are fairly

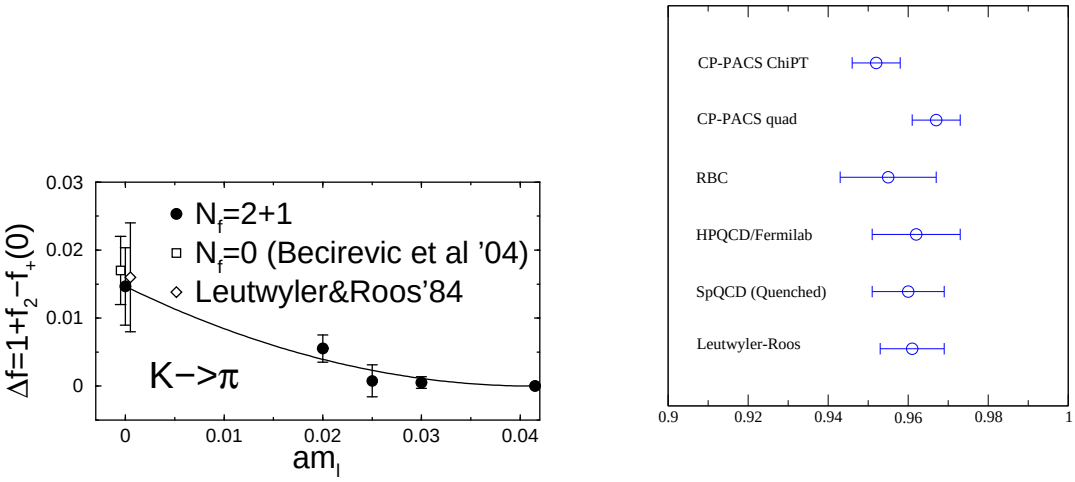

Figure 4: LEFT: The results of HPQCD/Fermilab collaboration RIGHT: The three new results collected together with the quenched results of Becirevic *et. al.* and the Leutwyler-Roos number.

large, and there is is no evidence from the data that the next-to-leading order chiral perturbation theory form is preferred, it is probably best to take the difference between these numbers as an indication of the potential size of the systematic errors in the extrapolation.

Finally, in [13], a calculation performed on the MILC 2+1 flavour gauge configurations by the HPQCD/Fermilab collaboration was reported. This used naive staggered quarks for the u- and d-quarks, and an improved Wilson formulation for the s-quark was reported. In contrast to the previous two calculations described, no attempt was made to extract the q^2 dependence of the form factors. Instead, this extrapolation was made by taking the pole form for the q^2 behaviour and the experimental number for the free coefficient, $\lambda^{(pol)}$. While not ideal, since the experimental value, $0.0278(7)$ [17], the quenched lattice value, $0.026(2)$ [10], and the two-flavour CP-PACS work presented at this conference, $0.021(2)$ [12], this seems a reasonable approximation for a preliminary extraction. The left-hand panel of Figure 4 – taken from [13] – shows the extrapolation of Δf in the light quark mass as

$$\Delta f = (A + B m_l)(m_s - m_l)^2 \,. \tag{2.14}$$

The final answer quoted is $f_+(0) = 0.962(6)(9)$.

In the right-hand panel of Figure 4, I summarise all the results covered. As can be seen, the various results are all fairly consistent both in central value, and in error-bar. This is true both amongst the various lattice results and between the lattice results and the Leutwyler-Roos number. It should be noted, however, that all the lattice results presented should be considered as preliminary. Non of the studies so far have presented a continuum extrapolation or a finite volume study. Only one of the studies includes the correct number of dynamical fermions, and that work took the parameters for the extrapolation to $q^2 = 0$ from experiment. Nevertheless, the fact that such different lattice studies are in good agreement with each other may – perhaps – be taken as

a sign that these systematic errors are dominated by the errors stemming from the extrapolations to the physical masses and momenta. This gives some hope that, using the lighter masses now possible with the latest generation of supercomputer, an improved number may be possible in the near future. It should be noted, however, that as the values of the masses are reduced, the extrapolation in momenta will become more problematic, as the calculation with the particles at rest will be further away from the $q^2 = 0$ point. One possible solution to this problem is to make use of (partially) twisted boundary conditions [18, 19, 20]. By adding a quark species dependent phase at the boundary of the lattice, a meson carrying any momentum may be constructed, and so the extrapolation distance reduced. Such a twist involves a breaking of flavour symmetry at the lattice boundary. Sachrajda and Villadoro [18] have studied this in chiral perturbation theory, coming to the conclusion that such flavour symmetry breaking effects are exponentially suppressed with the volume for physical quantities without final state interactions. More importantly, this remains the case for partial twisting (in which the valence quarks only are twisted), allowing many different momenta for one gauge background.

3. The Kaon B-parameter

B_K is the low energy matrix element relevant to indirect CP-violation in the $K^0 - \overline{K^0}$ system. After integrating out the W-boson and top, bottom, and charm quarks the experimentally measured measure of indirect cp-violation in the in the $K^0 - \overline{K^0}$ system, ε , can be written as

$$|\varepsilon| = C_\varepsilon A^2 \lambda^6 \overline{\eta} \left[-\eta_1 S(x_c) + \eta_2 S(x_t)(A^2\lambda^4(1-\overline{\rho}) + \eta_3 S(x_c, x_t) \right] \hat{B}_K \tag{3.1}$$

In this equation, C_ε, η_i ($i \in 1,2,3$) $S(x_c)$, $S(x_t)$ and $S(x_c, x_t)$ are perturbatively calculable factors containing the effect of the particles that have been integrated out. $\hat{B}_K$ is a hadronic matrix element – to be defined later – to be calculated in three-flavour QCD. Finally, A, λ, $\overline{\eta}$ and $\overline{\rho}$ are a priori unknown parameters of the CKM matrix in the Wolfenstein parameterisation. Since A and λ are well constrained from other experimental results, calculations of B_K are used to determine an ellipse in the $\overline{\rho} - \overline{\eta}$ plane.

When neither flavour nor chiral symmetry is explicitly broken there is one operator that contributes to B_K. It is of the form:

$$O_\Gamma = \bar{s}\Gamma_i d \; \bar{s}\Gamma_i d \tag{3.2}$$

with the gamma structure:

$$VV + AA \equiv \gamma_\mu \otimes \gamma_\mu + \gamma_\mu\gamma_5 \otimes \gamma_\mu\gamma_5 \; ; \tag{3.3}$$

which is simply the parity conserving part of $(V-A) \otimes (V-A)$. B_K itself is defined as

$$B_K = \frac{\langle \overline{K}^0 | O_{VV+AA} | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2} \tag{3.4}$$

$$= \frac{\langle \overline{K}^0 | O_{VV+AA} | K^0 \rangle}{\frac{8}{3} \langle \overline{K}^0 | A_0 | 0 \rangle \langle 0 | A_0 | K^0 \rangle} \tag{3.5}$$

where A_0 is the time component of the – partially conserved – axial current. B_K is most often quoted either renormalised in the NDR,$\overline{MS}$ scheme at 2 GeV, or as, the renormalisation group invariant, $\hat{B}_K$ ($\hat{B}_K \sim 1.4 B_K^{NDR,\overline{MS}}$).

In the past year there has been several new or updated results in the quenched approximation, together with a few preliminary results using dynamical fermions. While the quenched approximation is uncontrolled and so has an associated systematic error which is difficult to estimate with any confidence, it is useful for a comparison of various approaches to the calculation of B_K. For that reason I will first start with a review of the status of B_K in the quenched approximation, comparing the advantages and disadvantages of the various lattice actions, and discussing what we know of the common systematics such as finite volume and mass extrapolations.

3.1 Quenched Results

3.1.1 Staggered Fermions: broken flavour symmetry

Staggered Fermions, break both flavour symmetry and chiral symmetry, although in the latter case a residual $U(1)$ subgroup of this symmetry remains unbroken at finite lattice spacing. Even choosing the external states to be the mesons corresponding to this exact symmetry (the so-called lattice goldstone-boson), the main challenge of calculating B_K on the lattice using staggered fermions is that of resolving the operator mixing problem. If only symmetry arguments are used to constrain the various operators that could mix, then the breaking of flavour symmetry would lead to a huge number of such operators (a rough estimate is that there are $O(16^4)$). However, when working to a given order in perturbation theory, the problem becomes much more tractable. The standard (one-loop) calculation of the renormalisation factors involves a mixing between four different operators.

The most careful application of this procedure using naive staggered quarks is that due to Aoki *et al* [21]. This work has long been viewed as the benchmark quenched calculation, as 7 different lattice spacings were used (ranging from 0.025fm to 0.04fm) for lattice volumes $\sim (2.5\text{fm})^4$. At two of these lattice spacing a finite volume study was made. The results of this study are shown in the left-hand panel of Figure 5 (taken from [21]), plotted versus the rho-meson mass in lattice units. The two sets of points are from different operator definitions: one gauge invariant, one not. These should be equivalent up to errors due to the truncation of perturbation theory in the calculation of the mixing/renormalisation factors ($O(\alpha^2)$) and lattice artifact effects ($O(a^2)$). The fact that this is not the case is clear evidence that such effects are large, and the final results of

$$B_K(NDR, 2\text{GeV}) = 0.628(42) \tag{3.6}$$

is gained from a fit which includes both $O(a^2)$ and $O(\alpha^2)$ terms and is constrained to give the same continuum limit for both methods. As may be noted the final quoted error is much larger than the statistical errors on the individual points, this is due to the $O(\alpha^2)$ terms being both large and badly constrained[2]; The $O(a^2)$ are large, but statistically well resolved. In the following I will often compare newer results to these two data-sets, referred to as staggered one (invariant) and staggered two (non-invariant).

3.1.2 Improved Staggered Fermions

In recent years there has been much success using the approach of improved staggered fermions [22, 23]. While such improvements do not improved either the number of operators which may mix,

[2] this seems to because the relevant coeffi cient in the fi t is mainly constrained by the upwards "hook" of the invariant data, only noticeable at small lattice spacing.

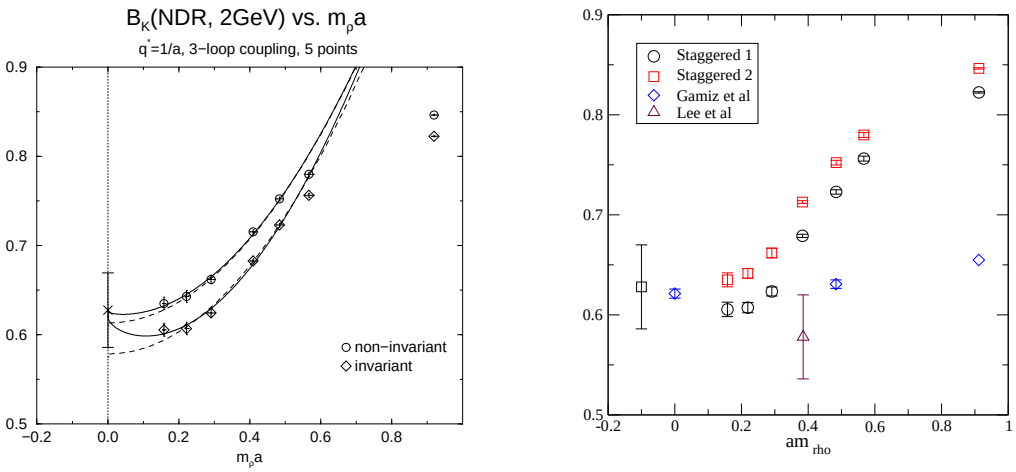

Figure 5: LEFT: $B_K(\overline{MS}, 2\text{GeV})$ versus the rho-meson mass in lattice units for the traditional (un-improved) staggered fermion calculation of the CP-PACS collaboration. RIGHT: The improved staggered results for $B_K(\overline{MS}, 2\text{GeV})$ together with the un-improved staggered results versus the rho-meson mass in lattice units.

or the order at which such mixing occurs, they are expected to reduce the size of such mixing and have better scaling with the lattice spacing. There are currently two results in the literature using this approach:

1. Gamiz *et. al.*[24], who used a Hyp smeared [25] fermion action, albeit with thin-link operators, using perturbative renormalisation factors calculated to one-loop. The results are shown in the right-hand panel of Figure 5, represented by the diamonds.

2. W. Lee *et al* [26, 27], who again used the Hyp smeared fermion action, with renormalisation factors calculated to one-loop in perturbation theory. Their single lattice spacing is shown in the right-hand panel of Figure 5, represented by the triangle.

I will concentrate on the (consistently renormalised) final result of W. Lee *et al*, which is

$$B_K(\overline{MS}, 2\text{GeV}) = 0.578 \pm 0.018 \pm 0.042 \tag{3.7}$$

The first error is from statistics plus the chiral extrapolation, while the latter is an estimate of the error due to the use of one-loop renormalisation factors. It is worth understanding the large size of this last error. The perturbative calculation used

$$\alpha_s(q^* = 1/a) = 0.192\,. \tag{3.8}$$

Simply estimating the truncation error to be of order $\pm 1 \times (\alpha_s(q^*))^2$ gives a $\sim 4\%$ error. The actual error quoted is around a factor of two larger, due to taking the largest deviation due to separately varying the coefficient of the four different operators by terms of order $\pm 1 \times (\alpha_s(q^*))^2$. While this is in no way a systematic way to calculate this error, it is a warning that the operator mixing problem with staggered fermions may magnify the error due to the truncation of the perturbative expansion.

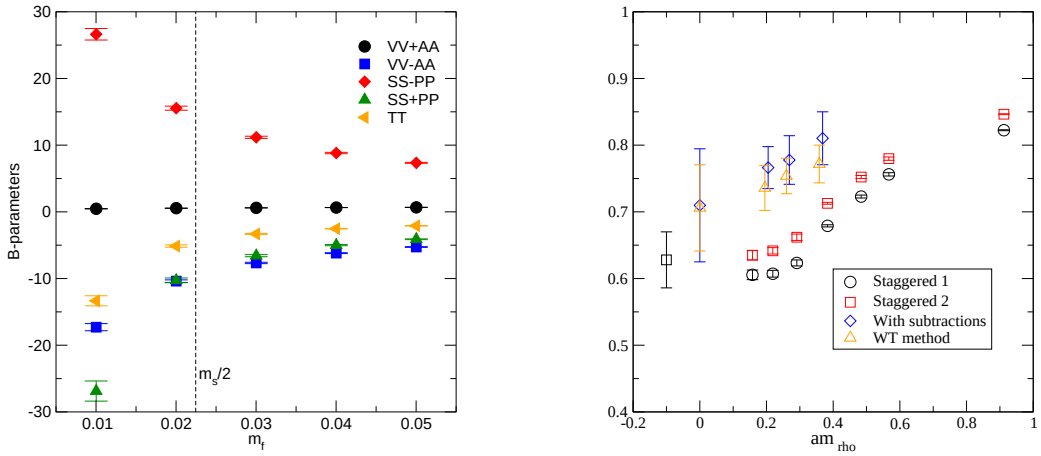

Figure 6: LEFT: The B-parameters the operators corresponding to all five possible chiral structures RIGHT: results for $B_K(\overline{MS}, 2\text{GeV})$ versus the rho-meson mass in lattice units for the Wilson action.

3.2 Wilson Fermions: Broken Chiral Symmetry

When chiral symmetry is (fully) broken then four other operators can mix under renormalisation with that in Equations 3.2 and 3.3: the other four possible gamma structures. The renormalised operator can then be written as:

$$\langle \overline{K}^0 | O_{VV+AA} | K^0 \rangle_{\text{latt}} = \frac{1}{Z_{VV+AA}} \left(\langle \overline{K}^0 | O_{VV+AA} | K^0 \rangle_{\text{ren}} + \sum_{i=1}^{3} c_i \langle \overline{K}^0 | O_{MIX,i} | K^0 \rangle_{\text{ren}} \right) \tag{3.9}$$

While this is a much smaller number of operators than for staggered fermions, this mixing problem is still very hard to control using perturbation theory due to the fact that the operator that mix have a different chiral structure. In fact, first order chiral perturbation theory predicts

$$\langle \overline{K}^0 | O_{VV+AA} | K^0 \rangle \propto M_k^2 \tag{3.10}$$

while

$$\langle \overline{K}^0 | O_{\text{MIX,i}} | K^0 \rangle \propto \text{constant}\,, \tag{3.11}$$

and so, for small enough mass these wrong chirality operators will dominate the matrix element measured on the lattices, requiring an extremely accurate determination of the mixing coefficients. Of course, for the extraction of B_K, we will be working with meson masses around that of the physical kaon. However, even at this relatively large mass, the matrix elements of the wrong chirality are large. The left-hand panel of Figure 6 shows a comparison of the various B-parameters taken from [28], with the wrong chirality ones being up to 30 times larger than the matrix element of interest around the masses we are interested in (where each quark has a mass of approximately half the strange quark mass).

Nevertheless, it is possible to attempt to extract B_K using Wilson fermions. In particular, I will here cover the work of Becirevic *et. al.* [29, 30, 31]. They take two approaches to the

problem. The first approach is to take into account the wrong-chirality operators, calculating the mixing coefficients using the non-perturbative renormalisation method of the Rome-Southampton group [32]. The second approach is to use the chiral Ward-Takahashi identities to relate the matrix element of interest to its parity partner:

$$2\langle KO_{VV+AA}K\rangle = 2m\int d^4x\langle P_5(x)KO_{VA+AV}K\rangle + O(a) + \cdots \tag{3.12}$$

B_K can then be calculated in term of the operator O_{VA+AV}, which renormalises multiplicatively due to the discrete CPS symmetry which remains unbroken even at finite lattice spacing. The matrix element on the right-hand side of Equation 3.12 is, however, much more challenging to calculate on the lattice. Figure 6 shows results for both methods for three different lattice spacings, ranging from $a^{-1} \sim 2$GeV to $a^{-1} \sim 3.5$GeV. As can be seen, while the results are consistent with each other (and the benchmark staggered result), the (statistical) error-bar for both methods is very large. While it seems impractical to reduce either of these error-bars to the point that these approaches would be competitive merely by increasing the statistics, the idea of relating the matrix element of interest to that of the opposite parity operator which is protected from operator mixing by discrete symmetries that are exact at finite lattice spacing is one that proves very useful when using twisted mass fermions.

3.2.1 Twisted-Mass Fermions

Twisted mass fermions [33, 34, 35], preserve neither flavour nor chiral symmetry. However, enough discrete symmetries are extant at finite lattice spacing that many of the nice properties that we would usually think of as consequences of chiral symmetry are, in fact, true. Examples of this are the absence of lattice spacing errors linear in the lattice spacing when working at full twist (although see the discussion [36, 37]) , and, as I will discuss, the multiplicative renormalisation of the operator relevant to the calculation of B_K.

Let us first consider the example of twisted u- and d-quarks, combined with a standard (Wilson) s-quark. Working in continuum, the action for this may be written as

$$\mathscr{L}_f = \overline{\psi}\left(\not{D} + m + i\mu_q\gamma_5\tau^3\right)\psi + \bar{s}\left(\not{D} + m_s\right)s \tag{3.13}$$

where $\psi = (u,d)$. If this is written in terms of twisted fields, $\Psi' = e^{i\alpha\gamma_5\tau^3/2}\Psi$ this action takes the more usual form. Of course, on the lattice the starting point is the Wilson action with a twisted mass term. When re-writing the action in terms of twisted fields, the action will take the standard continuum form plus a twisted Wilson term. It is the discrete symmetries of this lattice action that are exploited to ease the calculation of B_K. However, for the overview I will give here I will neglect these details, which can be found in [33]. The relationship between the operator relevant to B_K in the twisted and non-twisted bases is

$$O'_{VV+AA} = \cos(\alpha)O_{VV+AA} - i\sin(\alpha)O_{VA+AV} \tag{3.14}$$

which, for maximal twist ($\alpha = \pi/2$), reads

$$O'_{VV+AA} = -iO_{VA+AV} \tag{3.15}$$

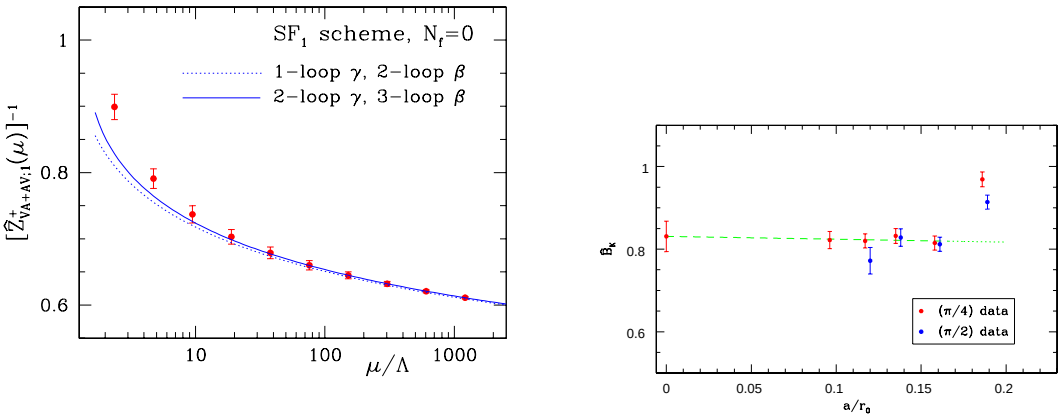

Figure 7: LEFT: results from the Schrodinger functional step scaling study RIGHT: preliminary results for $\hat{B}_K$ from both methods for twisted mass fermions.

The right-hand side of this equation is the same operator as used in the Ward-Takahashi identity method discussed in the previous sections, and is similarly protected from mixing under renormalisation by discrete symmetries that are exactly preserved at finite lattice spacing.

As second approach that can be taken is to twist the s- and d- quarks. Since this means twisting all four quark fields in the expression for O_{VV+AA}, rather than the two that were twisted in the previous method, the relationship between the operators in the two bases now reads:

$$O'_{VV+AA} = \cos(2\alpha)O_{VV+AA} - i\sin(2\alpha)O_{VA+AV} \tag{3.16}$$

Taking $\alpha = \pi/4$, gives multiplicative renormalisation. However, it should be noted that as this is not maximal twist, the $O(a)$ discretisation error again do not cancel, and that twisting the quarks in this way forces them to have equal mass. Finally, a third approach is put forward by Frezzotti and Rossi in [38]. They consider a theory with active charm. With this additional quark species they are able to construct combinations of twists such that both multiplicative renormalisation and $O(a)$ improvement are possible.

Moving to numerical results, the Alpha collaboration has presented preliminary results from a calculation using both methods one and two [39]. Combined with the calculation of the matrix elements, they have also completed a non-perturbative calculation of the renormalisation factors using the Schrodinger functional approach [40]. Such a calculation requires both a non-perturbative step-scaling study, matching different lattice volumes until a volume is reached for which $\mu = 1/L$ is firmly in the perturbative regime, and a perturbative matching calculation between this finite volume scheme and the more familiar $\overline{MS}$-scheme. These are provided in [41] and [42] respectively, and are discussed at this conference in the talk by Stefan Sint [43]. One notable advantage of this approach compared to the non-perturbative renormalisation technique of the Rome-Southampton group[32] is that it only requires gauge-fixing for the perturbative calculation, not on the lattice, and thus it avoid the lattice Gribov-copy problem. The left-hand panel of Figure 7 shows some

results of this step scaling study, compared to the predicted running of perturbation theory. The right-hand panel shows the results for $\hat{B}_K$ (again taken from [39]) versus lattice spacing divided by the Sommer parameter for both the methods. These data-points correspond to lattice spacings in the range 0.1fm to 0.05fm. The lattices used were of size $16^3 \times 48$ to $32^3 \times 72$, and were generated using the Wilson gauge action. As can be seen, aside from a strange jump for the largest lattice spacing (which the authors ascribe to known large discretisation error), the results seem to scale very well with lattice spacing, leading to a preliminary result for B_K of

$$B_K(\overline{MS}, 2\text{GeV}) = 0.604(27)\,. \tag{3.17}$$

3.2.2 Domain Wall Fermions

Domain wall fermions [44, 45], have exact vector symmetry and greatly suppressed chiral symmetry breaking at the expense of adding an extra, fifth, dimension to the lattice action. Since the matrix elements of the wrong chirality operators are as much as 30 times larger than the matrix element of interest it is necessary to have some estimate of the precise size of this suppression. In particular, if the suppression factor is large enough that only a single operator needs to be calculated to measure B_K.

A simple model to estimate the order of magnitude of the chiral symmetry breaking effects is to associate a suppression factor for each "trip" through the bulk of the fifth dimension which separates the walls on which the right- and left-handed components of the quark field are localised. The additive shift in the mass caused by the fact that these two wall are not infinitely separated is quantified by the residual mass, am_{res}. This would naturally be thought to be dominated by the contribution from a single trip through the bulk, and so may be taken as an estimate of the size of the associated suppression. Moving to the problem at hand, it is simple to note that the operator of interest, due to it's $(V-A)^2$ spinor structure, is made up of four left-handed quark fields. The wrong chirality operator with which it can mix each consist of two right-handed and two left-handed quark fields. It therefore requires two flips of chirality to mix these operators, and the suppression factor due to mixing can be estimated to be $O\left((am_{\text{res}})^2\right)$. For the quenched data discussed, this is at most $O(10^{-6})$, enough that the mixing with wrong chirality operators can be neglected.

As pointed out by Golterman and Shamir, the above model is too simple [46, 47, 48]. Their approach is to consider the transfer matrix constructed in [44] to describe propagation in the fifth dimension. This may be written

$$T = \frac{1-H}{1+H} \tag{3.18}$$

where

$$H = \frac{H_W}{2+\gamma_5 H_W} \tag{3.19}$$

with $H_W = \gamma_5 D$ being the Hermitian Wilson Dirac operator. They identify two distinct source of chiral symmetry breaking associated with two different types of eigenvalue/eigenvector of H:

- Extended Modes with eigenvalues above a certain critical value which are spatially extended. This critical value is referred to as the mobility edge, λ_C, and the contribution of these modes to the chiral symmetry breaking is expected to fall off as $\exp(-\lambda_C L_s)$, when L_s – the size of the fifth dimension – is large.

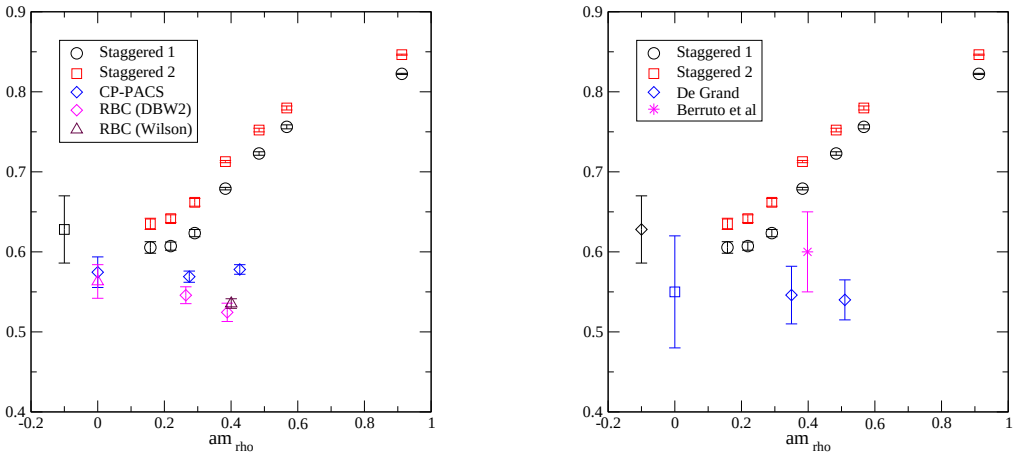

Figure 8: LEFT: Domain Wall Fermion results for $B_K(\overline{MS}, 2\text{GeV})$ versus the rho-meson mass in lattice units. RIGHT: Overlap Fermion results for $B_K(\overline{MS}, 2\text{GeV})$ versus the rho-meson mass in lattice units

- Localised modes, including zero-modes of H, which give rise to unsuppressed propagation in the fifth dimension. It is the presence of such modes, in particular, that calls into question the simple model presented above.

Both the value of λ_C and the density of localised modes is strongly dependent on both the lattice spacing and gauge action being used. For coarse enough lattice spacing the lattice will undergo a localisation transition (λ_C will go to zero) and the Domain Wall Mechanism will fail [48, 46]. These issues are addressed in two talks at this conference. The relative size of the two types of contribution for realistic (dynamical) lattices has been studied by Peter Boyle [49]. Furthermore, Norman Christ [50], has used the transfer matrix formulation of DWF to more concretely show how chiral symmetry breaking occurs. When applied to the case of the mixing problem for the calculation of B_K his conclusion is that the wrong chirality mixing is indeed suppressed by a factor of $O\left((am_{\text{res}})^2\right)$, making it numerically practical to neglect such operator mixings (as has been done in all Domain Wall Fermion calculations to date).

Two groups have performed calculations using domain wall fermions: the CP-PACS collaboration [51], and the RBC collaboration [28]. While using the same fermionic action, these two calculations are quite dissimilar: The RBC calculation uses the non-perturbative renormalisation method of the Rome-Southampton group [32], a combination of the Wilson and DBW2 [52] gauge actions, and a relatively small lattice spatial extent of ~ 1.5fm (using a combination of periodic and anti-periodic boundary condition for the fermion propagators to effectively "double" the lattice in the time direction). The CP-PACS calculation, however, uses a perturbative renormalisation calculation (performed in the infinite L_s limit), the Iwasaki gauge action [53], and larger lattice volumes of spatial extent ~ 2.5fm. They also performed studies of the dependence on lattice volume and extent of the fifth dimension. As can be seen in the right-hand panel of Figure 8, however, when the continuum limit of the two calculations are compared, there is good agreement.

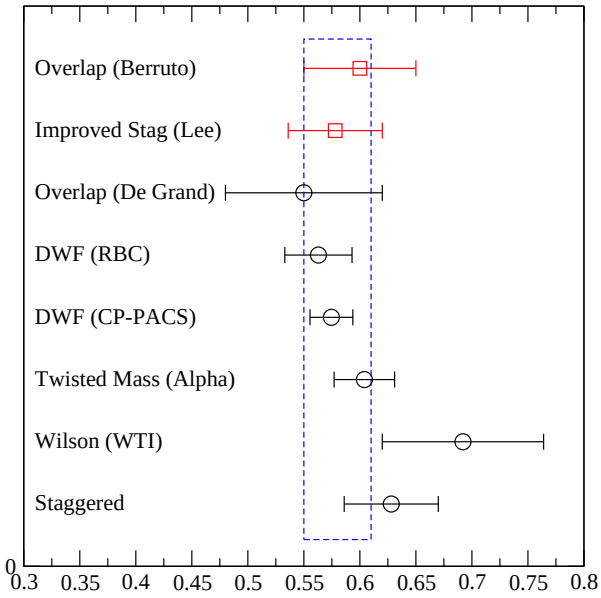

Figure 9: Summary plot for quenched, degenerate mass $B_K(\overline{MS}, 2\text{GeV})$

3.2.3 Overlap Fermions

Finally, I will quickly discuss two calculations which use overlap fermions [54, 55]. Overlap fermions have exact flavour and chiral symmetries at finite lattice spacing, and so have no operator mixing problem. As with Domain Wall Fermions, however, this convenience come at the expense of computer time. Two groups have undertaken calculation of B_K, and the right-hand panel of Figure 8 displays the results. The diamonds represent the work of DeGrand [56] which uses perturbative renormalisation, two lattice spacings of ~ 0.09fm and 0.13fm, on lattices of size $12^3 \times 36$ and $16^3 \times 48$. These lattices were generated using the Wilson gauge action, then Hyp smeared. In contrast the work of Berruto *et. al.*[57, 58] (represented by the star) used the non-perturbative renormalisation technique of the Rome-Southampton group, a single lattice spacing of ~ 0.1fm, and two lattice volumes of $16^3 \times 32$ and $18^3 \times 64$ (which give consistent results for the relevant region of quark masses). In comparison to the Domain Wall fermion results, the error-bars are quite large. However, this seems more likely to stem from the particular set-up used (choice of sources, number of configurations etc.), rather than some intrinsic problem with Overlap fermions.

3.2.4 Comparison

Figure 9 shows a comparison of the quenched, degenerate mass calculations for B_K I have mentioned. The points denoted by red squares are those calculations for which no continuum

limit has been taken, while for those denoted by black circles, some type of extrapolation was made. The error-bars plotted are from adding the quoted errors in quadrature. As can bee seen, the agreement amongst the various calculations is impressive. A simple error-weighted average of all the continuum extrapolated values gives $B_K(2\text{GeV}) = 0.587(13)$, while a restriction to those calculations that both include $O(a^2)$ terms in the continuum extrapolation and are published, gives $B_K(2\text{GeV}) = 0.582(17)$, both with a χ^2/dof of approximately one. Since the errors are not all statistical, it is sensible to assign a larger error-bar, and I would suggest a quenched average of

$$B_K(\overline{MS}, 2\text{GeV}) = 0.58(3) \tag{3.20}$$

(c.f. the value put together by Shoji Hashimoto for ICHEP 2004 [59]: $B_K(\overline{MS}, 2\text{GeV}) = 0.58(4)$). Besides the continuum extrapolation, there are two sources of systematic error that should, in particular, be mentioned: chiral extrapolation and finite volume effects. The numerical calculations are generally not performed at the needed masses, with an interpolation being needed to the physical point. It is natural to attempt to do this using (next-to-leading order) chiral perturbation theory. However, there is mixed success in fitting the lattice data to this form. While, for example, the Domain Wall Fermion calculation of the RBC group fits this form quite well, the Overlap calculation of Berruto *et. al.*does not. This could be due to different discretisation errors between the two approaches, or even different finite volume effects (the RBC work is at relatively small volume). Even though the RBC data can be fit to the chiral perturbation theory form, stronger evidence (from smaller masses on larger volumes, for example) is needed before it can be concluded that next-to-leading order chiral perturbation theory is a good description of B_K around the kaon mass. It should be noted, however, that the form used for the fitting has very little impact on the final result as – in the quenched approximation – we are able, in most calculations, to interpolate to the physical point. As for the size of finite volume effects around the physical mass-point, these are generally found to be small when the lattice spatial extent is above ~ 2fm.

3.3 Dynamical fermions

The dominant error on the value of B_K is a guess of the size of the quenching error, and it is vital to move beyond this approximation. Several groups have taken an important step towards this; I will show results from two groups who have published results in the two-flavour theory, and three groups who have presented work using $2+1$ flavours at this conference.

First, two-flavours of dynamical quark. The first calculation I will discuss is that of the UKQCD collaboration using Wilson fermions [60]. This work uses perturbatively calculated mixing factors, working with large quark masses in an attempt to work in a regime where the cancellation between the various matrix elements occurring in the mixing calculation is under better numerical control. This makes the extrapolation to the physical masses problematic, as chiral perturbation theory cannot be trusted in this regime. However, given the maturity of dynamical calculations for B_K, it is an interesting question to simply ask which direction B_K moves as the dynamical mass is decreased towards the physical value, with the evidence suggesting that B_K decreases as the dynamical (up and down) quark masses are decreased.

Also in the two-flavour theory (and therefore with a quenched strange quark) is the work of the RBC collaboration [14]. Similarly with the quenched case, this used domain wall fermions in

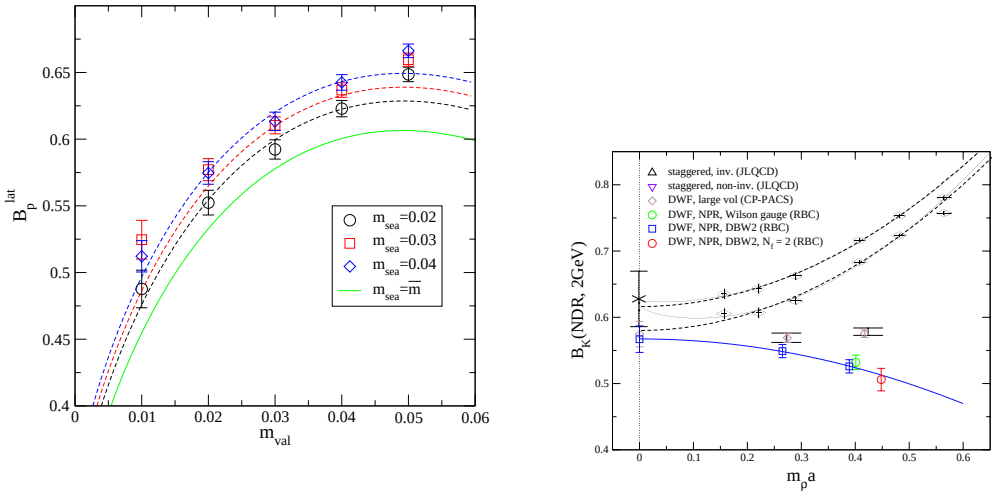

Figure 10: LEFT: Bare B_K versus the (degenerate) valence quark mass for the three dynamical ensembles RIGHT: $B_K(\overline{MS}, 2\text{GeV})$ for the two-flavour dynamical Domain Wall Fermions work, together with the quenched domain wall and unimproved staggered results.

combination with the non-perturbative renormalisation technique of the Rome-Southampton group. Three dynamical ensembles were used, with masses of $\sim m_s/2$, $\sim 3m_s/4$ and $\sim m_s$; a single lattice spacing of $a^{-1} \sim 1.7$GeV on a lattice volume of $16^3 \times 32$. In contrast to previous work, this study also used non-degenerate masses for the s- and d- quarks. The final result quoted was

$$B_K(\overline{MS}, 2\text{GeV}) = 0.495(18) \tag{3.21}$$

where the latter error is *statistical* only, and an analysis working only with degenerate quarks was a statistically well resolved 3% higher. While this number is dramatically lower than the quenched results, caution must be taken, as the quoted error is statistical only, and there are various sources of systematic error. Again, a firmer result may be the trend with dynamical mass, rather than the central value quoted. The left-hand panel of Figure 10 shows the degenerate valence quark mass values for B_K for each of the ensembles. As with the quenched work, chiral perturbation theory can be used to move to the physical point. However, in this case it must be used to extrapolate from $\sim m_s/2$ to the up and down quark masses, rather than interpolate over a small range. While there is no trend resolved for the heaviest two dynamical masses, the data for the lightest dynamical mass is clearly lower with the relevant chiral perturbation theory coefficient being non-zero by $\sim 2\sigma$. The right-hand panel of Figure 10 shows this $N_f = 2$ result, the results of the RBC using the same formalism in the quenched approximation, and the quenched staggered results of CP-PACS. The solid line shows the central value for the lattice spacing extrapolation of the quenched RBC data. As can be seen, the two-flavour point is consistent with the quenched data once we extrapolate that data to the same lattice spacing. While there is no reason for the quenched and two-flavour theories to scale with lattice spacing with the same coefficient, this is a good demonstration that there is a untreated systematic in the two-flavour calculation that could plausibly be of the same order as the shift in central value observed.

Moving to $2+1$ flavours, Saul Cohen [61] has presented preliminary results for a calculation of B_K on configurations generated with dynamical Domain Wall Fermions. The work presented was based upon lattices very similar to the RBC two-flavour lattices discussed previously: $16^3 \times 32$ lattices with a lattice spacing of ~ 0.12fm. As finite volume effects are expected to be larger for dynamical fermions this volume is a little small, and in many ways this calculation should be considered a warm-up for a later calculation of the large ($24^3 \times 64$) lattices now being jointly generated by the RBC and UKQCD collaborations. While the renormalisation has not yet been completed, using the two-flavour renormalisation factors as a guide shows the final result being roughly consistent with the two-flavour work at a comparable lattice spacing.

Taegil Bae, JongJeong Kim and Weonjong Lee [62, 63] have presented preliminary results of B_K on the MILC collaboration $2+1$ flavour ensembles [64] ($20^3 \times 32$ lattices with $a \sim 1.25$fm), using a mixed action approach: Hyp-smeared operators, a^2-tad background. This calculation is currently ongoing, with only a tree-level matching of the operators being currently used. A fit of this raw data to continuum next-to-leading order chiral perturbation theory shows a deviation from the predicted logarithm. This is consistent with the predictions of the next-to-leading order staggered chiral perturbation theory calculation to be discussed later, but certainly – considering the small size of the effect – it does not provide conclusive evidence that the chiral perturbation theory calculation is valid in this mass regime.

Elvira Gamiz [65] has presented the results of a calculation using improved staggered calculation. Again, these are based on the "coarse" MILC configurations [64] with $a \sim 0.125\,fm$, and have two different light dynamical masses of $\sim m_s/2$ and $\sim m_s/4$, degenerate s- and d-quark masses, and make use of the newly completed one-loop renormalisation calculation for the a^2-tad action [66]. The final result is

$$B_K(\overline{MS}, 2\text{GeV}) = 0.630(18)(15)(30)(130). \tag{3.22}$$

The errors quoted are from statistics, the chiral fit, discretisation errors (estimated by from the quenched approximation), and from the perturbative matching factors. Clearly, the latter error is the dominant one. While this was also the case for the quenched improved staggered calculation of Lee *et. al.*[26, 27], the error quoted here is much larger. The reason for this is that $\alpha_V(q^* = 1/a)$ is simply larger due to a combination of coarser lattices and the inclusion of dynamical quarks. It should also be noted that the error-bar of Lee *et. al.*was inflated by approximately a factor of two due to an attempt to take the effects of operator mixing into account, while no such attempt is made in this estimate.

To make an improved estimate of B_K using staggered quarks, the obvious next step is to perform either the required two-loop perturbative calculation, or apply some non-perturbative renormalisation technique[3]. Lacking (or in conjunction with) such a calculation, an alternative approach is fit for the higher order terms from the lattice data, and then subtract their contribution. It is now possible to do this in a systematic fashion thanks to a recent calculation of B_K at NLO in staggered chiral perturbation theory by Van de Water and Sharpe [67], presented at this conference by Ruth Van de Water [68]. For such a calculation to be possible it is first necessary to construct a power counting scheme to decide which operators can occur at a given order in the expansion. Since the

[3]Moving to a fi ner lattice will also help, but is not enough alone.

size of discretisation errors and mixing coefficients depend on the details of the lattice calculation, it is clearly impossible to construct a power counting scheme that will apply to any lattice calculation. Van de Water and Sharpe therefore construct a phenomenological power counting scheme adapted to the lattice parameters in current use. It may be expressed as

$$p^2 \sim m \sim \frac{\alpha}{4\pi} \sim \alpha^2 \sim a^2 \sim a_\alpha^2 \,. \tag{3.23}$$

Here, the first relationship – that p^2, the squared meson momenta, is of the same order as m, the quark mass – is simply the standard power counting of chiral perturbation theory. The known one-loop matching coefficients are all of $O(1) \times \frac{\alpha}{4\pi}$. The strong coupling constant does, of course, depend upon some choice of scale which, as we are studying a lattice artifact, is most suitably chosen to be $\alpha_V(q^* = 1/a)$. For the lattice parameters being considered, this is large enough that it should be included at NLO. No two-loop matching calculation has been performed, but naively such terms would be expected to be $O(\alpha_V(q^* = 1/a)^2)$. Since $\alpha_V(q^* = 1/a)$ is roughly in the range $0.2 - 0.6$ for present lattices, while suppressed by an extra factor of α, the two-loop terms should be considered at the same order in chiral perturbation theory as the one-loop terms (In fact, one may worry that three loop terms are also needed). The final two terms that appear in the power counting relationship are the standard discretisation errors coming from the action ($O(a^2)$) and the taste-breaking discretisation errors. The latter are nominally of order $a_\alpha^2 = a^2\alpha^2$ and so consistency would suggest they should be treated as NNLO. However, they are found to be numerically large and so, as is standard in staggered chiral perturbation theory, they are included at NLO.

The upshot is that 13 chiral operators contribute to B_K at NLO in this power counting scheme, some with multiple independent coefficients, corresponding to different orders in a and α. A full fit to lattice data from several lattice spacing would require the extraction of 37 free parameters. This is clearly a daunting task, but there are several way to make it simpler: Firstly, any term that is included in the matching calculation need not be included in the fit to chiral perturbation theory. One interesting result of the work of Van de Water and Sharpe is that, in their power-counting scheme, the taste-breaking operators that mix at one-loop in perturbation theory (usually neglected, as they are thought to be small) contribute at the same order as the taste-diagonal terms. Adding in these operators would be comparatively simple. A more challenging approach is to perform the full two-loop matching calculation. An alternate (or complimentary) approach, suggested by Van de Water and Sharpe, is to perform auxiliary calculations of wrong taste operators to pin down some of the coefficients. Working at fixed lattice spacing and for degenerate s- and d-quarks, this can reduce the number of free parameters to 4. Dropping any term which is "purely an error", this would leave a final – 4 parameter – fit to the lattice spacing dependence.

3.4 Conclusions

To summarise, I would say that B_K is in good control in the quenched approximation. Various groups have used different fermion actions, different renormalisation techniques, and different approaches to treating the systematic errors due to finite lattice spacing and finite volume, with the final results agreeing well between the various error-bars. A combined results for quenched B_K with degenerate s- and d-quark masses from the extractions described is

$$B_K(\overline{MS}, 2\text{GeV}) = 0.58(3) \,. \tag{3.24}$$

Several dynamical extractions were also described. However, these studies are not yet at the level of maturity of those in the quenched approximation. (For example, no calculation has yet to include more than one lattice spacing). We can, however, use these calculations to gleam some estimate of the systematic error due to quenching. To be precise, by comparing the $N_F = 2$ and quenched extractions of the RBC collaboration, which take exactly the same approach aside from the inclusion of dynamical quarks, we see no difference in the two values if we assume that the lattice spacing errors between the two studies are the same. While there is no reason for a quenched and dynamical calculation to have exactly the same scaling properties, it is reasonable to assume that the size of the lattice artifacts is roughly the same.

It is needless to point out that there is only one good way to estimate a quenching error: perform a dynamical calculation. The 2+1 flavour dynamical results presented at this conference, while preliminary, are good evidence that full QCD calculations, with good control over the systematic errors, should be available in the foreseeable future. However, until that time, it is sensible to take the degenerate mass, quenched average as a central value for B_K from the lattice. The degenerate and non-degenerate numbers from the RBC two-flavour work differ by 3%, so assigning a 5% error due to degeneracy seems reasonable. Assigning and error due to quenching is, of course, a black art. Here I will follow Steve Sharpe [69], and after noting that since the difference between quenched and dynamical for the RBC DWF work seems to be of the order of a plausible estimate of the errors due to finite lattice spacing, take the size of these errors is a conservative estimate of the systematic error due to quenching. This leads to a 10% error. Adding these errors in quadrature gives

$$B_K(\overline{MS}, 2\text{GeV}) = 0.58(3)(6)\,. \tag{3.25}$$

where the first error is on the degenerate, quenched value, and the second takes in account non-degeneracy and dynamical quark effects.

4. $K \to \pi\pi$

The calculation of $K \to \pi\pi$ decays has long being a goal of lattice QCD. To be precise, the decays of interest are: $K^+ \to \pi^+\pi^0$, $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$. The amplitudes for these decays can be split up into isospin 2 and isospin 0 final states:

$$\mathscr{A}\left(K^+ \to \pi^+\pi^0\right) = \sqrt{3/2}A_2 e^{i\delta_2} \tag{4.1}$$

$$\mathscr{A}\left(K^0 \to \pi^+\pi^-\right) = \sqrt{2/3}A_0 e^{i\delta_2} + \sqrt{1/3}A_2 e^{i\delta_2} \tag{4.2}$$

$$\mathscr{A}\left(K^0 \to \pi^0\pi^0\right) = \sqrt{2/3}A_0 e^{i\delta_0} + \sqrt{4/3}A_2 e^{i\delta_2}\,, \tag{4.3}$$

with δ_2 being a strong phase. It is of great interest to both reproduce the long observed enhancement of decays into the ispospin zero final state over the isospin one final state – the so-called $\Delta I = 1/2$ rule, – and determine the ratio of direct to indirect cp-violation in the kaon system in terms of the parameters of the CKM matrix by measuring ε'/ε, where

$$\varepsilon' = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]. \tag{4.4}$$

Calculating $K \to \pi\pi$ is a very difficult problem on the lattice for two main reasons: operator renormalisation, and the need to treat two particles in the final state. The standard approach to the latter is to actually calculate K to π and K to vacuum, then use chiral perturbation theory to move to the result for $K \to \pi\pi$. There has been progress in both these topics over the past year.

4.1 Operator mixing for Twisted mass QCD

After integrating out the weak force and top, bottom and charm quarks there are 7 dimension six (four quark) operators in the $\Delta S = 1$ effective Hamiltonian. These operators are a mixture of $\Delta I = 3/2$ and $\Delta I = 1/2$ pieces. To make a multiplet of the former requires four-quarks, the latter only two and so the $\Delta I = 1/2$ operators can mix with lower dimensional operators. This is a particularly serious problem, as the mixing coefficients for these operators can diverge like a power of $1/a$, requiring some sort of non-perturbative subtraction technique. Fortunately, all possible lower dimensional operators have a different chiral structure than the four-quark operators and so the mixing is suppressed by at least one factor of the mass. To be precise, with chiral symmetry, the only lower dimensional operator that can still have a power divergent coefficient takes the form:

$$(m_s + m_s)\,\bar{s}d + (m_s - m_d)\bar{s}\gamma_5 d \tag{4.5}$$

Without chiral symmetry the factor of $m_s + m_d$ lost. The factor $m_s - m_d$, however, remains, as it is also a prediction of the discrete CPS symmetry[70] which is remains unbroken. Furthermore, if a dynamical charm quark is included then the GIM mechanism can give an extra factor of

$$(m_c^2 - m_u^2)\ , \tag{4.6}$$

with a chirally symmetric actions,

$$(m_c - m_u) \tag{4.7}$$

without. Therefore, if chiral symmetry is only softly broken by the lattice action used, and a dynamical charm quark is included, then their are enough powers of the mass to make the contribution from this operator the same dimension as the four-quark operators. In this, clearly desirable, situation there is no power divergence. For this reason, action with good chiral properties have been vital for such calculations.

In [38], Frezzotti and Rossi have shown how to use the discrete symmetries of twisted mass QCD to enforce the above relationships (this is presented at this conference by Roberto Frezzotti [71]). They are able to construct a scheme that kills all the power divergent mixings for either parity for quantities protected by the GIM mechanism, while at the same time disallowing $O(a)$ terms. Obviously, for this to be possible, there must be four twisted mass quarks.

One possible stumbling block for this approach is that the determinant of the non-degenerate twisted mass QCD Dirac operator is not provably positive for the entire range of possible input bare masses [72]. It is not a priori clear that the bare masses corresponding to the physical quark masses are such that the determinant is positive (and so a dynamical simulation possible).

4.2 Calculating $K \to \pi\pi$

The state of the art calculations of $K \to \pi\pi$ are those of the RBC [73] and CP-PACS [74] collaborations, both using Domain Wall Fermions. The biggest systematics of these calculations are:

- The use of the quenched approximation. While the quenched approximation always introduces an uncontrolled systematic error into a calculation, for the $\Delta S = 1$ Hamiltonian, and, in particular, the $\Delta I = 1/2$ operators the problem is much worse.

- Using chiral perturbations to move from $K \to 0$ and $K \to \pi$, to $K \to \pi\pi$. This only works in leading order chiral perturbation theory. At next-to-leading order there is no way to move directly between these correlators. Instead some information is needed from $K \to \pi\pi$ lattice calculations, although it need not be at the physical kinematics.

The first of these points has been partially addressed by the talk of Junichi Noaki at this conference [75], in which he discusses a calculation of $K \to \pi\pi$ from $K \to 0$ and $K \to \pi$ on the RBC two-flavour Dynamical Domain Wall fermion ensembles, concentrating on the $\Delta I = 3/2$ matrix elements. As to the second point: The main problem with putting two particles in the final state of a lattice calculations is the so-called Maiani-Testa no-go theorem. Simply explained, this is the statement that the lowest energy state of the two pion interpolating operator is two-pions at rest. However, the physical decay we are looking for is $K \to \pi(p)\pi(-p)$, not $K \to \pi(0)\pi(0)$. Moreover, working in the continuum (but in Euclidean space), the spectrum of two pion states is continuous. Extracting the wished for state from this continuous spectrum using numerical data is not possible. A way around these problems was provided by Lellouch and Luscher in [76]. Firstly, they pointed out that in a finite volume the spectrum of two-particle states is discrete, not continuous, so the wished for decay can be extracted by using an excited state fit. Secondly, they provided a formula for connecting the decay measured in a finite volume to the infinite volume result in the centre of mass frame (total momentum zero).

Applying the technique of Lellouch and Luscher, however, is challenging due to the need to extract the excited state. An alternative approach is to work with a kinematic set-up for which the final state of interest is also the lowest energy state. This can be done by working in the lab frame (total momentum P). An example of such a calculation is that of Boucaud *et. al.*[77], which uses the SPQR kinematics[78, 79]: the kaon and one of the pions are at rest, with the other pion at finite spatial momentum. (For an alternate approach see [80, 81]). This work considered the $\Delta I = 3/2$ operators, worked within the quenched approximation, using the Wilson gauge and fermion actions with an inverse lattice spacing ~ 2GeV, on lattices of size $24^3 \times 48$, and made use of the non-perturbative renormalisation technique of the Rome-Southampton group [32]. Using data from these (unphysical) kinematics it is, in principle, possible to extract all the free parameters at next-to-leading order in chiral perturbation theory. With these in hand, chiral perturbation theory can then be used to extrapolate to the physical parameters. In practise, however, it was found that the data was not a good fit to NLO chiral perturbation theory. In fact, the best results are those which used a polynomial fit (i.e. dropping all the logs from the chiral perturbation theory fit).

Figure 11 (taken from [77]) shows the quality of these two fits. The left-hand panel shows the fit divided by the data for the NLO chiral perturbation theory fit, while the right-hand panel shows this for the polynomial fit. However, this bad convergence is not too surprising, as the smallest pion mass used in the simulations is ~ 500MeV. Using a hybrid polynomial fit and the Lellouch-Luscher formula, the final results are:

$$\langle \pi\pi \left| O_8^{3/2} \right| K^0 \rangle \; = \; 0.68 \pm 0.09 \text{GeV}^3 \tag{4.8}$$

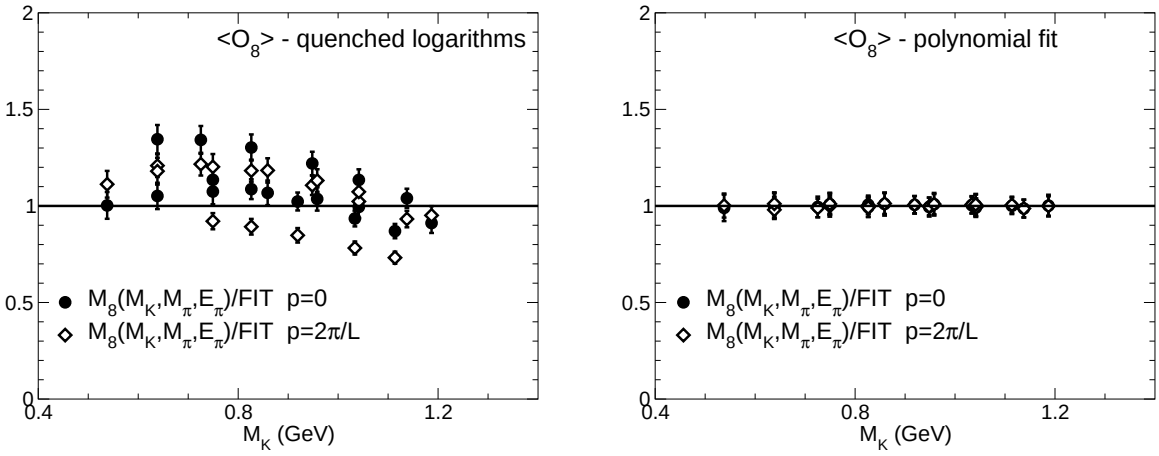

Figure 11: LEFT: data/fit for the fit to NLO chiral perturbation theory RIGHT: data/fit for the polynomial fit

$$\langle \pi\pi \left| O_7^{3/2} \right| K^0 \rangle \; = \; 0.12 \pm 0.02 \text{GeV}^3 \tag{4.9}$$

As noted, the Lellouch-Luscher formula was derived in the centre of mass frame, not the lab-frame. Nevertheless, this work is an important first step on the way to pinning down the NLO coefficients. With smaller masses, and an analogue of the Lellouch-Luscher formula for the lab-frame this approach looks very promising.

In fact, one of the important theoretical advances in the past year is the derivation by two groups [82, 83] just such a formula for the lab-frame. While the proofs are too technical to give here, it is worth mentioning exactly what such a proof entails. The proof in the centre of mass frame can be split into two halves:

1. First, the equivalence between the scattering problem in field theory, in the centre of mass frame, and corresponding two-particle scattering problem in quantum mechanics was shown [84, 85, 86].

2. Secondly, the normalisation problem was solved in two-particle quantum mechanics [76].

The second problem was solved for the lab-frame by Rummukainen and Gottlieb in 1995 [87]. However, this work was based on the assumption that, as with the centre of mass frame, there is a relationship between the problem in quantum field theory and a two-particle scattering problem in quantum mechanics. This (non-obvious) step is what has been added by [82, 83]. In fact, Takeshi Yamazaki has presented a preliminary result at this conference which uses this formula [88].

5. Summary

I have reviewed three topics: kaon beta-decay, $\overline{K^0} - \overline{K}$ mixing and $K \to \pi\pi$ decays. For the former, one of unquantified systematics – the use of the quenched approximation – has been addressed, with the results of three dynamical calculations reported: two performed in two-flavour QCD, one in full (three flavour) QCD. These results are basically consistent with each other, the

previous quenched work, and the estimate by Leutwyler and Roos. This suggests that the dominant systematic error is not due to quenching, lattice spacing dependence, or finite volume effects (as these things differ between the various calculations), but due to the requires momentum and mass extrapolations. The recent increases in available computing power should allow the latter of these to be addressed in the near future by simply moving to smaller masses.

For B_K there are now many different extractions in the quenched approximation. Perhaps the most challenging aspect of such calculations is the renormalisation of the operators in the effective weak Hamiltonian. Having flavour and chiral symmetries at finite lattice spacing is a great boon in this respect and both Domain Wall Fermion and the Overlap are well suited to this problem. New results for twisted mass QCD are also very impressive. Both Wilson and Staggered fermion calculations have relatively large errors due to trying to treat (or estimate the size of) the effects of operator mixing. In the latter case it is easier to see how to reduce these errors, and some combination of higher order (or non-perturbative) renormalisation calculations and the use of chiral perturbation theory calculations that include neglected mixings and lattice spacing effects should be effective. The dominant systematic error on B_K is that due to the use of the quenched approximation, and several groups have presented new results using dynamical fermions. While non of these calculations has yet reached the maturity of those in the quenched approximation, they are all part of ongoing projects which should bear fruit in the next few years. Nevertheless, applying what is known from the few dynamical calculations we have (that the addition of dynamical quarks doesn't seem to make a very large difference), I suggest a combined average of $B_K(\overline{MS}, 2\text{GeV}) = 0.58(3)(6)$ where the first error is on the degenerate, quenched value, and the second takes in account non-degeneracy and dynamical quark effects.

The calculation of $K \to \pi\pi$ decays on the lattice is a very challenging problem. The most successful approach to date has been to use lattice formulation that respect chiral symmetry (thus greatly simplifying the operator mixing problem) and calculate $K \to 0$ and $K \to \pi$, using leading order chiral perturbation theory to move to $K \to \pi\pi$. The reliance on leading order chiral perturbation theory and the use of the quenched approximation are the two major source of systematic error in this calculation. There have been a couple of advances recently. Firstly, it has been shown that it is possible to construct operators with twisted mass fermions for which the operator mixing problem is also simplified. Secondly, an attempt has been made to move beyond first order chiral perturbation theory. To do this, it is necessary to calculate $K \to \pi\pi$ directly on the lattice. To do this practically requires a kinematics for which the physical two-pion state has the lowest energy, and a formula for converting the finite volume result into the infinite volume result. I discussed briefly a calculation the SPQR kinematics for which the two-pion final state carries momentum (the lab-frame), and the new proof of a formula to relate the finite volume results in the lab-frame to infinite volume ones. As increases in computing power allow smaller masses to be simulated the combination of next-to-leading order chiral perturbation theory, a kinematics such as the SPQR, and the lab-frame conversion formula provide an attractive approach to studying $K \to \pi\pi$ decays on the lattice.

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