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Power Counting Regime of Chiral Extrapolation and Beyond

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Finite-range regularised chiral effective field theory is presented in the context of approximation schemes ubiquitous in modern lattice QCD calculations. Using FRR techniques, the power-counting regime (PCR) of chiral perturbation theory can be estimated. To fourth-order in the expansion at the 1% tolerance level, we find $0 \leq m_\pi \leq 180$ MeV for the PCR, extending only a small distance beyond the physical pion mass.

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1. Introduction

Most everyone is now familiar with the importance of accounting for chiral nonanalytic behavior in the quark-mass extrapolation of physical observables. The (pseudo) Goldstone bosons associated with dynamical chiral symmetry breaking in QCD couple strongly and give rise to a quark-mass dependence of hadron observables in which significant curvature is usually encountered in approaching the physical regime. The Adelaide Group has played a leading role in emphasizing the role of this physics [1] and establishing new approximation schemes to enable the extrapolation of today's lattice QCD results [2, 3].

The established, model-independent approach to chiral effective field theory is that of power counting, the foundation of chiral perturbation theory (χPT). However, this requires one to work in a regime of pion mass where the next term in the truncated series expansion makes a contribution that is negligible. As there is no attempt to model the higher-order terms of the chiral expansion, one simply obtains the wrong answer if one works outside this region. Knowledge of the power counting regime (PCR), where neglected higher-order terms are truly small, is as important as the chiral expansion itself.

Approximation schemes play a significant role in modern lattice QCD simulations. Consider for example the calculation of all-to-all propagators [4], where the most important low-lying eigenmodes of the Dirac operator are treated exactly in the inversion process. Having treated the dominant contributions precisely, the remainder of the matrix inverse is approximated using stochastic estimator techniques. The FRR approach to χEFT also has these characteristic features. Just as the low-lying eigenmodes are treated exactly, so is the chiral regime of the chiral expansion. FRR χEFT is mathematically equivalent to standard χPT to the finite order one is working.[1] Having treated the dominant contributions precisely, the remainder of the chiral expansion is approximated using FRR-induced resummation techniques.

Because terms of the chiral expansion beyond the finite order calculated are treated in an approximate manner, FRR χEFT is often regarded as a model. In the lattice community, models are usually eschewed at all costs, but the costs are high. Most chiral extrapolations presented this year at Lattice '05, are still of the most naive linear or polynomial form. Those performing extrapolations with traditional χPT are performing the extrapolations from well outside the PCR. The most common signature of this is that when higher-order terms are calculated, they are almost always found to be large, even in the favorable meson sector [5]. If one was working in the PCR to begin with, then the next order term of the expansion is small by definition! The reluctance to quantitatively determine the PCR undermines the integrity and credibility of lattice QCD predictions.

There continues to be a reluctant but growing recognition that some form of resummation of the chiral expansion is necessary in order to make contact with lattice simulation results of full QCD. The resummation of the chiral expansion induced through the introduction of a finite-range cutoff in the momentum-integrals of meson-loop diagrams is perhaps the best known resummation method [1, 2, 3]. Taylor expansions of FRR fits to lattice QCD results for magnetic moments indicate that terms to m_π^{26} are required to reach the first lattice data point at $m_\pi^2 = 0.2$ GeV^2 [6]. Given the astronomical number of low energy constants to be determined if such calculations were even possible in χPT, one must question if this really is the interesting physics.

[1] A survey of the literature reviewing χEFT illustrates that most practitioners are unaware of this fact.

As we will demonstrate in the following, the quark masses accessible with today's algorithms and supercomputers lie well outside the regime of baryon χPT in its standard form. This situation is unlikely to change significantly until it becomes possible to directly simulate QCD on the lattice within twice the squared physical mass of the pion and with suitably large lattice volumes. Still, one might wonder if the lattice techniques that would allow simulations at light masses within $2m_\pi^2$, might also allow a calculation directly at the physical pion mass, obliterating the chiral extrapolation problem altogether.

2. FRR χEFT is not a model in the PCR

To demonstrate that FRR χEFT is mathematically equivalent to χPT to the order calculated and alleviate the myth that the FRR χEFT approach is simply a model, we review the process of renormalisation in a minimal subtraction scheme and in FRR χEFT. To leading one-loop order

$$M_N = a_0 + a_2 m_\pi^2 + \chi_\pi I_\pi \,, \tag{2.1}$$

where $\chi_\pi = -3g_A^2/(32\pi f_\pi^2)$ is the LNA coefficient of the nucleon mass expansion, and I_π denotes the relevant loop integral. In the heavy baryon limit, this integral over pion momentum is given by

$$I_\pi = \frac{2}{\pi}\int_0^\infty dk \, \frac{k^4}{k^2 + m_\pi^2} \,. \tag{2.2}$$

This integral suffers from a cubic divergence for large momentum. The infrared behavior of this integral gives the leading nonanalytic correction to the nucleon mass. This arises from the pole in the pion propagator at complex momentum $k = i m_\pi$ and will be determined independent of how the ultraviolet behavior of the integral is treated. Rearranging Eq. (2.2) we see that the pole contribution can be isolated from the divergent part

$$I_\pi = \frac{2}{\pi}\int_0^\infty dk \left(k^2 - m_\pi^2\right) + \frac{2}{\pi}\int_0^\infty dk \, \frac{m_\pi^4}{k^2 + m_\pi^2} \,. \tag{2.3}$$

The final term converges and provides m_π^3. In the most basic form of renormalization we could simply imagine absorbing the infinite contributions arising from the first term in Eq. (2.3) into a redefinition of the coefficients a_0 and a_2 in Eq. (2.1). This solution is simply a minimal subtraction scheme and the renormalized expansion can be given without making reference to an explicit scale,

$$M_N = c_0 + c_2 m_\pi^2 + \chi_\pi m_\pi^3 \,, \tag{2.4}$$

with the renormalized coefficients defined by

$$c_0 = a_0 + \chi_\pi \frac{2}{\pi}\int_0^\infty dk\, k^2 \,, \quad c_2 = a_2 - \chi_\pi \frac{2}{\pi}\int_0^\infty dk \,. \tag{2.5}$$

Equation (2.4) therefore encodes the complete quark mass expansion of the nucleon mass to $\mathscr{O}(m_\pi^3)$. This result will be precisely equivalent to any form of minimal subtraction scheme, where all the ultraviolet behavior is absorbed into the two leading coefficients of the expansion. Such a minimal subtraction scheme is characteristic of the commonly implemented dimensional regularization.

We now describe the chiral expansion within finite-range regularization, where the cut-off scale remains explicit. In particular, we highlight the mathematical equivalence of FRR and dimensional regularization in the low energy regime. We introduce a functional cutoff, $u(k)$, defined such that the loop integral is ultraviolet finite,

$$I_\pi = \frac{2}{\pi} \int_0^\infty dk \, \frac{k^4 u^2(k)}{k^2 + m_\pi^2} \, . \tag{2.6}$$

To preserve the infrared behavior of the loop integral, the regulator is defined to be unity as $k \to 0$. For demonstrative purposes, we choose a dipole regulator $u(k) = (1 + k^2/\Lambda^2)^{-2}$, giving

$$I_\pi^{\rm DIP} = \frac{\Lambda^5 (m_\pi^2 + 4 m_\pi \Lambda + \Lambda^2)}{16 (m_\pi + \Lambda)^4} \sim \frac{\Lambda^3}{16} - \frac{5\Lambda}{16} m_\pi^2 + m_\pi^3 - \frac{35}{16\Lambda} m_\pi^4 + \dots \, , \tag{2.7}$$

The first few terms of the Taylor series expansion, as shown, provide the relevant renormalisation of the low-energy terms. The renormalized expansion in FRR is therefore precisely equivalent to Eq. (2.4) up to $\mathscr{O}(m_\pi^3)$ where the leading renormalized coefficients are given by

$$c_0 = a_0 + \chi_\pi \frac{\Lambda^3}{16} \, , \quad c_2 = a_2 - \chi_\pi \frac{5\Lambda}{16} \, . \tag{2.8}$$

As a_0 and a_2 are fit parameters, the value Λ takes is irrelevant and plays no role in the expansion to the order one is working; in this case m_π^3. Hence the suggestion, for example, that infrared regularization is somehow less model dependent than FRR is false and misleading. Within the PCR of χPT there is no physics in the regulator.

It is straight forward to extend this procedure to next-to-leading nonanalytic order, explicitly including all terms up to $m_q^2 \sim m_\pi^4$. Most importantly, there are nonanalytic contributions of order $m_\pi^4 \log m_\pi$ arising from the Δ-baryon and tadpole loop contributions. Details may be found in [1, 2].

3. Power-counting regime (PCR)

The PCR is the regime in which neglected higher-order terms of the standard expansion of χPT are small, because m_π is a small number raised to a high power. Since the chiral expansion of χPT is truncated with no attempt to estimate the contribution of higher-order terms, one simply obtains the wrong answer if one works outside the PCR.

As discussed in detail surrounding Eq. (2.8), the FRR chiral expansion is mathematically equivalent to that of χPT to the finite order one is working. In other words, these terms of the FRR expansion are independent of the regulator parameter, Λ. Thus FRR χEFT can be used to determine the power-counting regime by varying Λ and identifying the regime in pion mass in which the results are invariant to some level of precision.

Fig. 1 illustrates the fourth-order chiral expansion for various dipole regulator parameters Λ. Since the expansion to fourth order is automatically independent of Λ, the observed changes in the curves are simply a reflection of the changes in terms beyond fourth order. Fig. 2 displays the relative error between the two extremal regularization scales for the left (solid) and right (dashed) panels of Fig. 1. The regime where the curves agree within one percent is $m_\pi \leq 180$ MeV extending only 40 MeV beyond the physical mass. While this is excellent news for understanding experimental results within chiral perturbation theory, it also illustrates that today's naive application of χPT to the chiral extrapolation problem in lattice QCD is inappropriate.

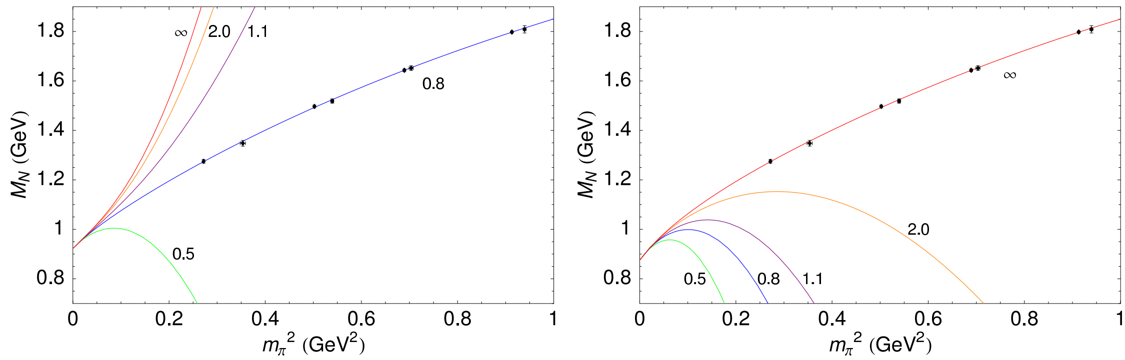

Figure 1: With the low-energy parameters c_0, c_2 and c_4 fixed to those obtained by the fit to lattice data with $\Lambda = 0.8$ (left) and the minimal subtraction limit $\Lambda \to \infty$ (right), the chiral expansion is shown for various values of the dipole regulator scale, $\Lambda = 0.5,\ 0.8,\ 1.1,\ 2.0$ and ∞ GeV.

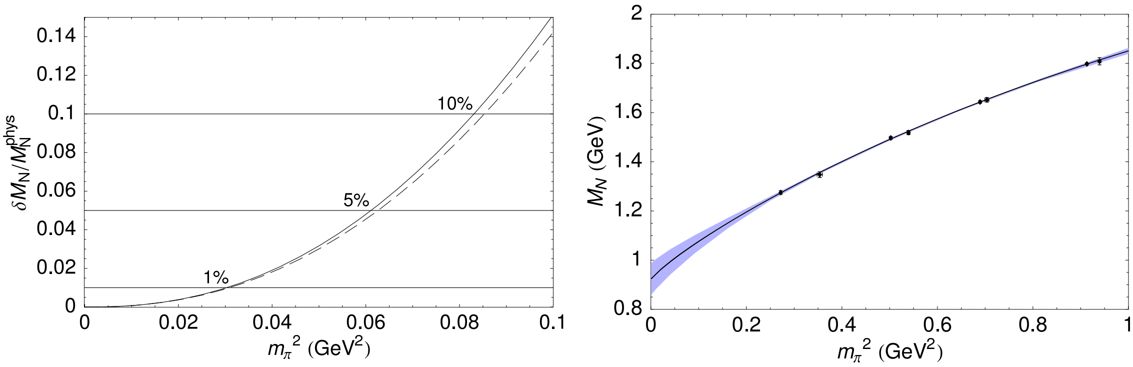

Figure 2: (left) For fixed low-energy coefficients c_0, c_2 and c_4, the relative difference in the nucleon mass expansion for two extremal regularization scales, where $\delta M_N = M_N(\Lambda = \infty) - M_N(\Lambda = 0.5)$. The solid and dashed curves correspond to the differences displayed in the left and right-hand panels of Fig. 1 where the c_i are determined with a dipole of scale $\Lambda = 0.8$ and ∞ GeV, respectively. (right) Extrapolation of CP-PACS collaboration simulation results [7] to the chiral limit using finite-range regularization [2]. Differences between the illustrated dipole, monopole, Gaussian and theta-function regulators cannot be resolved on this scale. The one-standard deviation error bound for the dipole extrapolation is also illustrated.

4. FRR χEFT as a solution to the chiral extrapolation problem.

To investigate the extent to which various regulators provide a model-independent estimator for the *sum* of higher-order terms of the chiral expansion, beyond the finite order calculated, the finite-range regulator $u(k)$ is taken to be either a sharp theta-function cut-off, a dipole, a monopole or finally a Gaussian. These regulators have very different shapes, with the only common feature being that they suppress the integrand for momenta greater than Λ. Figure 2 (right) displays the extrapolation [2] of full QCD simulation results of the nucleon mass from the CP-PACS collaboration [7] using FRR χEFT to fourth order in the expansion; *i.e.* to order $m_\pi^4 \log m_\pi$. The curves are indistinguishable and produce physical nucleon masses which differ by less than 0.1%.

The astonishing discovery in FRR chiral effective field theory, is that the term-by-term details of the higher-order chiral expansion are largely irrelevant in describing the chiral extrapolation of simulation results. The coefficients of the higher-order terms (m_π^5 and beyond) appearing in the FRR expressions differ significantly, yet the curves of Fig. 2 (right) are indistinguishable. Given the level of agreement between the curves associated with different regulators, and the fact that the lattice results are described perfectly, it is sufficient to approximate the remainder of the chiral expansion in terms of a single parameter, Λ.

5. Summary

So why does FRR χEFT work? The essential physics is that loop integrals vanish as the quark masses grow large. Exactly how zero is approached is governed by the regulator parameter, Λ, and in most cases Λ is constrained by lattice QCD simulation results. The contribution of any individual higher-order term is largely irrelevant. The only thing that really counts is that there are other terms that enter to ensure the sum of all terms of the loop integral approaches zero, in accord with what is observed in lattice QCD calculations. Of course, this beautiful feature of FRR expansions would be lost if one were to truncate the expansion at any finite order. Resummation of chiral effective field theory is essential to solving the chiral extrapolation problem.

The finite-range regularisation (FRR) approach to chiral effective field theory (χEFT) provides an approximation scheme that connects *today's* lattice simulation results to the physical world. It has been successfully applied to describe partially-quenched simulation results of the rho meson mass in a unified analysis incorporating both finite volume and finite lattice spacing artifacts [8]. The CSSM lattice collaboration has completed extensive simulations of baryon electromagnetic form factors. An associated quenched FRR χEFT analysis of the magnetic moments correcting finite-volume and quenched artifacts has led to the most precise determination of the nucleon's strange magnetic moment [9].

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