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A practical construction of U(1) chiral lattice gauge theories

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In the gauge-invariant construction of abelian chiral gauge theories on the lattice based on the Ginsparg-Wilson relation, the gauge anomaly is topological and its cohomologically trivial part plays the role of the local counter term. We give a prescription to solve the local cohomology problem within a finite lattice by reformulating the Poincaré lemma so that it holds true on the finite lattice up to exponentially small corrections. Moreover we perform a caluculation of the trivial part of the anomaly numerically, which is given by the solution of the local cohomology problem.

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1. Introduction

Recently it turned out that lattice gauge theory can provide a framework for non-perturbative study of chiral gauge theories, despite the well-known problem of the species doubling. The clue to this development is the construction of gauge-covariant and local lattice Dirac operators satisfying the Ginsparg-Wilson relation[1, 2, 3, 4],

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D. \tag{1.1}$$

This relation has made it possible to introduce Weyl fermions on the lattice and construct anomaly-free chiral gauge theories with exact gauge invariance[5, 6]. One of the crucial steps in the gauge-invariant construction of chiral lattice gauge theories is to establish the exact cancellation of the gauge anomaly at a finite lattice spacing.

In the case of U(1) chiral gauge theories[6], the exact cancellation has been achieved through the cohomological classification of the chiral anomaly $q(x)$ which is a topological field in the sense that

$$\sum_x \delta q(x) = 0 \tag{1.2}$$

under a local variation of the admissible gauge field. Such a field on the four-dimensional infinite lattice is classified uniquely in the following form:

$$q(x) \;=\; \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu k_\mu(x), \tag{1.3}$$

where $F_{\mu\nu}(x)$ is the gauge field tensor, $\alpha, \beta_{\mu\nu}$ and γ are constants and $k_\mu(x)$ is a gauge invariant local current[5, 7, 8]. For the multiplets that satisfy the anomaly cancellation condition of the U(1) charges e_α, it has been shown that the anomaly is cohomologically trivial,

$$\sum_\alpha e_\alpha q^\alpha(x) = \partial^*_\mu k_\mu(x), \qquad q^\alpha(x) = q(x)|_{U\to U^{e_\alpha}} . \tag{1.4}$$

Then the trivial part of the anomaly is used in the gauge-invariant construction of the Weyl fermion measure. In short, it plays the role of the local counter term in the effective action.

For the practical computation of observables in the abelian chiral lattice gauge theories, it is required to compute the Weyl fermion measure for every admissible configuration. However it seems difficult to follow the steps given in [6] literally. The problem is the use of the infinite lattice in order to make sure the locality property of the trivial part and therefore the current $k_\mu(x)$ was constructed through infinite dimensional procedures which would not be immediately usable for numerical calculation.

The purpose of this paper is to give a prescription to solve the local cohomology problem *within a finite lattice*. With this method, we will show that the current $k_\mu(x)$ can be obtained directly from the calculable quantities on the finite lattice. We first introduce the vector potential which represents the admissible U(1) gauge fields by separating the link variables into the instanton configuration and the part of the local and dynamical degrees of freedom around the magnetic flux (Section 2). We next show that the Poincaré lemma can be reformulated so that it holds true on the finite lattice up to exponentially small corrections of order $O(\mathrm{e}^{-L/2\rho})$, where L is the lattice size and ρ is the localization range of the differential forms in consideration (Section 3.1). Equipped with the modified Poincaré lemma and the vector potential for the admissible U(1) gauge fields, we will perform the cohomological analysis of the topological field (like as chiral anomaly) directly on the finite lattice (Section 3.2). We will compute $k_\mu(x)$ in two-dimensions numerically and check its locality properties (Section 4).

2. Admissible U(1) gauge fields on a finite lattice

Our first step is to formulate the vector-potential-representation of the link variables associated with an admissible U(1) gauge field on a finite lattice. Such a representation has been formulated in the original cohomological analysis in [5] on the infinite lattice. As has been shown in our previous paper[10], it is also possible to formulate the periodic vector-potential representation for the admissible gauge fields on the finite lattice.

We set the lattice spacing a to unity and consider U(1) gauge fields on a finite lattice of size L with periodic boundary conditions. The independent degrees of freedom are then the link variables at the points in the region $\Gamma_n = \left\{x \in \mathbb{Z}^n | -L/2 \leq x_\mu < L/2\right\}$ where L is assumed to be an even integer for simplicity. We impose the admissibility condition, $|F_{\mu\nu}(x)| < \varepsilon$. For $\varepsilon < \pi/3$ the gauge fields can be classified uniquely by the magnetic fluxes $m_{\mu\nu}$. In this respect, the following field (the instanton configuration) is periodic and have constant field tensor equal to $2\pi m_{\mu\nu}/L^2$:

$$V_{[m]}(x,\mu) = \mathrm{e}^{-\frac{2\pi i}{L^2}\left[L\delta_{\tilde{x}_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}\tilde{x}_\nu + \sum_{\nu<\mu} m_{\mu\nu}\tilde{x}_\nu\right]}, \tag{2.1}$$

where the abbreviation $\tilde{x}_\mu = x_\mu$ mod L has been used. Then any admissible U(1) gauge field in the topological sector with the magnetic flux $m_{\mu\nu}$ may be expressed as

$$U(x,\mu) = \tilde{U}(x,\mu)V_{[m]}(x,\mu). \tag{2.2}$$

We may regard $\tilde{U}(x,\mu)$ as the actual local and dynamical degrees of freedom in the given topological sector. This is because the magnetic flux $m_{\mu\nu}$ is invariant with respect to a local variation of the link field. As shown in [10], it is possible to establish the one-to-one correspondence between $\tilde{U}(x,\mu)$ and periodic vector potentials $\tilde{A}_\mu(x)$ which satisfy

$$\tilde{U}(x,\mu) = \mathrm{e}^{i\tilde{A}_\mu(x)}, \qquad \tilde{F}_{\mu\nu}(x) = \partial_\mu\tilde{A}_\nu(x) - \partial_\nu\tilde{A}_\mu(x), \tag{2.3}$$

where $\tilde{F}_{\mu\nu}(x) = F_{\mu\nu}(x) + \frac{2\pi m_{\mu\nu}}{L^2}$ and moreover $\tilde{A}_\mu(x)$ is indefinite up to a gauge function which takes values that are integer multiples of 2π.

As emphasized in [5], an important point is that the locality properties of gauge invariant fields should be the same independently of whether they are considered to be functions of the link variables or the vector potential.

3. Local cohomology problem

3.1 Poincare lemma on a finite lattice

The Poincare lemma is given in [5] on the infinite lattice for the difference operators. On the finite periodic lattice the lemma does not hold true any more, because the lattice is a n-dimensional torus and its cohomology group is non-trivial. However, the lemma can be reformulated so that it holds true up to exponentially small corrections of order $O(\mathrm{e}^{-L/2\rho})$ and for the form satisfying $\sum_{x\in\Gamma_n} f(x) = 0$, it holds exactly even on the finite lattice [10]. The latter result is the lattice counter part of the corollary of de Rham theorem known in the continuum theory. The precise statements are the following.

Lemma A (Modified Poincaré lemma)
Let f be a k-form which satisfies $d^* f = 0$ and $\sum_{x\in\Gamma_n} f(x) = 0$ if $k = 0$. Then there exist a $k+1$-form g and a k-form Δf such that $f = d^* g + \Delta f$. The tensor field $\Delta f_{\mu_1\cdots\mu_k}(x)$ linearly depends only on the values of the tensor field $f_{\mu_1\cdots\mu_k}(x)$ at the boundary, $\{f_{\mu_1\cdots\mu_k}(z)|\, z \in \partial\Gamma_n\}$.

Lemma B (Corollary of de Rham theorem)
Let f be a k-form which satisfies $d^* f = 0$ and $\sum_{x\in\Gamma_n} f(x) = 0$. Then there exists a $k+1$-form g such that $f = d^* g$.

The construction of the forms $g(x)$ and $\Delta f(x)$ is given explicitly in the proof of the lemmas. The coefficients of $g(x)$ and $\Delta f(x)$ are some linear combinations of the coefficients of $f(x)$ and the sizes of them are intimately related to that of $f(x)$. Now let us introduce norms of the forms by

$$\| f(x)\|_{x_0,p,\rho} = \max_{x\in\Gamma_n} \frac{|f_{\mu_1\cdots\mu_k}(x+x_0)|}{(1+\|x\|^p)\,\mathrm{e}^{-\|x\|/\rho}} \tag{3.1}$$

with a localization range ρ, an integer p and a reference point x_0 fixed. $\|x\|$ is the taxi driver distance from the origin to x. Then we can show the following bound for the norm of the form $g(x)$: $\|g(x)\|_{x_0,p,\rho} \le C\|f\|_{x_0,p,\rho}$ for some constant C independent of $f(x)$, x_0 and L. As for the form $\Delta f(x)$, we have $|\Delta f_{\mu_1\cdots\mu_k}(x)| \le C'\mathrm{e}^{-L/2\rho}$ for some constant C' independent of $f(x)$, x_0 and L. The proofs of these bounds are given in the appendix of [10].

3.2 A solution of the local cohomology problem

We now describe how to perform the cohomological analysis of the topological fields. Let us consider a gauge-invariant, exponentially local and topological field $q(x)$. As mentioned in section 2, We may regard it as the local function of the vector potential $\tilde{A}_\mu(x)$. Scaling vector potential by a parameter $t \in [0,1]$ and differentiating and integrating $q(x)$ with respect to t, we obtain

$$q(x) = \alpha + \sum_{y\in\Gamma_n} j_\mu(x,y)\tilde{A}_\mu(y), \qquad j_\mu(x,y) = \int_0^1 \mathrm{d}t\, \frac{\partial q(x)}{\partial \tilde{A}_\mu(y)}\bigg|_{\tilde{A}\to t\tilde{A}}, \tag{3.2}$$

where α is a constant that depends on $m_{\mu\nu}$ and four Wilson lines independent of the field tensor. From the topological property, the gauge invariance and the locality property of $q(x)$, we have the following constraints on the current $j_\mu(x,y)$ respectively,

$$\sum_{x\in\Gamma_n} j_\mu(x,y) = 0, \qquad j_\mu(x,y)\overleftarrow{\partial}{}^*_\mu = 0, \qquad \|j_\mu(x,y)\|_{y,p,\rho} \le C, \tag{3.3}$$

where $\overleftarrow{\partial}{}^*_\mu$ is the backward nearest-neighbor difference operator with respect to y and C is a constant.

We note that the above relation is exactly same as that obtained in the course of the original cohomological analysis[5]. Once we obtain this relation, we can immediately see that the same argument as in the original analysis by using the lemma A(B) and it leads to the same first-step result, lemma 6.1 in [5] up to exponentially small corrections, lemma 6.1 in [10]. The second-step result, lemma 6.2 in [10], can be also obtained by using the lemma A(B). As a consequence, we can show that

$$q(x) \;=\; \alpha+\beta_{\mu\nu}\tilde{F}_{\mu\nu}(x)+\gamma\,\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(x)\tilde{F}_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})+\partial^{*}_{\mu}h_{\mu}(x)+\Delta\tilde{q}(x) \tag{3.4}$$

where $\alpha,\beta_{\mu\nu}$ and γ are constants that depend on $m_{\mu\nu}$ and four Wilson lines independent of the field tensor, $h_{\mu}(x)$ is a gauge invariant local current and $|\Delta\tilde{q}(x)|\leq c\mathrm{e}^{-L/2\rho}$. When $q(x)$ is the covariant abelian gauge anomaly, for the anomaly free multiplets, the first three terms of eq.(3.4) are the order $O(\mathrm{e}^{-L/2\rho})$ and therefore $q(x)=\partial^{*}_{\mu}h_{\mu}(x)+\Delta q(x)$ where $|\Delta q(x)|\leq c\mathrm{e}^{-L/2\rho}$. The index theorem suggests us to conclude $\sum_{x\in\Gamma_n}\Delta q(x)=0$. From the lemma B, $\Delta q(x)=\partial_{\mu}\Delta k_{\mu}(x)$. Finally we obtain eq.(1.4) where $k_{\mu}(x)=h_{\mu}(x)+\Delta k_{\mu}(x)$.

4. Numerical calculation in two dimensions

We now describe our result of numerical computations of the local current $k_{\mu}(x)$ for the chiral anomaly, $q(x)=\mathrm{tr}\{\gamma_5(1-aD)(x,x)\}$ in two dimensions. Once the bi-local current $j_{\mu}(x,y)$ is computed, the cohomological analysis of the chiral anomaly can be performed numerically by the sequence of the applications of the lemmas. We may represent the chiral anomaly as follows:

$$q(x)=\mathscr{A}(x)+\partial^{*}_{\mu}h_{\mu}(x)+\Delta q(x), \tag{4.1}$$

where $\mathscr{A}(x)=\alpha+\gamma\varepsilon_{\mu\nu}\tilde{F}_{\mu\nu}(x)$. We consider the lattice sizes $L=8,10,12$. Admissible gauge fields are generated by Monte Carlo simulation using the action $S=\beta\sum_{\Box}F_{\Box}^2/\{1-F_{\Box}^2/\varepsilon^2\}$. As reported in [9], the topological charge is preserved during the Monte Carlo updates with this type of action, even when ε is set to π. We adopt this option and check the locality of the topological field numerically for several values of β. We consider the topological sectors with $m_{12}=0,1$ and the initial configuration is chosen as $V_{[m]}(x,\mu)$ with a given m_{12}. Here the original topological field $q(x)$ is constructed by computing all eigenvalues and eigenvectors of H_{w} for the given admissible gauge field $U(x,\mu)$. We found that the topological field has typically the values of order $O(10^{-2})-O(10^{-3})$ and $|\Delta q(x)|\simeq O(10^{-5})\simeq O(\mathrm{e}^{-L/2\rho})$ $(L=12)$. The integer topological charge $Q=\sum_{x\in\Gamma_2}q(x)=m_{12}$ is reproduced within the error of order $O(10^{-12})-O(10^{-13})$. We use the Gaussian Quadrature formula with the degree $N_g=20$ to approximate the parameter integral in eq.(3.2). The other details are given in [11].

In order to check the locality properties of the fields, $q(x)$ and $h_{\mu}(x)$, we apply a small local variation to the gauge field as $U(x,\mu)\to\mathrm{e}^{i\eta_{\mu}(x)}U(x,\mu)$ where $\eta_{\mu}(x)=0.05\times2\pi\,\delta_{x,x_0}\delta_{\mu,1}$ For each variation of the fields, we define $\delta_{\eta}f(r)=\max\left\{|\delta_{\eta}f(x)|\,\middle|\,r=\parallel x-x_0\parallel\right\}$ and see the locality properties of the fields by plotting $\delta_{\eta}f(r)$ against $r=\parallel x-x_0\parallel$. The results are shown in figure. The variations of the topological field $q(x)$ and the current $h_{\mu}(x)$ are shown with the vanishing magnetic flux $m_{12}=0$. The locality of the topological field $q(x)$ is clearly seen. It also shows that the current $h_{\mu}(x)$ has the same locality property as the topological field $q(x)$ has and thus is an exponentially local current. We can read the locality range as $\rho\simeq0.9$. The maximum value of the field $\Delta q(x)$ is also shown in the same figure. The locality properties of the fields are also confirmed for topologically non-trivial gauge fields.

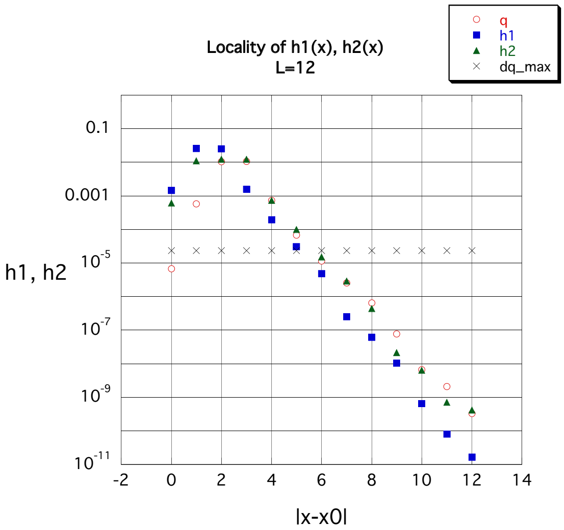

Figure 1: open circle:$\delta_\eta q(r)$, cross:max of $|\Delta q(x)|$, filled square and triangle: $\delta_\eta h_\mu(r)$.

We next examine the cancellation of the gauge anomaly for the so-called 11112 model which consists of four Left-handed Weyl fermions with unit charge and one Right-handed Weyl fermion with charge two in two-dimensions. The gauge anomaly cancellation condition is satisfied as follows: $\sum_{i=1}^{4} e^2 - (2e)^2 = 0$. We also check that $\sum_\alpha e^\alpha \mathscr{A}^\alpha(x) \simeq O(10^{-4}) \simeq O(\mathrm{e}^{-L/2\rho}) \quad (L=10)$. Then the current $k_\mu(x)$ can be computed as explained in section 3.2 and thus it is an exponentially local current. It can reproduce the original topological charge Q within the deviation of order $O(10^{-15})$. The gauge invariance of the current is also maintained within the error.

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