

Series representation; Pade' approximants and critical behavior in QCD at nonzero T and μ

Maria Paola Lombardo*

Laboratori di Frascati dell' Istituto Nazionale di Fisica Nucleare, I-00044 Frascati(RM), Italy

E-mail: lombardo@lnf.infn.it

We discuss the analytic continuation beyond $\mu/T \simeq 1$ in QCD at nonzero T and μ by use of the Pade' approximants. The slope of the critical line obtained in this way increases at large μ with respect to the second order Taylor result. In the hot phase Pade' and Taylor approximants coincide, suggesting a very large, and possibly infinite, radius of convergence of the Taylor series in this thermodynamic region.

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1. Introduction

Results obtained at imaginary μ_B can be analytically continued to real μ_B [1, 2, 3, 4]. In principle, rigorous arguments guarantee that the analytic continuation of a function can be done within the entire analytic domain. In practice, the exact analytic form is not known, and a systematic procedure relying on the Taylor expansion is only valid within the circle of convergence of the series itself. In this talk we discuss how to implement the analytic continuation of the critical line and of thermodynamics observables beyond the circle of convergence of the Taylor series in a controlled way.

Let us remind ourselves that an analytic function is locally representable as a Taylor series. The convergence disks can be chosen in such a way that they overlap two by two, and cover the analytic domain. Thus, one way to build the analytic continuation is by connecting all of these convergence disks. The arcs of the convergence circles which are within the region where f is analytic have a pure geometric meaning, and by no means are an obstacle to the analytic continuation. Assume now that the circle of convergence about $z = (0,0)$ has radius unit, i.e. is tangent to the lines which limit the analytic domain; take now a z value, say $z_1 = (0, a)$, $1/2 < a < 1$ inside the convergence disk as the origin of a new series expansion, which is explicitly defined by the rearrangement $(z - z_0)^n = (z - z_1 + z_1 - z_0)^n$. As the radius of convergence of the new series will be again one, this procedure will extend the domain of definition of our original function (the two series define restrictions of the same function to the intersection between the two disks), and by 'sliding' the convergence disk we can cover all the analytic strip.

We have sketched above the standard theoretical argument to demonstrate the feasibility of analytic continuation beyond the radius of convergence, and we will show that the Pade' series is one practical way to accomplish it.

2. Naive, but maybe useful examples

Let us consider a function $f(z)$ of a complex variable z which is analytic within a strip limited by $z = \pm i$, and let us assume that the radius of convergence of the Taylor expansion about $z = 0$ is given by the distance from the nearest singularity (it could be larger, see discussions below). This is the kind of pattern expected for temperatures above the end point of the chiral transition line, the nearest line of singularities being associated to the Roberge–Weiss transition. A possible function realizing this pattern is $f_1(z) = \frac{1}{z^2+1}$. For real $z = 1$ the Taylor approximants the partial sums S_n are $S_n = 1$ for n even, and $S_n = 2$ for n odd, demonstrating the lack of convergence of the series. On the other hand, obviously, the Pade' approximants $P[N, M]$ not only converge, but become exact as soon as $M > 1$ on the entire real axes. Similar exercises can be repeated for less trivial example, for instance the function $f_2(z) = \frac{e^{-z}}{z^2+1}$ is well represented by the Pade' approximants $P[N, N]$ for $N > 2$ well beyond the convergence disk.

3. The Critical Line beyond $\mu/T \simeq 1$

We can measure the critical line for imaginary μ , and our goal is to analytically continue the results on the righthand side. The radius of convergence of the Taylor representation of the critical

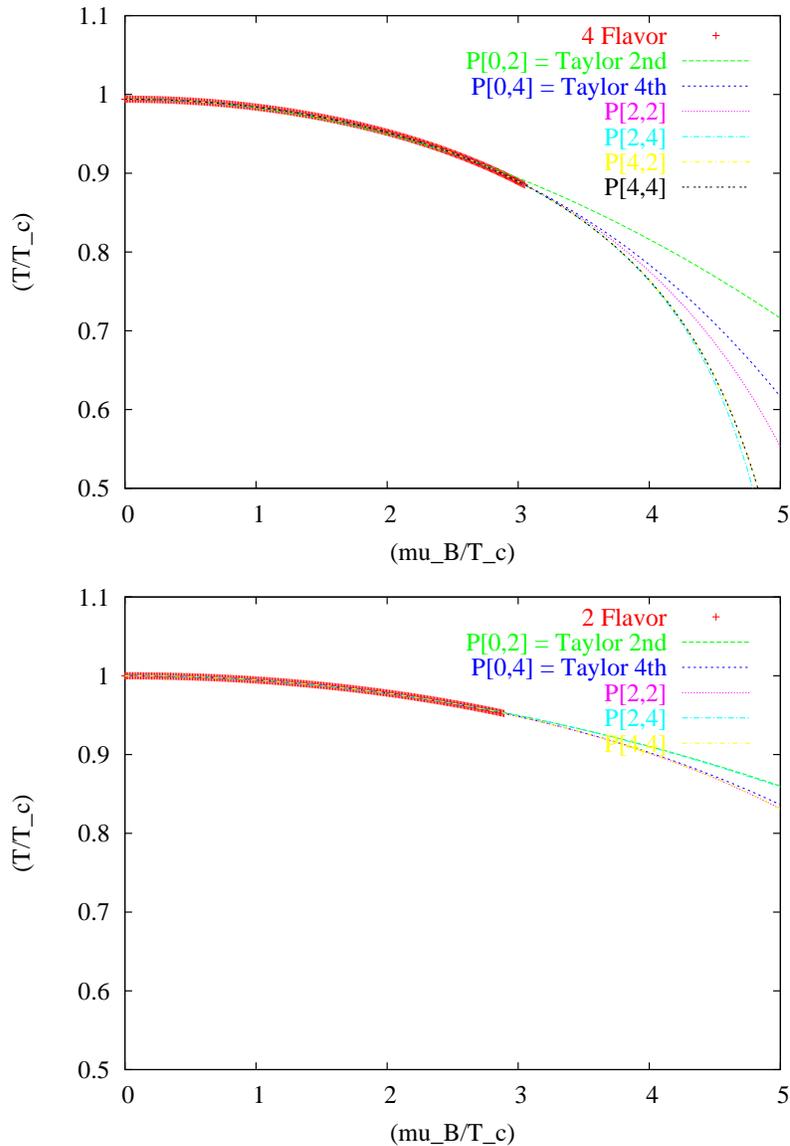


Figure 1: Pade' approximants for the critical line of four (top) and two flavor QCD. The red, bold line extends up to the radius of convergence of the Taylor series $\mu/T \simeq 1$, and has been fitted to the Pade' approximants, which converge well beyond it.

line might well be limited by the Roberge Weiss singularities. However, as explained before, the Pade' approximation is not.

In Fig. 1 we present the Pade' analysis of data for four [4] and two flavor [3]. Results seem stable beyond $\mu_B = 500MeV$ ($\mu_B/T \simeq 1$), with the Pade' analysis in good agreement with Taylor expansion for smaller μ values. At larger μ the Taylor expansion seems less stable, while the Pade' still converges, giving a slope of the critical line larger than the naive continuation of the second order Taylor approximations. The same behavior is suggested by recent results within the canonical approach[10] and the DOS method [11].

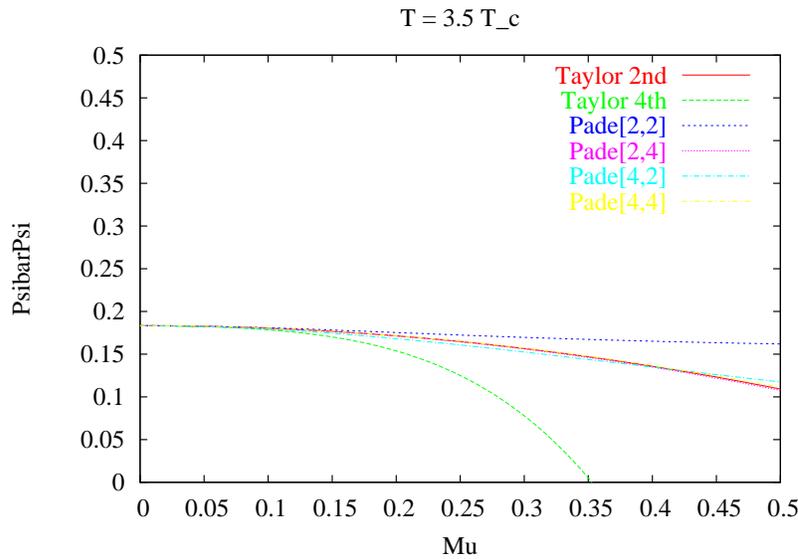


Figure 2: Analytic continuation via Pade' and Taylor approximants for the chiral condensate in the hot phase for $T = 3.5T_c$: note that both series seem to converge to the same result, suggesting a very large radius of convergence of the Taylor series in this thermodynamic region.

We underscore that the possibility of analytically continue the results beyond the radius of convergence of the Taylor series by no means imply that one can blindly extrapolate a lower order approximation! Even when it is possible to achieve convergence – via Pade' approximants, or within the convergence radius of the Taylor series – one has always to cross check different orders of approximation to make sure that convergence has indeed been achieved. For instance, it would then be interesting to repeat the comparisons between the second order results shown in [5] by extending the Taylor series to fourth order and/or by use of Pade' approximants.

4. Interlude – Radius of convergence and critical behavior

I mentioned en passant that the radius of convergence of a Taylor expansion about the origin might well be larger than the distance of the origin itself from the nearest singularity. While complex analysis textbooks offer a full discussion of this point, I would like to single out here three cases which might be encountered in usual critical behavior: $f_1(z) = A_1(z)(1 - z/z_c)^{-\lambda}$; $f_2(z) = A_2(z)\theta(z - z_c)$; $f_3(z) = A_3(z)\theta(z - z_c)(1 - z/z^*)^{-\lambda}$, where $A_n(z)$ is an analytic function.

Case 1 corresponds to an usual critical behavior (second order or larger). Case 2 represents a strong first order phase transition. Case 3 is intermediate between the two, a weak first order transition at z_c , and a spinodal point at z^* . Correspondingly, we have different radius of convergence of the Taylor series: $r_1 = |z_c|$, $r_2 = \infty$, $r_3 = z^*$: in conclusion the radius of convergence of the Taylor expansion for the critical line and thermodynamics observables might be infinite as well a finite, depending on the nature of the Roberge Weiss transition. Conversely, if the nature of the phase transition is known, one can infer from it the radius of convergence of the Taylor series, as done by Gavai and Gupta [6], which in turn locates the critical point.

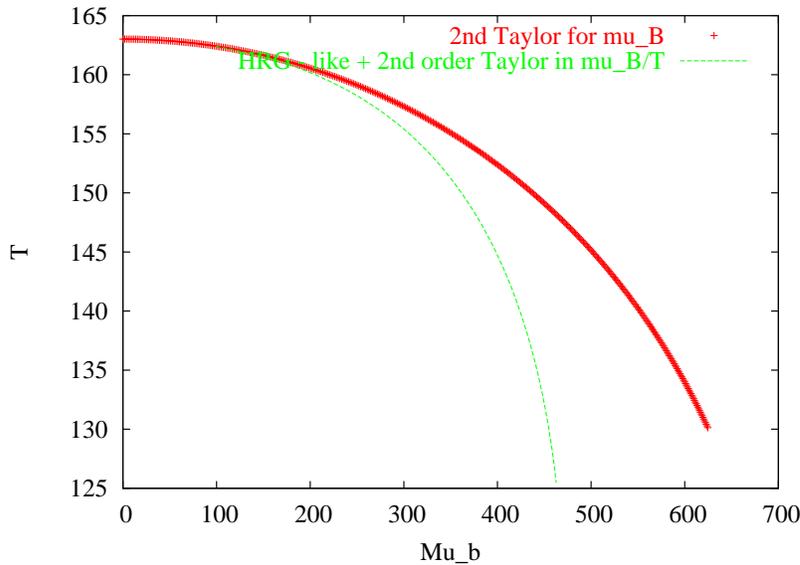


Figure 3: The critical line of the four flavor model computed by use of the simplified parametrization of the HRG model discussed in the text.

5. Beyond $\mu/T \simeq 1$ in the Hot Phase

The Pade' approximants to the results for the chiral condensate in the hot phase are shown in Fig. 2, where I used four flavor data [7].

Again we see that the Pade' analysis seems capable to produce stable results. We should also note that the Taylor expansion seems stable as well, which might indicate a large (infinite?) radius of convergence in this range of temperature. Indeed, as noted in [7] the radius of convergence should tend to infinite in the infinite temperature limit, and indeed it has been estimated to be large by the Bielefeld–Swansea collaboration[8]. A detailed investigation at imaginary μ of the region closer to the critical temperature is in progress [9].

6. The critical line from the Hadron Gas

An alternative way to analytically continue the results relies on phenomenological modeling. The Hadron Resonance Gas model might provide a description of QCD thermodynamics in the confined, hadronic phase of QCD [12, 8, 7], and can be used to determine the critical line as well.

The critical temperature as a function of μ_B is determined by lines of constant energy density: $\varepsilon \simeq 0.5 - 1.0 \text{ GeV/fm}^3$ [13] A continuation of the critical line using the HRG ansatz plus a fixed energy (or any other quantity determined at $\mu = 0$) criterion suggests the implicit form for the critical line $T = f(T)\cosh(\mu_B/T)$ with $\lim_{\mu_B/T \rightarrow 0} f(T)\cosh(\mu_B/T) = 1 - k\mu^2$. We have naively approximated $f(T) = 1 - k\mu^2$, and used the resulting form to fit the data in the $\mu/B < 1$ range. According to the above discussion, this again can be continued beyond this limit, and also in this case we get a critical line whose slope increases with increasing μ (Fig. 3).

7. Summary

For $T > T_c$ Pade' approximants are a viable tool to analytically continue thermodynamic results obtained at imaginary chemical potential to the entire analytic domain, beyond the radius of convergence of the Taylor expansion.

The stability of the Taylor partial sums for $T \simeq 3.5T_c$ hints at an infinite radius of convergence of the Taylor series at high temperature, according with a naive expectations of a strong first order Roberge Weiss transition at high T .

For $T < T_c$ the analytic continuation is best performed via a Fourier series which converge (one term alone) to the Hadron Resonance Gas model. In turn, the HRG parametrisation might be used to constrain the analytic form of the critical line.

Pade' analysis, and an HRG parametrisation of the critical line, are viable tools to continue the critical line beyond $\mu/T \simeq 1$. The results suggest that the curvature of the critical line increases at lower temperature.

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