

Non-perturbative plaquette in 3d pure SU(3)

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We present a determination of the elementary plaquette and, after the subsequent ultraviolet subtractions, of the finite part of the gluon condensate, in lattice regularization in three-dimensional pure SU(3) gauge theory. Through a change of regularization scheme to \overline{MS} and a matching back to full four-dimensional QCD, this result determines the first non-perturbative contribution in the weak-coupling expansion of hot QCD pressure.

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1. Introduction

The asymptotic freedom of QCD guarantees a small coupling constant g at large temperatures T . While observables can be expressed in a generalized power series in g , the loop expansion is not applicable to an arbitrary order in g , because of the so-called ‘‘infrared wall’’, as pointed out by Linde [1] (see also ref. [2]). For every observable there exists an order of the perturbative expansion to which an infinite number of Feynman diagrams contributes. For the pressure this order is $g^6 T^4$.

No way of resumming these infinitely many diagrams has been found, so a different approach is needed. The problem arises due to infrared divergences in the dynamics of zero Matsubara frequency modes of gauge fields. Because these modes are three-dimensional (3d) we can construct an effective 3d pure gauge theory called Magnetostatic QCD (MQCD) which accounts for the non-perturbative contribution [3, 4, 5]. QCD and MQCD can be matched to each other by using perturbation theory.

To obtain the non-perturbative contribution we perform lattice measurements in MQCD [6]. The observable we consider is the elementary plaquette expectation value. The theory being superrenormalisable, one can match the lattice regularization scheme exactly to the $\overline{\text{MS}}$ scheme. This requires a perturbative 4-loop computation on the lattice, however, which has not been completed yet: the missing ingredient is specified below. (A certain perturbative 4-loop computation in full QCD remains also to be carried out.)

2. Relation between $\overline{\text{MS}}$ and lattice regularization schemes

The euclidean pure SU(N_c) Yang-Mills action reads

$$S_E = \int d^d x \mathcal{L}_E, \quad \mathcal{L}_E = \frac{1}{2g_3^2} \sum_{k,l} \text{Tr}[F_{kl}^2], \quad (2.1)$$

where $d = 3 - 2\varepsilon$, g_3^2 is the gauge coupling, $k, l = 1, \dots, d$, $F_{kl} = i[D_k, D_l]$, $D_k = \partial_k - iA_k$, $A_k = A_k^a T^a$, and T^a are hermitean generators of SU(N_c) normalised such that $\text{Tr}[T^a T^b] = \delta^{ab}/2$. The vacuum energy density is defined as

$$f_{\overline{\text{MS}}} \equiv - \lim_{V \rightarrow \infty} \frac{1}{V} \ln \left[\int \mathcal{D}A_k \exp(-S_E) \right]_{\overline{\text{MS}}}, \quad (2.2)$$

where V denotes the d -dimensional volume. The use of the $\overline{\text{MS}}$ dimensional regularization scheme removes any $1/\varepsilon$ poles from the expression. In fact, using dimensional regularization the perturbative result vanishes, because there are no mass scales in the propagators. However, for dimensional reasons, the non-perturbative form of the answer is

$$f_{\overline{\text{MS}}} = -g_3^6 \frac{d_A N_c^3}{(4\pi)^4} \left[\left(\frac{43}{12} - \frac{157}{768} \pi^2 \right) \ln \frac{\bar{\mu}}{2N_c g_3^2} + B_G + \mathcal{O}(\varepsilon) \right], \quad (2.3)$$

where $d_A = N_c^2 - 1$. The logarithmic term has been calculated by introducing a mass scale m_G^2 for gluon and ghost propagators and sending $m_G^2 \rightarrow 0$ after the computation [7, 8].

Using standard Wilson discretization, we can write the same theory on the lattice as

$$S_a = \beta \sum_{\mathbf{x}} \sum_{k < l} \left(1 - \frac{1}{N_c} \text{ReTr}[P_{kl}(\mathbf{x})] \right), \quad (2.4)$$

where P_{kl} is the plaquette, a is the lattice spacing and $\beta \equiv 2N_c/(ag_3^2)$. Hence the continuum limit is taken by $\beta \rightarrow \infty$. Dimensionally, the vacuum energy density consists of terms of the form $g_3^{2n} a^{n-3}$. Thus, approaching the continuum limit, we can relate f_a and $f_{\overline{\text{MS}}}$ as follows:

$$\Delta f \equiv f_a - f_{\overline{\text{MS}}} \quad (2.5)$$

$$= C_1 \frac{1}{a^3} \left(\ln \frac{1}{ag_3^2} + C'_1 \right) + C_2 \frac{g_3^2}{a^2} + C_3 \frac{g_3^4}{a} + C_4 g_3^6 \left(\ln \frac{1}{a\bar{\mu}} + C'_4 \right) + \mathcal{O}(g_3^8 a). \quad (2.6)$$

Taking derivatives of eq. (2.5) with respect to g_3^2 and using 3d rotational and translational symmetries on the lattice, we obtain the master relation

$$8 \frac{d_A N_c^6}{(4\pi)^4} B_G = \lim_{\beta \rightarrow \infty} \beta^4 \left\{ \left\langle 1 - \frac{1}{N_c} \text{Tr}[P] \right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4} (\ln \beta + c'_4) \right] \right\}. \quad (2.7)$$

This quantity may be called the finite part of the gluon condensate in lattice regularization (in certain units).

The values of the constants c_1, \dots, c'_4 are trivially related to those of C_1, \dots, C'_4 . When $N_c = 3$, the numerical values for the c_i 's are

$$c_1 = \frac{d_A}{3} \approx 2.666666667, \quad (2.8)$$

$$c_2 = 1.951315(2), \quad (2.9)$$

$$c_3 = 6.8612(2). \quad (2.10)$$

Here c_1 results from a straightforward 1-loop computation, while c_2 and c_3 have been calculated in refs. [9] and [10], respectively. Because there is no $\bar{\mu}$ -dependence in f_a , the value of c_4 is determined by $f_{\overline{\text{MS}}}$ in eq. (2.3). Consequently,

$$c_4 \approx 2.92942132. \quad (2.11)$$

Because the constant $c'_4 = C'_4 - 1/3 - 2\ln(2N_c)$ is still unknown, we are not able to fully determine B_G . We can, however, determine the non-perturbative input needed for it. In order to evaluate c'_4 a 4-loop lattice perturbation theory calculation is required; alternatively, it can be evaluated non-diagrammatically by means of numerical stochastic perturbation theory [11].

3. Lattice measurements

We need the plaquette expectation value $\langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a$ as a function of β so that the extrapolation $\beta \rightarrow \infty$ in eq. (2.7) can be carried out. For each β the infinite-volume extrapolation is needed. Due to the non-perturbative mass gap of the theory, the finite-volume effects are exponentially small if the size of the box $L = Na$ is large compared with the inverse confinement scale g_3^{-2} . In practice finite-volume effects are invisible as soon as $\beta/N < 1$. This is demonstrated in Fig. 1. In Fig. 2 the infinite-volume extrapolated values of $\langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a$ are plotted as a function of $1/\beta$.

A major difficulty in the simulations is the significance loss caused by the subtractions in eq. (2.7). The dominant term c_1/β is about six orders of magnitude larger than the effect we are

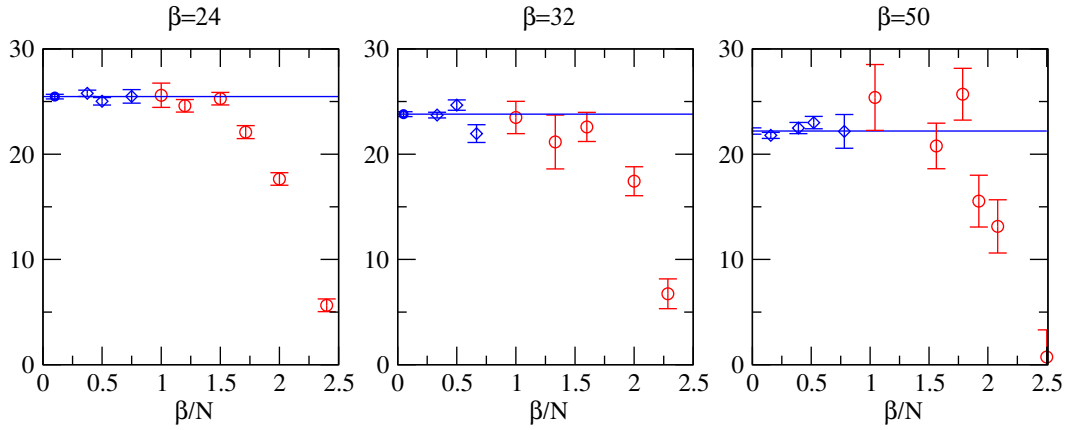


Figure 1: Finite-volume values for $\beta^4 \{ \langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a - [c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln \beta / \beta^4] \}$ as a function of the box size. The leftmost symbols indicate the infinite-volume estimates, obtained by fitting a constant to data in the range $\beta/N < 1$ (blue diamonds). Points denoted by red circles are omitted from the extrapolation.

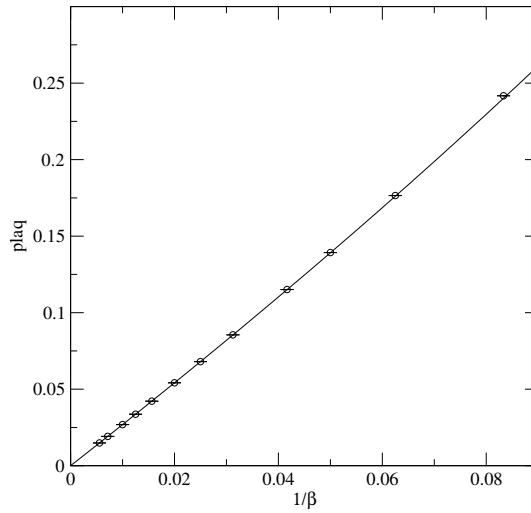


Figure 2: The plaquette expectation value $\text{plaq} \equiv \langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a$ as a function of $1/\beta$. Statistical errors are much smaller than the symbol sizes. The solid line contains the four known terms $c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln \beta / \beta^4$. The effect we are looking for is the difference between the data and the line.

interested in, namely $\sim 1/\beta^4$, if $\beta \sim 100$. Therefore the relative error of our lattice measurements should be smaller than one part in a million. The effect of the subtractions is illustrated in Fig. 3.

Given the infinite-volume limits, we extrapolate the data to the continuum limit, $\beta \rightarrow \infty$. In Fig. 4 we show two functions: $\beta^4 \{ \langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a - [c_1/\beta + c_2/\beta^2 + c_3/\beta^3] \}$ and $\beta^4 \{ \langle 1 - \frac{1}{3} \text{Tr}[P] \rangle_a - [c_1/\beta + c_2/\beta^2 + c_3/\beta^3 + c_4 \ln \beta / \beta^4] \}$. Even the 4-loop logarithmic term is visible in the data. For $1/\beta \leq 0.01$ the significance loss grows rapidly and the error bars become quite large, so that these data points have little effect on the fit.

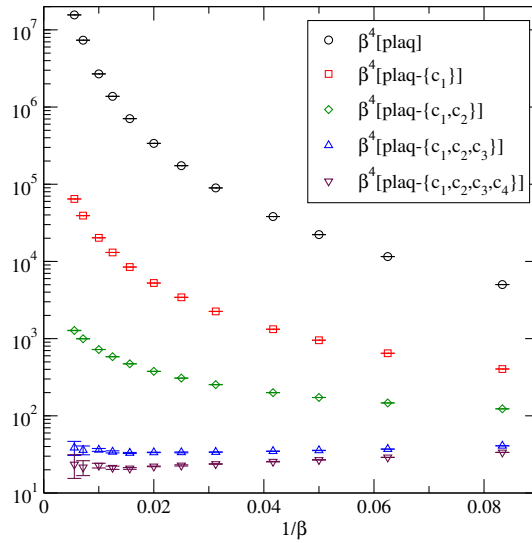


Figure 3: The significance loss due to the subtraction of divergent lattice contributions. Here again $\text{plaq} \equiv \langle 1 - \frac{1}{3}\text{Tr}[P] \rangle_a$, and the symbols c_i indicate which subtractions of eq. (2.7) have been taken into account.

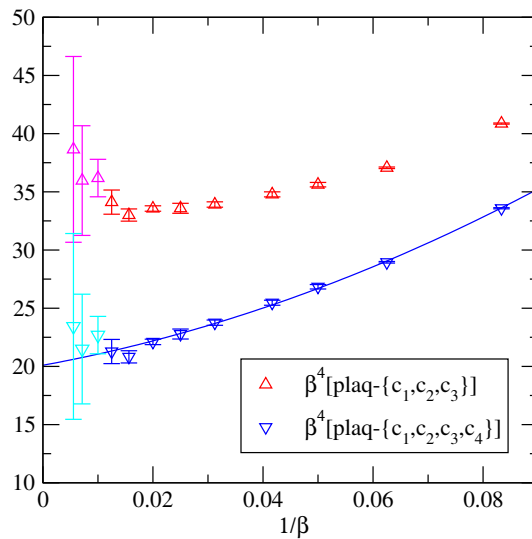


Figure 4: The infinite-volume extrapolated data. The effect of the 4-loop logarithmic divergence is to cause additional upwards “curvature” in the upper data set. The solid blue line gives the continuum extrapolation. Points with lighter colors have so large errors that they are insignificant as far as the fit is concerned.

The continuum extrapolation is carried out by fitting a function $d_1 + d_2/\beta + d_3/\beta^2$ to the infinite-volume extrapolated data, from which all the divergences ($\{c_1, c_2, c_3, c_4\}$) have been subtracted, in the range $0.01 < 1/\beta < 0.10$. The fitted values are $d_1 = 20.0(7)$, $d_2 = 86(24)$ and $d_3 = 909(192)$ with $\chi^2/\text{dof} = 5.8/6$. The error limits are the projections of the 68% confidence level contour onto the various axes. The systematic errors from the effect of higher order terms are inside these errors.

Substituting this to eq. (2.7) we obtain the final result,

$$B_G + \left(\frac{43}{12} - \frac{157}{768} \pi^2 \right) c'_4 = \frac{4\pi^4}{36} \times 20.0(7) = 10.7(4). \quad (3.1)$$

4. Conclusions

We have studied the expectation value of the elementary plaquette in 3d pure SU(3) theory and outlined how the 3d vacuum energy density in the $\overline{\text{MS}}$ scheme can be extracted from it. However, to achieve this, the constant c'_4 should be determined (cf. eqs. (2.3), (3.1)). This can be accomplished by a 4-loop matching computation, with the techniques discussed in ref. [11]. The full QCD pressure of order $g^6 T^4$ can be obtained by a further 4-loop matching computation.

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