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Spatial string tension revisited

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The spatial string tension, a classic non-perturbative probe for the convergence of the weak-coupling expansion at high temperatures, can be determined in full QCD as well as in a dimensionally reduced effective theory. Comparing both approaches, we find surprisingly good agreement almost down to the critical temperature of the deconfinement phase transition.

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1. Introduction

The interest in QCD at temperatures T larger than (a few) hundred MeV is triggered not only by purely theoretical reasons, but also by ongoing heavy ion collision experiments, and by cosmology. Given asymptotic freedom, a weak coupling expansion of this high-temperature phase seems well within reach. In practice, however, this expansion converges only slowly, and even shows a non-trivial analytic structure in the gauge coupling g^2.

By now, the problematic degrees of freedom have been identified. They are *soft* gauge-field modes with typical momenta $p \sim gT$, which give rise to odd powers in g, as well as *ultrasoft* modes $p \sim g^2T$, which enter the series via non-perturbative coefficients. For parametrically small values of the coupling g, these scales are well separated, such that an effective field theory treatment becomes feasible.

The general picture is that perturbation theory should work fine for parametrically *hard* scales $p \sim 2\pi T$, while soft and ultrasoft scales need improved analytic schemes, or non-perturbative treatment. We will work within dimensionally reduced effective theories, in order to treat these different physical contributions separately, in a consistent scheme with controllable errors.

It appears mandatory to give quantitative evidence for the general picture sketched above. To this end, the strategy is to pick some simple observables and compare, as a function of T, *full* results (e.g. from 4d lattice QCD simulations [1]) with *predictions* from the soft/ultrasoft effective theory setup, which should be exact for asymptotically large temperatures. This has been done for e.g. static correlation lengths [2], and in general agreement was found down to $T \sim 2T_\mathrm{c}$, where T_c is the deconfinement phase transition temperature.

As another concrete example of an observable allowing for an unambiguous comparison, we discuss the spatial string tension σ_s in this paper. It is defined in a manifestly gauge invariant way as the coefficient in the area law of a large rectangular Wilson loop $W_s(R_1,R_2)$ in (x_1,x_2) plane,

$$\sigma_s \equiv -\lim_{R_1\to\infty}\lim_{R_2\to\infty}\frac{1}{R_1R_2}\ln W_s(R_1,R_2)\,. \tag{1.1}$$

It has been measured in SU(3) on the 4d lattice, as a function of the temperature T (e.g. Ref. [1]),

$$\frac{\sqrt{\sigma_s}}{T} = \phi_a\left(\frac{T}{T_\mathrm{c}}\right)\,. \tag{1.2}$$

Our aim here is to get the effective theory prediction for σ_s, and to compare it with the lattice data, in order to assess the performance of the effective theory setup [3]. In the following two sections, we sketch the 2-step perturbative matching process of 4d QCD onto 3d M(agnetostatic)QCD, and discuss convergence properties. In section 4, we take existing data on σ_s from 3d lattice MQCD, match it to 4d QCD, and compare with the 4d lattice data.

2. Effective theory setup: QCD → EQCD

At high temperatures, all QCD dynamics is contained in a simpler, three-dimensional effective field theory called EQCD,

$$\mathscr{L}_\mathrm{E} = \frac{1}{2}\mathrm{Tr}F_{kl}^2 + \mathrm{Tr}[D_k,A_0]^2 + m_\mathrm{E}^2\mathrm{Tr}A_0^2 + \lambda_\mathrm{E}^{(1)}(\mathrm{Tr}A_0^2)^2 + \lambda_\mathrm{E}^{(2)}\mathrm{Tr}A_0^4 + \ldots\,, \tag{2.1}$$

where $F_{kl} = i[D_k, D_l]/g_\mathrm{E}$, $D_k = \partial_k - ig_\mathrm{E}A_k$ with the dimensionful 3d gauge coupling g_E, and the dots represent higher-order operators. In order to correctly describe all contributions from hard and soft scales, the parameters of 3d EQCD have to be regarded as matching coefficients, and are therefore related to the parameters of full QCD (being g^2, T, N_c, N_f, μ_q, m_q). Perturbative matching [4] gives, schematically,

$$m_\mathrm{E}^2 = T^2\left\{\#g^2 + \#g^4 + ...\right\}, \tag{2.2}$$

$$\lambda_\mathrm{E}^{(1),(2)} = T\left\{\#g^4 + \#g^6 + ...\right\}, \tag{2.3}$$

$$g_\mathrm{E}^2 = T\left\{g^2 + \#g^4 + \#g^6 + ...\right\}, \tag{2.4}$$

where all coefficients symbolized by "#" above are known. Most can be conveniently read from e.g. Ref. [5], while the g^6 term in the last line has been obtained only recently [3]. Higher-order coefficients could be obtained straightforwardly from the next order in the loop expansion.

There are also higher-order operators [6] in EQCD which become important at some point. In general, their relative magnitude can be estimated as [5]

$$\delta\mathscr{L}_\mathrm{E} \sim g^2\frac{D_kD_l}{(2\pi T)^2}\mathscr{L}_\mathrm{E} \sim g^2\frac{(g^2T)^2}{(2\pi T)^2}\mathscr{L}_\mathrm{E}, \tag{2.5}$$

where we assumed to be considering an observable dominated by the ultrasoft scale $p \sim g^2T$. Thus, the relative magnitude is at most $\sim g^6$, smaller than any known terms in Eqs. (2.2)–(2.4).

At this point, having the first few terms of the perturbative series of, say, $g_\mathrm{E}^2 = g_\mathrm{E}^2(g^2,T)$ at hand, one may ask about its convergence properties. In practice, renormalization is needed of course: let $g^2 = g^2(\bar\mu)$ be the (4d QCD) $\overline{\mathrm{MS}}$ coupling. From the solution of the 2-loop renormalization group equation, we define the $\overline{\mathrm{MS}}$ scale parameter as usual, and find the 2-loop running coupling as a function of $\bar\mu/\Lambda_{\overline{\mathrm{MS}}}$,

$$\Lambda_{\overline{\mathrm{MS}}} \equiv \lim_{\bar\mu\to\infty}\bar\mu\Big[b_0g^2(\bar\mu)\Big]^{-b_1/2b_0^2}\exp\Big[-\frac{1}{2b_0g^2(\bar\mu)}\Big], \tag{2.6}$$

$$\frac{1}{g^2(\bar\mu)} \approx 2b_0\ln\frac{\bar\mu}{\Lambda_{\overline{\mathrm{MS}}}} + \frac{b_1}{b_0}\ln\left(2\ln\frac{\bar\mu}{\Lambda_{\overline{\mathrm{MS}}}}\right), \tag{2.7}$$

where $b_0 \equiv -\beta_0/2(4\pi)^2$, $b_1 \equiv -\beta_1/2(4\pi)^4$ are coefficients of the QCD beta function. Hence, we can now write $g_\mathrm{E}^2 = g_\mathrm{E}^2(\bar\mu,\Lambda_{\overline{\mathrm{MS}}},T) = T\,\phi_b(\bar\mu/T, T/\Lambda_{\overline{\mathrm{MS}}})$ as a function of two dimensionless variables.

Formally, the renormalization scale dependence is of higher order, while numerically, there is $\bar\mu$ dependence due to our truncation of the perturbative series. We are free to choose some optimization procedure, e.g. the *principle of minimal sensitivity*, according to which we choose $\bar\mu_\mathrm{opt}$ as the extremum of the 1-loop expression for g_E^2. This leaves us $g_\mathrm{E}^2 = T\,\phi_c(T/\Lambda_{\overline{\mathrm{MS}}})$ as a function of one variable only, which is plotted in the left panel of Fig. 1 for $N_\mathrm{f} = 3$. Comparing 1-loop and 2-loop expressions (the gray band shows the effect of a scale variation within $\bar\mu = (0.5...2.0)\times\bar\mu_\mathrm{opt}$), note that the process of perturbative matching shows very comforting convergence properties: corrections are in the 10-20% range, and scale dependence gets significantly reduced.

In the right panel of Fig. 1, we show the effective gauge coupling $\alpha_\mathrm{s}^\mathrm{eff} \equiv g_\mathrm{E}^2/4\pi T$ of EQCD, for several N_f, in a much smaller temperature interval close to the phase transition temperature $T_\mathrm{c} \sim \Lambda_{\overline{\mathrm{MS}}}$. Noting that this 3d effective coupling is reasonably small even at these low temperatures, we are led yet again to observe that treating the hard modes perturbatively appears well justified.

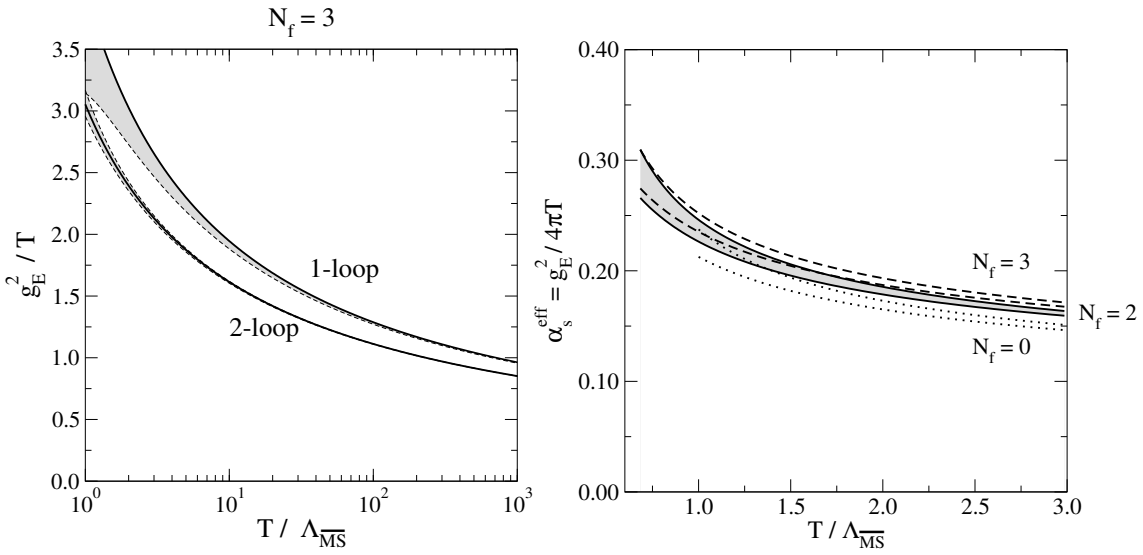

Figure 1: Left panel: *The 1- and 2-loop values for g_E^2/T. For each T, the scale $\bar{\mu}$ has been fixed to $\bar{\mu}_{opt}$ as explained in the text. The gray band corresponds to a variation of $\bar{\mu}_{opt}$ within a factor of two.* Right panel: *The effective 2-loop gauge coupling of EQCD, $\alpha_s^{eff} \equiv g_E^2/4\pi T$, for various values of N_f. Scale dependence results from $\bar{\mu}_{opt}$ variation as before. Note the different ranges of temperatures on the horizontal axes.*

3. Effective theory setup: EQCD → MQCD

The low-energy behaviour of 3d EQCD is contained in another three-dimensional effective field theory, called MQCD,

$$\mathscr{L}_{\rm M} = \frac{1}{2}{\rm Tr}F_{kl}^2 + \dots \,. \tag{3.1}$$

As before, the dots stand for higher-order operators, while the matching coefficients can be determined perturbatively [7, 3]

$$g_{\rm M}^2 = g_{\rm E}^2 \left\{ 1 + \# \frac{g_{\rm E}^2}{m_{\rm E}} + \# \frac{g_{\rm E}^4}{m_{\rm E}^2} + \# \frac{g_{\rm E}^2 \lambda_{\rm E}^{(1),(2)}}{m_{\rm E}^2} + \dots \right\} . \tag{3.2}$$

Let us note here – without showing the corresponding plot – that this expansion converges extremely well, even close to $T_{\rm c}$. Hence, we can safely ignore higher loop corrections for $g_{\rm M}^2$.

The higher-order operators of MQCD,

$$\delta\mathscr{L}_{\rm M} \sim g_{\rm E}^2 \frac{D_k D_l}{m_{\rm E}^3} \mathscr{L}_{\rm M} \sim g_{\rm E}^2 \frac{(g^2 T)^2}{m_{\rm E}^3} \mathscr{L}_{\rm M} \,, \tag{3.3}$$

give a relative contribution parametrically smaller than any of the known terms in Eq. (3.2), and will be neglected in the following.

4. Results

We are now in a position to write down the effective theory prediction for the spatial string tension σ_s, Eq. (1.1). The observable σ_s exists not only in 4d QCD, but also in 3d SU(3) gauge

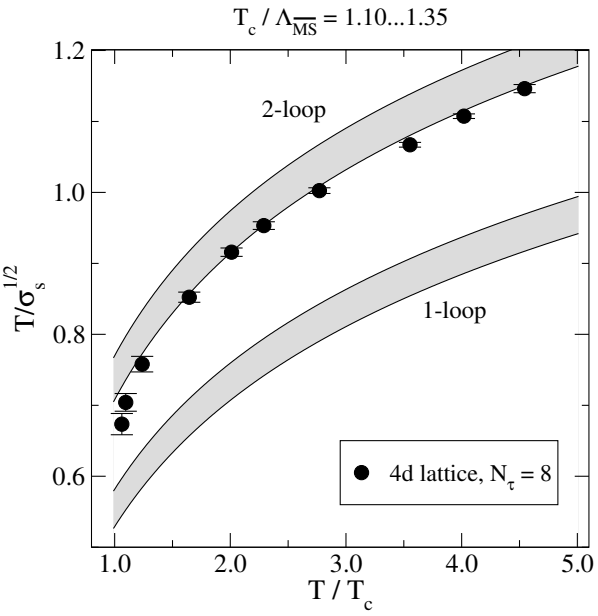

Figure 2: *Comparison of 4d lattice data for the spatial string tension [1] with expressions obtained by combining 1-loop and 2-loop results for g_E^2 together with Eq. (3.2) and the non-perturbative value of the string tension of 3d SU(3) gauge theory, Eq. (4.1). The upper edges of the bands correspond to $T_c/\Lambda_{\overline{\rm MS}} = 1.35$, the lower edges to $T_c/\Lambda_{\overline{\rm MS}} = 1.10$.*

theory, which is nothing but MQCD, Eq. (3.1). Since the 3d gauge coupling is dimensionful, and furthermore is the only scale that MQCD possesses, naive dimensional analysis dictates $\sigma_s = \# g_{\rm M}^4$. The proportionality constant is non-perturbative, and can be measured by 3d lattice simulations. Taking most recent lattice data [8],

$$\frac{\sqrt{\sigma_s}}{g_{\rm M}^2} = 0.553(1)\,. \tag{4.1}$$

To compare with the 4d lattice results of the form shown in Eq. (1.2) (see Fig. 2), we need to relate $g_{\rm M}^2$ and T. First, using Eq. (3.2) and Eqs. (2.2)–(2.4),

$$\frac{\sqrt{\sigma_s}}{T} = 0.553(1)\,\frac{g_{\rm M}^2}{g_{\rm E}^2}\,\frac{g_{\rm E}^2}{T} = \phi_d\left(\frac{T}{\Lambda_{\overline{\rm MS}}}\right)\,. \tag{4.2}$$

Next, we need to relate $\Lambda_{\overline{\rm MS}}$ and T_c. This is in fact a classic problem in (4d) lattice QCD. One line of measurements [9] employs the $T = 0$ string tension to get [10]

$$\frac{T_c}{\Lambda_{\overline{\rm MS}}} = \frac{T_c/\sqrt{\sigma}}{\Lambda_{\overline{\rm MS}}/\sqrt{\sigma}} = 1.16(4)\,, \tag{4.3}$$

while another possibility is to go via the Sommer scale [11]

$$\frac{T_c}{\Lambda_{\overline{\rm MS}}} = \frac{r_0 T_c}{r_0 \Lambda_{\overline{\rm MS}}} = 1.25(10)\,. \tag{4.4}$$

To be conservative, we will consider the interval $T_c/\Lambda_{\overline{\rm MS}} = 1.10\ldots1.35$, which also incorporates the result of Ref. [12].

In Fig. 2, we finally compare the 3d effective theory prediction for σ_s (gray bands) with the 4d lattice data (black dots). As a caveat, note that the lattice data has not been extrapolated to the continuum limit. On the other hand, we stress that the comparison is parameter-free. We may take the excellent agreement of the 2-loop prediction with the lattice data as support for hard/soft+ultrasoft picture of thermal QCD.

To conclude, we have given yet another example of a static observable in thermal QCD, for which the program of dimensional reduction works well, even down to temperatures $T \sim 2T_c$.

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