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Large-N_f behavior of the Yukawa model: analytic results

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We investigate the Yukawa model in which N_f fermions are coupled with a scalar field ϕ through a Yukawa interaction. The phase diagram is rather well understood. If the fermions are massless, there is a chiral transition at T_c: for $T < T_c$ chiral symmetry is spontaneously broken. At $N_f = \infty$ the transition is mean-field like, while, for any finite N_f, standard arguments predict Ising behavior. This apparent contradiction has been explained by Kogut et al., who showed by scaling arguments and Monte Carlo simulations that in the large-N_f limit the width of the Ising critical region scales as a power of $1/N_f$, so that only mean-field behavior is observed for N_f strictly equal to infinity. We will show how the results of Kogut et al. can be recovered analytically in the framework of a generalized $1/N_f$ expansion. The method we use is a simple generalization of the method we have recently applied to a two-dimensional generalized Heisenberg model.

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Finite-temperature transitions in quantum field theory models have been the object of many theoretical studies. Here we investigate the transition in the 2+1 Yukawa model (but the arguments can be generalized to generic (d+1) models, $d \leq 4$) in which a scalar field is coupled to N_f degenerate fermions by a Yukawa interaction. As discussed in Ref. [1] this model shows a peculiar behavior for $N_f \to \infty$. For finite N_f, dimensional reduction predicts that the finite-temperature transition, if continuous, belongs to the two-dimensional Ising universality class. On the other hand, for $N_f = \infty$ an explicit calculation gives mean-field behavior [2]. These two apparently contradictory results were explained in [1] in terms of a *critical-region suppression*. A similar behavior was observed in a generalized $O(N)$ σ model in [3] and an explanation was provided in [4]. The same techniques developed in [4] can be applied here to obtain an analytic description of the crossover from mean-field to Ising behavior in the large-N_f limit.

We will investigate the Yukawa model with action [1]

$$\mathscr{S}[\phi,\overline{\psi},\psi] = \int d^3\mathbf{r}\left[\frac{N_f}{2}\left(\partial\phi\right)^2 + N_f\frac{\mu}{2}\phi^2 + N_f\frac{\lambda}{4!}\phi^4 + \sum_{f=1}^{N_f}\overline{\psi}_f\left(\partial\!\!\!/ + g\phi + M\right)\psi_f\right], \qquad (1)$$

where the integral is over $\mathbb{R}^2 \times [0,1/T]$ and we use periodic (antiperiodic) boundary conditions for the boson (fermion) in the “temporal" direction. We imagine the theory to be somehow regularized with a cutoff that sets the energy scale. The chosen regularization is not relevant for the discussion, and, for instance, the reader may imagine using the lattice action with staggered fermions studied in [1].

In order to determine the behavior for $N_f = \infty$ we integrate out the fermionic degrees of freedom. Starting from (1) we obtain

$$e^{-N_f\tilde{\mathscr{S}}_{\rm eff}[\phi]} = \int\prod_{f=1}^{N_f}{\rm d}\overline{\psi}_f{\rm d}\psi_f\, e^{-\mathscr{S}[\phi,\overline{\psi},\psi]} \qquad (2)$$

$$\tilde{\mathscr{S}}_{\rm eff}[\phi] = \frac{1}{2}\left(\partial\phi\right)^2 + \frac{\mu}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - {\rm tr}\log\left(\partial\!\!\!/ + g\phi + M\right). \qquad (3)$$

For $N_f = \infty$ we can perform an expansion around a translation-invariant saddle point $\overline{\phi}$. The stationarity condition gives the gap equation

$$\overline{\mu}m + \tfrac{1}{6}\overline{\lambda}m^3 = (m+M)T\sum_{n\in\mathbb{Z}}\int^{\Lambda}\frac{d^2\mathbf{p}}{(2\pi)^2}\,\frac{1}{p^2+(m+M)^2+(2n+1)^2\pi^2T^2}. \qquad (4)$$

Here we have defined $m \equiv g\overline{\phi}$, $\overline{\mu} \equiv \mu g^{-2}$, and $\overline{\lambda} \equiv \lambda g^{-4}$. With this choice, the dependence on g disappears. Note that the gap equation is symmetric under $m \to -m$, and $M \to -M$ so that it is not restrictive to consider $m \geq 0$.

We first analyze the model for $T = 0$. A simple analysis of the gap equation shows that there are two possibilities for $M = 0$. There is a critical value $\overline{\mu}_c$ such that for $\overline{\mu} < \overline{\mu}_c$ the relevant solution of the gap equation is such that $m \neq 0$ (chiral symmetry is broken), while in the opposite case we have $m = 0$ (no chiral symmetry breaking). For $M \neq 0$ the behavior is smooth in the parameters. In the following we will only be interested in the case in which the zero-temperature theory shows chiral symmetry breaking. Thus, we shall assume $\overline{\mu} < \overline{\mu}_c$. Parameters $\overline{\lambda}$ and $\overline{\mu}$ do not play any additional role in the model and thus in the following we will not consider explicitly the dependence on them.

Let us now consider the behavior for $T \neq 0$. We restrict the discussion to $\overline{\mu} > 0$; in this case symmetry is restored for $T \to \infty$ and $M = 0$ and therefore, there exists a value T_c such that for $T < T_c$, m is a nonvanishing function of T (chiral symmetry is broken), while for $T \geq T_c$, $m = 0$ (chiral symmetry is restored). Explicitly, by using (4) we obtain the relation

$$\overline{\mu} = T_c \sum_{n\in\mathbb{Z}} \int^{\Lambda} \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{1}{p^2 + (2n+1)^2\pi^2 T_c^2}. \tag{5}$$

If $M \neq 0$, the behavior is smooth in T. Thus, $T = T_c$, $M = 0$ is a critical point (CP) for model (1).

Close to the CP one can define a thermal scaling field u_t and a magnetic scaling field u_h. Using the gap equation it is easy to see that we can take $u_t = (T - T_c)/T_c$ and $u_h = M/T_c$. Then, for $u_t, u_h \to 0$ at fixed $x \equiv u_t u_h^{-2/3}$, one obtains the mean-field equation of state

$$m/T_c = u_h{}^{1/3} f(x) \qquad\qquad a f(x)^3 + b x f(x) + 1 = 0 \tag{6}$$

with $a > 0$. From (6), if $u_t = 0$, we obtain $m \sim u_h{}^{1/3}$, so that $\delta = \delta_{\rm MF} = 3$; if $u_h = 0$, we have $m \sim u_t{}^{1/2}$ so that $\beta = \beta_{MF} = 1/2$. Thus, for $N_f = \infty$, the behavior is of mean-field type, in agreement with previous results [2]. Note that the condensate $\Sigma = \langle\overline{\psi}\psi\rangle$ is proportional to $M + m \approx u_h{}^{1/3} f(x)$ in the scaling limit. A completely analogous discussion holds for the staggered lattice model.

In order to perform the $1/N_f$ calculation, we expand action (3) around the saddle-point solution, writing $\phi = \overline{\phi} + \widehat{\phi}/\sqrt{N_f}$. We obtain the expansion

$$\begin{aligned}
\tilde{\mathscr{S}}_{\rm eff}[\phi_n] = {} & \frac{1}{2}\sum_n \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\, \phi_n(\mathbf{p})\, \Delta_n^{-1}(\mathbf{p})\, \phi_{-n}(-\mathbf{p}) \\
& + \frac{1}{3!\sqrt{N_f}} \sum_{n,m} \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\, \tilde{V}^{(3)}(\mathbf{p},n;\mathbf{q},m;-\mathbf{p}-\mathbf{q},-m-n) \\
& \qquad\qquad \phi_n(\mathbf{p})\, \phi_m(\mathbf{q})\, \phi_{-n-m}(-\mathbf{p}-\mathbf{q}) \\
& + \frac{1}{4!N_f} \sum_{n,m,t} \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\, \tilde{V}^{(4)}(\mathbf{p},n;\mathbf{q},m;\mathbf{k},t;-\mathbf{p}-\mathbf{q}-\mathbf{k},-m-n-t) \\
& \qquad\qquad \phi_n(\mathbf{p})\, \phi_m(\mathbf{q})\, \phi_t(\mathbf{k})\, \phi_{-n-m-t}(-\mathbf{p}-\mathbf{q}-\mathbf{k}),
\end{aligned} \tag{7}$$

where the neglected terms are of order $1/N_f^{3/2}$. Here $\phi_n(\mathbf{p})$ is the Fourier transform of $\widehat{\phi}(x,\mathbf{r})$ ($x \in [0, T^{-1}]$, $\mathbf{r}, \mathbf{p} \in \mathbb{R}^2$):

$$\widehat{\phi}(x,\mathbf{r}) = \sum_{n\in\mathbb{Z}} e^{2\pi i x n T} \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\, \phi_n(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{8}$$

In the standard approach, $1/N_f$ expansions are obtained by performing a perturbative expansion of theory (7) in powers of $1/N_f$. This is possible here only far from the CP, since at the CP the field ϕ_0 is massless and the expansion is plagued by infrared divergences. Indeed, starting from the explicit expression

$$g^{-2}\Delta_0^{-1}(\mathbf{p} = \mathbf{0}) = \overline{\mu} + \frac{1}{2}\overline{\lambda} m^2 - T \sum_{n\in\mathbb{Z}} \int^{\Lambda} \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{q^2 + (2n+1)^2\pi^2 T^2 - (m+M)^2}{[q^2 + (2n+1)^2\pi^2 T^2 + (m+M)^2]^2} \tag{9}$$

and using (5), we find that close to the CP

$$\Delta_0^{-1}(\mathbf{0}) \sim m^2, (m+M)^2 \tag{10}$$

No singularity arises for the other modes ϕ_n, since $\Delta_n^{-1}(\mathbf{p}=0) = (2\pi nT)^2$ at the CP. The infrared singularity of $\Delta_0(\mathbf{p})$ is of course expected: $\Delta_n^{-1}(\mathbf{p})$ is proportional to the square of the mass of the boson field that should vanish at the CP (at the CP the correlation length diverges).

In order to deal with this singularity we use the technique we have recently applied in [4] to a generalized Heisenberg model in two dimensions. We first integrate out the massive modes and obtain an effective action for the zero mode:

$$e^{-\mathscr{S}_{\text{eff}}[\varphi]} = \int \prod_{n\neq 0} \mathrm{d}\phi_n e^{-\tilde{\mathscr{S}}_{\text{eff}}[\phi]}. \tag{11}$$

The effective action $\mathscr{S}_{\text{eff}}[\varphi]$ has an expansion in powers of $1/N_f$ of the form

$$\begin{aligned}\mathscr{S}_{\text{eff}}[\varphi] = & H\varphi(\mathbf{0}) + \frac{1}{2}\int_{\mathbf{p}}[K(\mathbf{p})+r]\varphi(\mathbf{p})\varphi(-\mathbf{p}) \\ & + \frac{\sqrt{u}}{3!}\int_{\mathbf{p}}\int_{\mathbf{q}} V^{(3)}(\mathbf{p},\mathbf{q},-\mathbf{p}-\mathbf{q})\varphi(\mathbf{p})\varphi(\mathbf{q})\varphi(-\mathbf{p}-\mathbf{q}) \\ & + \frac{u}{4!}\int_{\mathbf{p}}\int_{\mathbf{q}}\int_{\mathbf{s}} V^{(4)}(\mathbf{p},\mathbf{q},\mathbf{s},-\mathbf{p}-\mathbf{q}-\mathbf{s})\varphi(\mathbf{p})\varphi(\mathbf{q})\varphi(\mathbf{s})\varphi(-\mathbf{p}-\mathbf{q}-\mathbf{s}),\end{aligned} \tag{12}$$

where $\varphi = a\phi_0 + b$, $u = c/N_f$, and a, b, c are functions of T and M fixed by the following normalization conditions:

$$\begin{aligned} & K(\mathbf{p}) = \mathbf{p}^2 + O(\mathbf{p}^4) \qquad \text{for } \mathbf{p}\to 0 \\ & V^{(4)}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 1 \\ & V^{(3)}(\mathbf{0},\mathbf{0},\mathbf{0}) = 0 \end{aligned} \tag{13}$$

for any T and M close to the CP. The last condition is not trivial and can be imposed because the same property holds at the CP for $\tilde{V}^{(3)}(\mathbf{0},0;\mathbf{0},0;\mathbf{0},0)$. Again, in (12) we have neglected higher-order terms in $1/N_f$.

At this point, the origin of the mean-field–to–Ising crossover is quite clear. For $N_f = \infty$ ($u = 0$) the zero mode is a free field and thus the model shows mean-field behavior. On the other hand, for finite N_f, one must consider the full theory (12), which is nothing but a generalized φ^4 theory whose critical behavior belongs to the Ising universality class. Note that in (12) we have discarded higher-order vertices φ^n that are multiplied by higher powers of $1/N_f$: one can show that they do not play any role in the crossover limit we shall describe below [4].

We wish now to compute the crossover behavior. Formally, the momentum dependence of the vertices is irrelevant (one may think of the formal continuum limit in lattice theories) and thus we can simply consider (this relation becomes exact in the infrared limit, in which one only considers the long-distance behavior; for a proof in a lattice framework, see [4])

$$\mathscr{S}_{\text{cont}}[\varphi] = \int d^2\mathbf{r}\, \left[H\varphi + \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}r\varphi^2 + \tfrac{1}{4!}u\varphi^4\right]. \tag{14}$$

Then, we define $\chi(\mathbf{r}) = \varphi(\mathbf{r}/\sqrt{u})$ and note that we can rewrite the action as

$$\mathscr{S}_{\text{cont}}[\chi] = \int d^2\mathbf{r}\, \left[\frac{H}{u}\chi + \frac{1}{2}(\partial\chi)^2 + \frac{r}{2u}\chi^2 + \frac{1}{4!}\chi^4\right]. \tag{15}$$

In terms of χ, the action is a function of H/u and r/u. Then, consider the zero-momentum connected correlation function χ_n. We have

$$\chi_n = \int d^2\mathbf{r}_2 \ldots d^2\mathbf{r}_n \, \langle \varphi(\mathbf{0})\varphi(\mathbf{r}_2)\ldots\varphi(\mathbf{r}_n)\rangle^{\text{conn}}$$
$$= u^{n-1}\int d^2\mathbf{r}_2 \ldots d^2\mathbf{r}_n \, \langle \chi(\mathbf{0})\chi(\mathbf{r}_2)\ldots\chi(\mathbf{r}_n)\rangle^{\text{conn}} = u^{n-1} f_n(H/u, r/u), \tag{16}$$

i.e. $u^{1-n}\chi_n$ is a scaling function of H/u and r/u. Unfortunately, the derivation is not correct, since we have not taken into account the presence of the cutoff that breaks scale invariance. However, in two dimensions (and, in general, for $d < 4$) only a mass renormalization (a redefinition of the parameter r) is needed in order to take care of divergencies. By a proper treatment [5] one can show that there is a function $r_c(u)$ (in two dimensions the determination of this function requires only a one-loop computation since the only diverging diagram is the tadpole) such that, for $t \equiv r - r_c(u) \to 0$ (infrared limit), $H \to 0$, and $u \to 0$ (weak-coupling limit), the correlation function χ_n satisfies the scaling relation $\chi_n = u^{n-1} f_n(H/u, t/u)$. These results extend to theory (12) although the presence of odd powers of φ requires also a renormalization of H. In [4] we showed that one can find functions $r_c(u)$ and $H_c(u)$, such that, for $t \equiv r - r_c(u) \to 0$, $h \equiv H - H_c(u) \to 0$, $u \to 0$ at fixed h/u and t/u, the correlation function χ_n scales as

$$\chi_n \sim u^{1-n} f_n(h/u, t/u). \tag{17}$$

Functions $f_n(x,y)$ are universal: they do not depend on the explict form of the vertices and of $K(\mathbf{p})$ and can be computed directly in the continuum theory. They are the crossover functions that relate mean-field and Ising behavior. Consider, for instance, the case $h = 0$. For t fixed and $u \to 0$ we obtain the standard perturbative expansion; thus, $t/u \to \infty$ corresponds to the mean-field limit. On the other hand, for $t \to 0$ at u fixed, Ising behavior is obtained; $t/u = 0$ is the nonclassical limit. By varying t/u between 0 and ∞ one obtains the full universal crossover behavior. The universality of the crossover implies that these functions can be computed in completely different settings: one can use field theory [5, 6], generalized Heisenberg models such as those considered in [4], or medium-range models (see [6, 7] and references therein). For instance, crossover curves for the effective exponents for $M = 0$ can be found in [6] (field theory) and in [8] (in this case the correspondence is $R^2 = bN_f$, where the constant b can be computed by matching the corresponding perturbative expansions at one loop).

The analysis of [4] can be simplified in the present model since chiral symmetry is preserved by regularization. In this case the relations

$$H = 0 \qquad\qquad V^{(3)}(\mathbf{p},\mathbf{q},\mathbf{r}) = 0 \tag{18}$$

hold at the CP. They imply $H_c(u) = 0$.

In theory (12) r, H, $r_c(u)$, and u are functions of T, M, N_f. Thus, the next step consists in determining how these quantities should scale close to the CP in order to keep $x_t \equiv N_f(r - r_c(N_f))$ and $x_h \equiv N_f(H - H_c(N_f))$ fixed. A calculation gives [4]

$$T_c(N_f) - T_c(N_f = \infty) \approx \frac{a + b\log N_f}{N_f}, \tag{19}$$

where a and b can be explicitly computed. Moreover, we have

$$\frac{T - T_c(N_f)}{T_c(N_f)} \sim \frac{x_t}{N_f} \qquad \frac{M}{T_c(N_f)} \sim \frac{x_h}{{N_f}^{3/2}}. \tag{20}$$

Ising behavior is observed for $x_t, x_h \ll 1$. This confirms the critical-region suppression predicted in [1]: The width of the Ising critical region scales as N_f^{-1} in the thermal direction and as $N_f^{-3/2}$ in the magnetic one. For $M = 0$ one can also characterize the crossover in terms of the bosonic mass $M_{\rm bos}$. Indeed, similar arguments (see [6]) allow us to predict

$$\frac{M_{\rm bos}}{T_c} \sim \frac{1}{N_f^{1/2}} f_M[x_t \equiv N_f(T - T_c)/T_c], \tag{21}$$

where $f_M(x)$ is a crossover function that behaves as x for $x \ll 1$ ($\nu = 1$ in the Ising theory) and as $\sqrt{x}$ for $x \gg 1$ ($\nu = 1/2$ for a Gaussian field). This relation shows that fixing x_t is essentially equivalent to fixing $x_M = N_f^{1/2} M_{\rm bos}/T_c$: Ising behavior is observed for $x_M \ll 1$, mean-field behavior in the opposite case. This is exactly the scaling condition discussed in [1].

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