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# Lattice and renormalons in heavy quark physics

**Antonio Pineda**[*†]
*Dept. d'Estructura i Constituents de la Matèria,*
*U. Barcelona, Diagonal 647, E-08028 Barcelona, Catalonia, Spain*
*E-mail:* pineda@ecm.ub.es

Perturbative expansions of QCD observables in powers of $\alpha_s$ are believed to be asymptotic and non-Borel summable due to the existence of singularities in the Borel plane (renormalons). This fact is connected with the factorization of scales (which is inherent to QCD and asymptotic freedom) and jeopardizes the convergence of the perturbative expansion and the accurate determination of power-suppressed corrections. This problem is more acute for physical systems composed by one or more heavy quarks. In lattice regulations, it reflects on the appearance of power-like divergences in the inverse of the lattice spacing for a series of quantities ($\bar{\Lambda}$, gluelump masses, the singlet and hybrid potentials, ...) making that the continuum limit can not be reached for them. Nevertheless, all these problems are solved within the framework of effective field theories with renormalon substraction. This allows us to obtain convergent perturbative series and to unambiguously define power corrections. In particular, one can connect with lattice results. Remarkably enough the dependence on the lattice spacing can be predicted by perturbation theory. This framework has been applied to the prediction of the gluelump masses and the singlet and octet (hybrid) potentials at short distances, as well as to their comparison with lattice simulations. Overall, very good agreement with data is obtained.

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[*]Speaker.
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## 1. Introduction

The perturbative series relating the pole and the $\overline{\rm MS}$ mass:

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n(\nu)\alpha_s^{n+1}(\nu)\,, \tag{1.1}$$

suffers from renormalon ambiguities, which makes this series asymptotic and non-Borel summable. The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform:

$$B[m_{\rm OS}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}}\left(1 + c_1(1-2u)c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}), \tag{1.2}$$

where ($u = \frac{\beta_0 t}{4\pi}$)

$$m_{\rm OS} = m_{\overline{\rm MS}} + \int_0^{\infty} {\rm d}t\, e^{-t/\alpha_s}\, B[m_{\rm OS}](t)\,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}. \tag{1.3}$$

The asymptotic behavior of the coefficients of the perturbative series then reads

$$r_n^{as} \overset{n\to\infty}{=} N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right). \tag{1.4}$$

Quite remarkable, it is possible to obtain:
**a)** the coefficients $b$, $c_1$, $c_2$,.., exactly through the use of the renormalization group [1, 2] (actually they will depend on the coefficients of the beta function: $\beta_0$, $\beta_1$, ...).
**b)** approximate determinations of the normalization constant, $N_m$, [2] by defining new functions with improved analitical properties in the Borel plane [3], such that for those it is possible to perform an analytic expansion in the Borel parameter $u$. The determination of $N_m$ is quite solid and survives a series of checks, see [2, 4]:

- Good convergence of the perturbative series in $u$ that determines $N_m$.
- Mild scale dependence of $N_m$.
- Consistency with the determination of $N_{V_s}$, the normalization constant of the infrared renormalon of static singlet potential $V_s$. $2N_m + N_{V_s} \simeq 0$.
- Agreement of the absolute value and scale dependence of the exact and asymptotic estimates of the coefficients of the perturbative series. See Fig. 1.

## 2. Applications

As we can see in Fig. 1, in heavy quark physics the asymptotic behavior sets in at quite low orders in perturbation theory, and the powers of $\ln[\nu/m]$ effectively exponentiate, becoming a linear power-like divergence in the factorization scale[1]:

$$r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!}\,. \tag{2.1}$$

[1] At this stage the similitude with a $1/a$ power-like divergence one would find in lattice computations is evident.

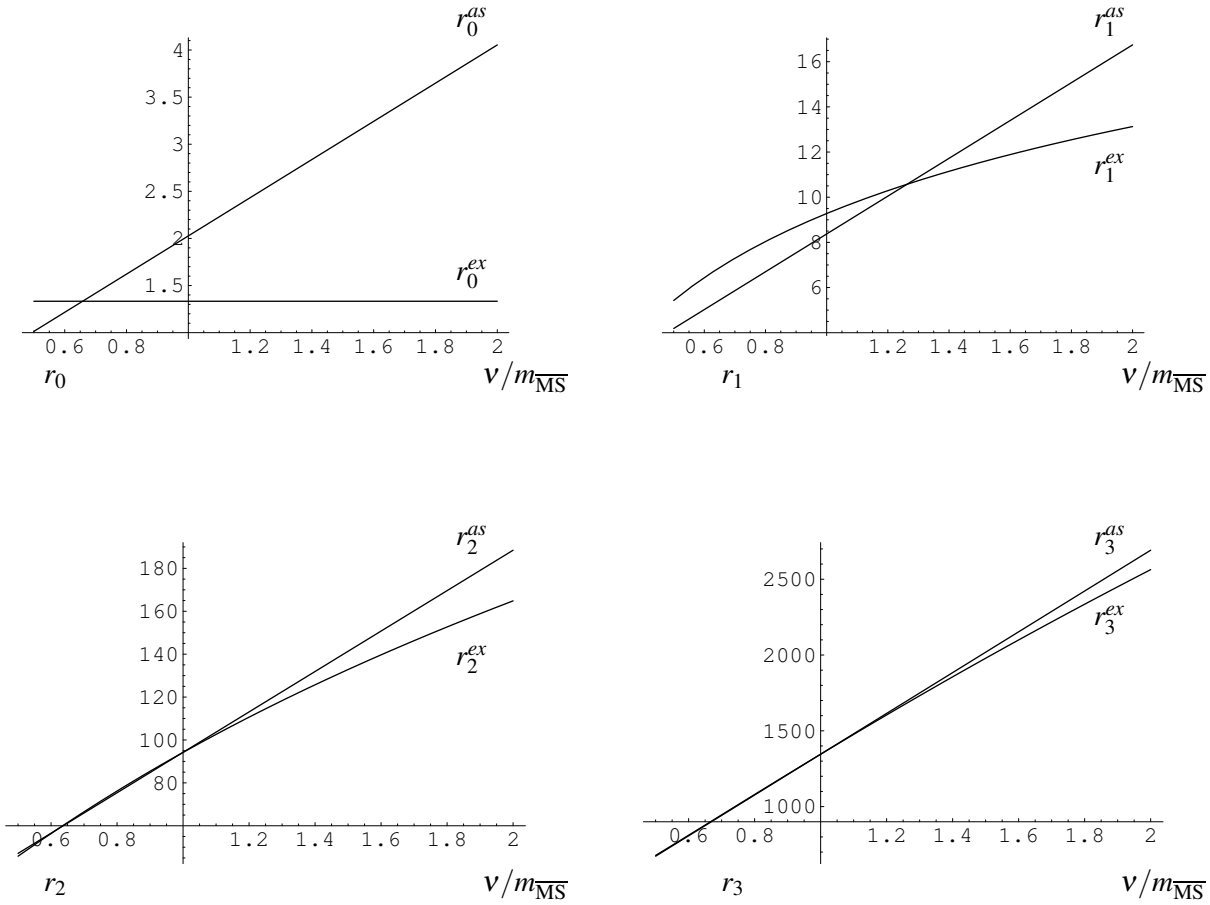


**Figure 1:** Plots of the exact ($r_n^{ex}$) and asymptotic ($r_n^{as}$) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\rm MS}}$. The scale dependence of $r_3^{ex}$ is known exactly. The constant term has been fixed using renormalon dominance.

The associated lack of convergence of the perturbative series at low orders in perturbation theory becomes a problem in applications to heavy quark physics. The solution proposed in Ref. [2] was to shift the $n!$ factorial behavior from the perturbative series to the low energy matrix elements, where it properly belongs, since this behavior is associated to low energy dynamics. For the heavy quark mass, this implies to work with the RS mass:

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \delta m_{\rm RS} = m_{\rm OS} - \sum_{n=0}^{\infty} N_m \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}\,. \qquad (2.2)$$

This framework of renormalon subtraction can be applied to any effective theory with heavy quarks: HQET, NRQCD, pNRQCD, ... (see [5, 6] for reviews). In this scheme some parameters become dependent on the scale, $\nu_f$, and scheme of renormalon subtraction. Here we will focus on a series of (quasi-) observables that can be studied in the static limit:

$$\langle M_{B/D}\rangle = m_{b/c,\rm RS}(\nu_f) + \bar{\Lambda}_{\rm RS}(\nu_f) + \mathcal{O}(1/m_{b/c,\rm RS})\,, \qquad (2.3)$$

$$E_s(r) = 2m_{\rm RS}(\nu_f) + V_{s,\rm RS}(r;\nu_f) + \mathcal{O}(r^2) \qquad V_s = -C_F\alpha_s/r + \cdots\,, \qquad (2.4)$$

$$E_H(r) = 2m_{\rm RS}(\nu_f) + V_{o,\rm RS}(r;\nu_f) + \Lambda_H^{\rm RS}(\nu_f) + \mathcal{O}(r^2) \qquad V_o = 1/(2N_c)\alpha_s/r + \cdots\,, \qquad (2.5)$$

where $H$ labels the hybrid/gluelump state at short distances, and the label RS for $V_{s/o}$, $\bar{\Lambda}$, $\Lambda_H$ means that the leading infrared renormalon has been subtracted (added) from the perturbative series. The use of this scheme significantly improves the convergence of the perturbative series and the agreement with experiment and lattice simulations, when available, see [2, 4, 7]. Examples include the determination of the 1S bottomonium mass and the agreement with the singlet and hybrid potentials computed in the lattice at short distances. Another nice example is the accurate description of the dependence on the lattice spacing of the static singlet and hybrid potential, $\bar{\Lambda}$ and $\Lambda_H$ using perturbation theory. This is possible because one may also consider to do the renormalon subtraction in a scheme different from the one presented here (as far as the subtracted quantity has the same non-analytic behavior in the Borel plane this would be legitimate). Here we would like to connect with the lattice scheme. In practice, this means to perform the replacements: $\{\mathrm{RS} \to \mathrm{L}, \nu_\mathrm{f} \to 1/\mathrm{a}\}$. The relation between both schemes is renormalon free and governed by a convergent series as we shall next see.

## 3. Perturbative running of $1/a$ and scheme dependence

Due to the existence of renormalon ambiguities ($1/a$ divergences in the lattice language), it is not possible to get the continuum limit of the static singlet and octet potential, gluelump masses, $\Lambda_H$ and $\bar{\Lambda}$. Nevertheless, their dependence on $1/a$ is free of renormalons and can be predicted by perturbation theory:

$$2(\bar{\Lambda}_L(1/a) - \bar{\Lambda}_L(1/a')) = V_{s,L}(r;1/a) - V_{s,L}(r;1/a') = C_F \left(\frac{1}{a} - \frac{1}{a'}\right) v_1 \alpha_s + \cdots . \tag{3.1}$$

The coefficients $v_1$, $v_2$, $v_3$ are known in pure gauge theory with Wilson action [8, 9]. A similar renormalon-free equality can be constructed for the octet potential and gluelump masses.

$$\begin{aligned} \Lambda_H^L(1/a) - \Lambda_H^L(1/a') &= [V_{o,L}(r;1/a) - V_{s,L}(r;1/a)] - \left[V_{o,L}(r;1/a') - V_{s,L}(r;1/a')\right] \\ &= \frac{C_A}{2} \left(\frac{1}{a} - \frac{1}{a'}\right) v_1' \alpha_s + \cdots , \end{aligned} \tag{3.2}$$

where $v_1'$, $v_2'$ are exactly known and $v_3'$ in the large $N_c$ limit.

We show the plot corresponding to Eq. (3.1) in Fig. 2, where the static potential data has been normalized to agree with $\bar{\Lambda}$ at one specific value of the lattice spacing

$$\overline{\Lambda}^L_{\mathrm{pot}}(a) = \frac{1}{2} V_s^L(r_0;a) + \Delta. \tag{3.3}$$

We can see how nicely the perturbative prediction in the lattice scheme (continuous green line in the first plot in Fig. 2) agrees with the lattice data [10, 8, 11]. Significantly, only for one collaboration the slope of the lattice data points slightly differs from the prediction of perturbation theory [8]. We can also see the convergence of the perturbative series relating the lattice and the RS scheme, since it is also renormalon free. The final value for $\bar{\Lambda}_{\mathrm{RS}}$, $\bar{\Lambda}_{\mathrm{RS}}(\nu_f = 2.5 r_0^{-1}) = 1.17 r_0^{-1}$, agrees within errors with the result obtained directly from experiment using a combined analysis of the $\Upsilon(1S)$ and the $B$ meson mass [2, 7], $\bar{\Lambda}_{\mathrm{RS}}(\nu_f = 2.5 r_0^{-1}) = 0.92 r_0^{-1}$.

A similar analysis can be performed for the octet potential and gluelump masses [7]. In this case there is less statistics but the slope predicted by perturbation theory nicely agrees with the

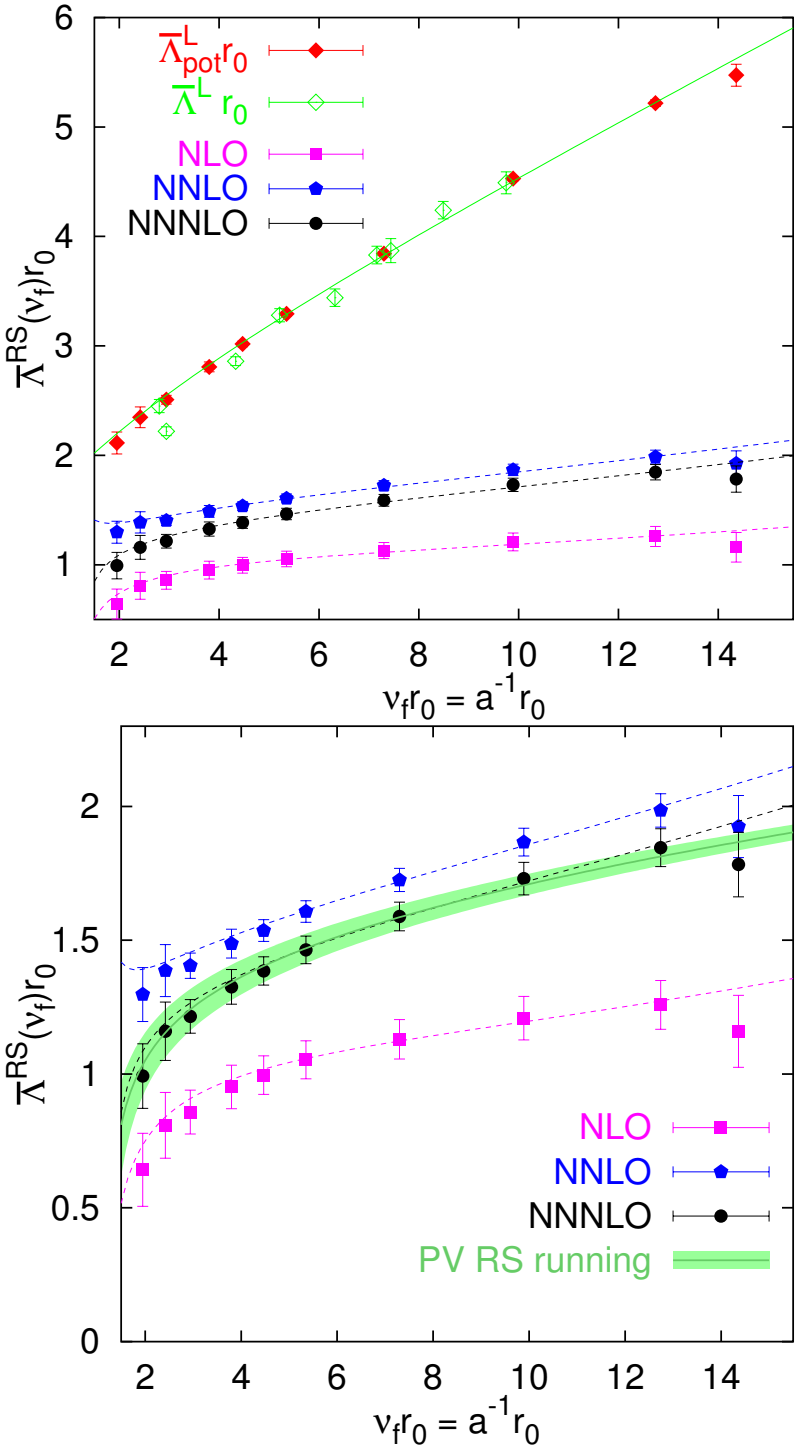


**Figure 2:** The first figure shows the binding energy $\overline{\Lambda}^L_{\rm pot}$ (full diamonds [10]), in comparison with $\overline{\Lambda}^L$ (open diamonds [8, 11]). NLO, NNLO and NNNLO refer to transformations of $\overline{\Lambda}^L_{\rm pot}$ into the RS scheme to different orders in perturbation theory. The solid line corresponds to the NNNLO evaluation in the lattice scheme with the central value $\overline{\Lambda}^{\rm RS}(\nu_f = 9.76\, r_0^{-1}) = 1.70\, r_0^{-1}$. The second figures shows $\overline{\Lambda}^L_{\rm pot}$ translated into the RS scheme at NLO (squares), NNLO (pentagons) and NNNLO (circles). The solid line corresponds to the principal value running in the RS scheme. The error band corresponds to the prediction $\overline{\Lambda}^{\rm RS}(9.76\, r_0^{-1}) = (1.70 \pm 0.04)\, r_0^{-1}$, and includes the uncertainty due to $\Lambda_{\overline{\rm MS}} = (0.602 \pm 0.048)\, r_0^{-1}$. For details see [7].

one obtained from the gluelump mass $\Lambda_B^L$ measured at different lattice spacings [12]. Moreover, the perturbative series governing the change to the RS scheme converges well. Finally, it is quite comforting that the same value, within errors, is obtained for $\Lambda_B$ from either the hybrid potential, $\Lambda_B^{\rm RS}(\nu_f = 2.5 r_0^{-1}) = 2.25 r_0^{-1}$, or from the direct lattice determination, $\Lambda_B^{\rm RS}(\nu_f = 2.5 r_0^{-1}) = 2.31 r_0^{-1}$. All these findings can be found summarized in Fig. 14 in [7].

In brief, we can relate processes computed with different scales/schemes using well-behaved (renormalon free) perurbative series:

$$\left.\begin{array}{ccc} \bar\Lambda_L(1/a) & \Longleftrightarrow & \bar\Lambda_L(1/a') \\ \Updownarrow\,(\nu_f = 1/a) & & \Updownarrow\,(\nu_f' = 1/a') \\ \bar\Lambda_{\rm RS}(\nu_f) & \Longleftrightarrow & \bar\Lambda_{\rm RS}(\nu_f') \end{array}\right\} \text{The circle can be closed using perturbation theory}$$

and a similar circle applies for $V_s^L(r;1/a)$, $V_o^L(r;1/a)$ and $\Lambda_H^L(1/a)$.

## 4. Conclusions

We have accumulated a lot of evidence in favour of the renormalon dominance in heavy quark physics. A proper handle of these effects appears to be crucial to accurately describe either lattice or experiment. We point out that the dependence on the lattice spacing can be obtained from perturbation theory with good accuracy and well controlled errors. Therefore, lattice simulations in heavy quark physics can be peformed with quite coarse lattices and yet obtain accurate results. This may have important consequences to diminish errors in lattice simulations.

## References


[1] M. Beneke, Phys. Lett. B **344**, 341 (1995) [arXiv:hep-ph/9408380].

[2] A. Pineda, JHEP **0106**, 022 (2001) [arXiv:hep-ph/0105008].

[3] T. Lee, Phys. Rev. D **56**, 1091 (1997) [arXiv:hep-th/9611010].

[4] A. Pineda, J. Phys. G **29**, 371 (2003) [arXiv:hep-ph/0208031].

[5] M. Neubert, Phys. Rept. **245**, 259 (1994) [arXiv:hep-ph/9306320].

[6] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. *in press* [arXiv:hep-ph/0410047].

[7] G. S. Bali and A. Pineda, Phys. Rev. D **69**, 094001 (2004) [arXiv:hep-ph/0310130].

[8] A. Duncan, E. Eichten, J. Flynn, B. Hill, G. Hockney and H. Thacker, Phys. Rev. D **51**, 5101 (1995).

[9] G. Martinelli and C. T. Sachrajda, Nucl. Phys. B **559**, 429 (1999); V. Gimenez, L. Giusti, G. Martinelli and F. Rapuano, JHEP **0003**, 018 (2000); F. Di Renzo and L. Scorzato, JHEP **0102**, 020 (2001); H. D. Trottier, N. H. Shakespeare, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **65**, 094502 (2002); G. S. Bali and P. Boyle, arXiv:hep-lat/0210033.

[10] G. S. Bali and K. Schilling, Phys. Rev. D **46**, 2636 (1992); Phys. Rev. D **47**, 661 (1993); Int. J. Mod. Phys. C **4**, 1167 (1993); G. S. Bali, K. Schilling and A. Wachter, Phys. Rev. D **56**, 2566 (1997).

[11] C. R. Allton *et al.* [APE Collaboration], Nucl. Phys. B **413**, 461 (1994); Phys. Lett. B **326**, 295 (1994); Nucl. Phys. Proc. Suppl. **42**, 385 (1995); A. K. Ewing *et al.* [UKQCD Collaboration], Phys. Rev. D **54**, 3526 (1996); C. Michael and J. Peisa [UKQCD Collaboration], Phys. Rev. D **58**, 034506 (1998).

[12] M. Foster and C. Michael [UKQCD Collaboration], Phys. Rev. D **59**, 094509 (1999).