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Perturbative renormalisation of quark bilinear operators for overlap fermions with and without stout links and improved gauge action

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We calculate lattice renormalisation constants of local and one-link quark operators for overlap fermions and improved gauge actions in one-loop perturbation theory. For the local operators we stout smear the SU(3) links in the fermionic action. Using the popular tadpole improved Lüscher-Weisz actions at $\beta = 8.45$ and $\beta = 8.0$ we present numerical values for the Z factors in the $\overline{MS}$ scheme (partly as function of the stout smearing strength). We compare various levels of mean field (tadpole) improvement which have been applied to our results.

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1. Introduction

Lattice calculations at small quark masses require an action with good chiral properties. The same is true for calculations of matrix elements of certain operators, which otherwise mix with operators of opposite chirality. Ginsparg-Wilson fermions [1] have an exact chiral symmetry on the lattice [2], and thus are well suited for these tasks. A further advantage is that they are automatically $O(a)$ improved [3]. Overlap fermions [4, 5, 6] provide a four-dimensional realisation of Ginsparg-Wilson fermions.

It is well known that computer simulations with overlap fermions suffer from large computational costs due to the large condition number. One possibility to reduce this number is by using improved gauge actions [7] and hence there is a need for the corresponding perturbative calculations. In this paper we use the tadpole improved Lüscher-Weisz action [8, 9, 10]

$$S_G^{TILW} = \frac{6}{g^2}\left[c_0 \sum_{\text{plaquette}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1-U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1-U_{\text{rectangle}}) + c_3 \sum_{\text{parallelogram}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1-U_{\text{parallelogram}})\right]. \tag{1.1}$$

The parameters c_1 and c_3 weight the contributions of the corresponding six-links loops. It is customary to impose the normalisation condition

$$c_0 + 8c_1 + 8c_3 = 1. \tag{1.2}$$

Defining the lattice coupling β by

$$\beta = \frac{6}{g^2}c_0 \tag{1.3}$$

we choose in accordance with numerical simulations performed by the QCDSF collaboration [11, 12] the following values [13]

β	c_1	c_3
8.45	-0.154846	-0.0134070
8.0	-0.169805	-0.0163414

(1.4)

A further step to reduce the computational costs consists of smearing the gauge link variables in the fermionic action. The resulting fat links tend to reduce the density of eigenmodes speeding up the inversion of the fermion Dirac operator [14]. Morningstar and Peardon [15] have proposed an analytic smearing method (stout link method) which can be applied in Hybrid Monte Carlo (HMC) simulations. As by construction the stout links remain in the SU(3) group, this enables the force term in the equations of motion for HMC to be easily determined.

The calculations which we will present in the next sections have been performed for $r = 1$ and the overlap parameter $\rho = 1.4$ (for the definition see [16]). The influence of stout links is investigated for local operators only.

2. Renormalisation

To obtain continuum results from lattice calculations of hadron matrix elements, the underlying operators have to be renormalised. A non-perturbative determination of the corresponding renormalisation factors would be preferable. However, often perturbative renormalisations are done first. Especially, if this calculation is performed analytically it provides useful information about the intrinsic singular structure and possible complicated mixing properties. We use a *Mathematica* program which has been developed for one-loop lattice perturbative calculations [17] and has been extended to overlap fermions with improved gauge actions.

We define renormalised operators $\mathscr{O}$ by

$$\mathscr{O}^{\mathscr{S}}(\mu) = Z_{\mathscr{O}}^{\mathscr{S}}(a,\mu)\mathscr{O}(a)\,, \tag{2.1}$$

where $\mathscr{S}$ denotes the renormalisation scheme. $Z_{\mathscr{O}}^{\mathscr{S}}(a,\mu)$ is the renormalisation factor connecting the lattice operator $\mathscr{O}(a)$ with the renormalised operator $\mathscr{O}^{\mathscr{S}}(\mu)$ at scale μ. We use the MOM scheme by computing the amputated Green function $\Lambda_{\mathscr{O}}$ of the operator $\mathscr{O}$ and define the $Z_{\mathscr{O}}$ via

$$\frac{Z_{\mathscr{O}}^{MOM}(a,\mu)}{Z_{\psi}^{MOM}(a,\mu)}\,\Lambda_{\mathscr{O}}\,\big|_{p^2=\mu^2} = \Lambda_{\mathscr{O}}^{\text{tree}} + \text{ other Dirac structures}\,, \tag{2.2}$$

where $Z_{\psi}^{MOM}(a,\mu)$ is the quark wave function renormalisation factor. The renormalisation constants can be converted to the $\overline{MS}$ scheme,

$$Z_{\psi,\mathscr{O}}^{\overline{MS}}(a,\mu) = Z_{\psi,\mathscr{O}}^{\overline{MS},MOM} Z_{\psi,\mathscr{O}}^{MOM}(a,\mu)\,, \tag{2.3}$$

where the conversion factors $Z_{\psi,\mathscr{O}}^{\overline{MS},MOM}$ are calculable in continuum perturbation theory.

3. One-link operators

Let us first consider the one-link operators

$$\mathscr{O}_{\mu\nu} = \frac{i}{2}\,\bar{\psi}(x)\gamma_\mu \overleftrightarrow{D}_\nu\,\psi(x) - \text{traces}\,, \tag{3.1}$$

$$\mathscr{O}^5_{\mu\nu} = \frac{i}{2}\,\bar{\psi}(x)\gamma_\mu\gamma_5 \overleftrightarrow{D}_\nu\,\psi(x) - \text{traces}\,, \tag{3.2}$$

which are related to the first moments of unpolarised and polarised nucleon structure functions, where $\overleftrightarrow{D}_\nu$ is the left-right covariant lattice derivative. The chiral properties of overlap fermions imply that matrix elements of $\mathscr{O}_{\mu\nu}$ and $\mathscr{O}^5_{\mu\nu}$ give identical results. Therefore, we restrict our calculations to the unpolarised case, i.e. to $\mathscr{O}_{\mu\nu}$. For a detailed discussion see [18].

The amputated one-loop Green function $\Lambda_{\mu\nu}$ obtained from (3.1) has the form

$$\begin{aligned}\Lambda_{\mu\nu}(a,p) = {} & \gamma_\mu p_\nu + \frac{g^2 C_F}{16\pi^2}\Bigg\{\left[\left(\frac{1}{3}+\xi\right)\log(a^2p^2) - 4.29201\,\xi + b_1\right]\gamma_\mu p_\nu \\ & + \left[\frac{4}{3}\log(a^2p^2) + \frac{1}{2}\,\xi + b_2\right]\gamma_\nu p_\mu + \left[-\frac{2}{3}\log(a^2p^2) - \frac{1}{2}\,\xi + b_3\right]\delta_{\mu\nu}\,\not{p} \\ & + b_4\,\delta_{\mu\nu}\gamma_\nu p_\nu + \left(-\frac{4}{3}+\xi\right)\frac{p_\mu p_\nu}{p^2}\,\not{p}\Bigg\}\,,\end{aligned} \tag{3.3}$$

with ξ as gauge parameter (Feynman gauge: $\xi = 0$) and $C_F = 4/3$. The constants b_i are linear combinations of finite lattice integrals [17] and depend on the used lattice fermionic and gauge actions. For the chosen actions and parameters we obtain

Action	b_1	b_2	b_3	b_4	b_Σ
$\beta = 8.45$	-5.6115	-3.8336	2.7793	0.3446	-16.180
$\beta = 8.0$	-5.2883	-3.7636	2.7310	0.3331	-15.733

(3.4)

In (3.4) we have added the contribution of the quark self energy b_Σ which is needed for the calculation of the wave function renormalisation.

From (2.2), (2.3) and (3.3) we determine the Z factors in the $\overline{MS}$ scheme for the commonly used representations under the hypercubic group

$$\tau_3^{(6)}: \quad \mathcal{O}_{v_{2a}} \equiv \frac{1}{2}\left(\mathcal{O}_{14} + \mathcal{O}_{41}\right), \qquad \tau_1^{(3)}: \quad \mathcal{O}_{v_{2b}} \equiv \mathcal{O}_{44} - \frac{1}{3}\left(\mathcal{O}_{11} + \mathcal{O}_{22} + \mathcal{O}_{33}\right).$$

They have the form

$$Z_{v_i}^{\overline{MS}}(a,\mu) = 1 - \frac{g^2 C_F}{16\pi^2}\left[\frac{16}{3}\log(a\mu) + B_{v_i}(c_k,\rho)\right] \tag{3.5}$$

with

$$B_{v_i}(c_k,\rho) = \frac{40}{9} + b_{v_i} + b_\Sigma, \quad b_{v_{2a}} = b_1 + b_2, \quad b_{v_{2b}} = b_1 + b_2 + b_4. \tag{3.6}$$

It is well known that the naive perturbative results suffer from lattice artefacts. Therefore, mean field (tadpole) improvement [19] has been proposed to rearrange the perturbative series. In case of overlap fermions the tadpole improved Z factor is given by [16]

$$Z_{\mathcal{O}}^{TI} = Z_{\mathcal{O}}^{MF}\left(\frac{Z_{\mathcal{O}}}{Z_{\mathcal{O}}^{MF}}\right)_{\text{pert}}, \tag{3.7}$$

where $Z_{\mathcal{O}}^{MF}$ is the mean field approximation of $Z_{\mathcal{O}}$. For overlap fermions we have

$$Z_{\mathcal{O}}^{MF} = \frac{\rho}{\rho - 4(1-u_0)}, \quad Z_{\mathcal{O}\text{pert}}^{MF} = 1 + \frac{g_{TI}^2 C_F}{16\pi^2}\frac{4}{\rho}k_u^{TI}. \tag{3.8}$$

Here u_0 denotes the mean value of the link. The boosted parameters are chosen as [16]

$$g_{TI}^2 = g^2/u_0^4, \quad c_0^{TI} = c_0, \quad c_i^{TI} = u_0^2 c_i\,(i = 1,3), \quad C_0^{TI} = c_0 + 8c_1^{TI} + 8c_3^{TI}. \tag{3.9}$$

k_u^{TI} is the one-loop contribution of the perturbative expansion for u_0 with c_i^{TI} inserted for the corresponding gauge actions ($k_u^{TI} = 5.3625/5.0835$ for $\beta = 8.45/8.0$).

In case of overlap fermions one needs to improve the quark propagator as well which leads to a mean field improved ρ parameter

$$\rho^{TI} = \frac{\rho - 4(1-u_0)}{u_0}. \tag{3.10}$$

Results obtained with ρ^{TI} are denoted as fully tadpole improved (FTI). The Z factors have the form

$$Z_{v_i}^{TI,\overline{MS}} = Z_{\mathcal{O}}^{MF}\left\{1 - \frac{g_{TI}^2 C_F}{16\pi^2}\left[\frac{16}{3C_0^{TI}}\log(a\mu) + B_{v_i}^{TI}(c_k^{TI},\rho)\right]\right\}, \tag{3.11}$$

$$Z_{v_i}^{FTI,\overline{MS}} = Z_{\mathcal{O}}^{MF}\left\{1-\frac{g_{TI}^2 C_F}{16\pi^2}\left[\frac{16}{3C_0^{TI}}\log(a\mu)+B_{v_i}^{TI}(c_k^{TI},\rho^{TI})\right]\right\}. \tag{3.12}$$

The following table shows numerical results for the various levels of improvement at $a = 1/\mu$

Operator	β	B	$Z^{\overline{MS}}$	B^{TI}	$Z^{TI,\overline{MS}}$	B^{FTI}	$Z^{FTI,\overline{MS}}$
v_{2a}	8.45	-22.430	1.315	0.502	1.393	-0.077	1.411
v_{2b}	8.45	-22.085	**1.311**	0.793	**1.384**	0.230	**1.401**
v_{2a}	8.0	-22.036	1.310	0.603	1.390	-0.108	1.412
v_{2b}	8.0	-21.703	1.305	0.892	1.381	0.199	1.402

(3.13)

It can be read off from Table (3.13) that the one-loop corrections B for the improved perturbative Z factors become smaller as expected. Thus the perturbative series is better behaved. For $\beta = 8.45$ and representation v_{2b} we can compare the perturbative Zs (bold faced numbers in (3.13)) with a quenched Monte Carlo simulation [20] giving $Z^{MC,\overline{MS}} = 1.98(3)$. Using the stout smearing procedure (see Section 4) the resulting factors are $Z_{1-smear}^{MC,\overline{MS}} = 1.47(4)$ and $Z_{2-smear}^{MC,\overline{MS}} = 1.34(3)$ with the value of the smearing parameter $\omega = 0.15$.

4. Local operators and stout smearing

Z factors for local fermionic operators for overlap fermions and a set of improved gauge actions have been determined in [16]. A recalculation has been performed by Ioannou and Panagopoulos [21]. In this section we show the influence of stout smearing on the perturbative Z factors.

By construction a stout smearing step is performed on a gauge link variable $U_\mu(x)$ as [15]

$$U_\mu^{(n+1)}(x) = e^{iQ_\mu^{(n)}(U,\omega_{\mu\nu})}U_\mu^{(n)}(x), \tag{4.1}$$

where n denotes the step of smearing. The parameters $\omega_{\mu\nu}$ characterise the strength of smearing: they are the weights of the perpendicular staples associated to the link $(x, x+\hat{\mu})$. For our perturbative calculation we have assumed the isotropic case $\omega_{\mu\nu} = \omega$ and a small value of ω. Various investigations suggest values of $0.1 < \omega < 0.3$. Therefore, we have expanded (4.1) to first order in ω. Furthermore, we have restricted ourselves to $n = 1$. The resulting stout link has been inserted into the fermionic action modifying the corresponding Feynman rules for the quark-gluon vertices. The corresponding results are obtained in powers of ω. As a possible choice in the tadpole improvement we assumed that our approximate stout smearing has been done for the mean field rescaled links (this does not change $Z_{\mathcal{O}}^{MF}$ and does not rescale ω).

As examples we have calculated the Z factors in $\overline{MS}$-scheme for the scalar and axial vector operators for the TILW action at $\beta = 8.45$ and the perturbative improvement levels discussed in the previous section. For the scalar operator we find ($a = 1/\mu$)

$$\begin{aligned} Z_S &= 1.168 - 0.248\,\omega - 0.154\,\omega^2, \\ Z_S^{TI} &= 1.309 - 0.488\,\omega - 0.239\,\omega^2, \\ Z_S^{FTI} &= 1.359 - 0.241\,\omega - 0.685\,\omega^2. \end{aligned}$$

These numbers can be compared with Z^{MC} obtained from a quenched MC simulation at $\omega = 0.15$ and a single smearing [20]:

$$Z_S = 1.127\,, \quad Z_S^{TI} = 1.230\,, \quad Z_S^{FTI} = 1.307\,; \quad Z^{MC}_{S,0-smear} = 1.36(1)\,, \quad Z^{MC}_{S,1-smear} = 1.13\,.$$

The same has been done for the axial vector operator. We get in this case

$$\begin{aligned} Z_A &= 1.156 - 0.475\,\omega + 0.092\,\omega^2\,, \\ Z_A^{TI} &= 1.268 - 0.860\,\omega + 0.179\,\omega^2\,, \\ Z_A^{FTI} &= 1.303 - 0.560\,\omega - 0.346\,\omega^2\,. \end{aligned}$$

The comparison with MC gives for $\omega = 0.15$

$$Z_A = 1.087\,, \quad Z_A^{TI} = 1.114\,, \quad Z_A^{FTI} = 1.211\,; \quad Z^{MC}_{A,0-smear} = 1.42(1)\,, \quad Z^{MC}_{A,1-smear} = 1.16\,.$$

Contrary to the non-perturbative case the perturbative stout smearing decreases only mildly the renormalisation factors.

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