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Large-N QCD at strong transverse lattice gauge coupling

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We had previously obtained an integral equation for mesons in transverse lattice QCD, in the limit of large number of colours and strong transverse lattice gauge coupling [1]. This equation is a generalisation of the 't Hooft equation [2], by inclusion of the spin degrees of freedom. We analyse this equation to extract spectral properties and light-front wavefunctions of mesons. We also extend the method to study baryon properties in the same limit.

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1. The formalism

One of us had pointed out that transverse lattice QCD (i.e. QCD with two continuous dimensions on light-front and two transverse dimensions on lattice), can be solved analytically in a closed form in the combined $N \to \infty$ and strong transverse gauge coupling limits [3]. Although this limiting theory differs from real QCD, we hope that the results will be particularly useful for understanding deep-inelastic scattering structure functions dominated by valence quarks. In this case, the scaling behaviour implies that the transverse momenta do not contribute to the leading order, and so distorting them using a lattice may not be a severe drawback as long as we treat the light-front components exactly.

We study the limit of QCD defined by the action (g is held fixed as $N \to \infty$):

$$S = a_\perp^2 \sum_{x_\perp} \int d^2x \Big[-\frac{N}{4g^2} \sum_{\mu\nu a} F^a_{\mu\nu}(x) F^{\mu\nu a}(x) + \overline{\psi}(x) \Big(i \sum_\mu \gamma^\mu \partial_\mu - \sum_\mu \gamma^\mu A_\mu - m \Big) \psi(x)$$
$$+ \frac{\kappa}{2a_\perp} \sum_n \Big\{ \overline{\psi}(x)(1+i\gamma^n) U_n(x) \psi(x+\hat{n}a_\perp) + \overline{\psi}(x+\hat{n}a_\perp)(1-i\gamma^n) U_n^\dagger(x) \psi(x) \Big\} \Big] . \quad (1.1)$$

Here μ, ν label the light-front directions, and n labels the lattice directions. In the $g_\perp \to \infty$ limit, the transverse lattice spacing $a_\perp = O(\Lambda_{QCD})$. For the fermions, we follow the Wilson prescription in the transverse directions. The anisotropy parameter κ has to be determined non-perturbatively by demanding as much restoration of rotational symmetry as possible, and the metric is

$$\tfrac{1}{2}\{\gamma_\alpha, \gamma_\beta\} = g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (1.2)$$

In analysing this theory, the order of limits is important, because various limits do not commute. To obtain the correct phase of the theory, we first let $g_\perp \to \infty$, then $N \to \infty$, and then $m \to 0$.

Starting with the above action, we eliminate the gauge degrees of freedom completely by exact functional integrations over A^- (after choosing $A^+ = 0$) and $U_n(x)$. We then trade off the fermion fields in favour of non-local boson fields, and obtain an effective action in terms of $\sigma_{\alpha\beta}(x,y) \equiv \overline{\psi}_\alpha(x)\psi_\beta(y)$ [1]. The stationary point value of this effective action, $V_{\text{eff}}(\overline{\sigma}; J)$, yields the generating functional for the connected Green's functions, as $N \to \infty$.

For Wilson fermions. the projection operator structure of the fermion hopping term simplifies the formulae, and the results are simple modifications of those for the 't Hooft model. The transverse tadpole insertions renormalising the quark mass vanish, and the chiral limit of the theory remains at $m = 0$ [1]. The chiral condensate, obtained using split-point regularisation and operator formulation [4], is

$$\langle \overline{\psi}\psi \rangle_{4d} = \frac{2}{a_\perp^2} \langle \overline{\psi}\psi \rangle_{2d} \xrightarrow[m\to 0]{} -\frac{N}{a_\perp^3} \sqrt{\frac{g^2}{3\pi}} . \quad (1.3)$$

2. Meson states

The wavefunction for the meson state with spin-parity structure Γ is defined as

$$\phi_\Gamma(p,q) = \langle \overline{\psi}(p-q) \Gamma \psi(q) | \text{Meson}_\Gamma(p) \rangle . \quad (2.1)$$

It satisfies a homogeneous Bethe-Salpeter equation with two types of quark-antiquark interactions: gluon exchange in the longitudinal direction, and bilinear fermion hopping in the transverse directions. In the reference frame with momentum components $p = (p^+ = 1, p^- = M^2/2, p_\perp = 0)$, the interactions are independent of the "$-$" and "$\perp$" components, and the Bethe-Salpeter equation is easily projected on to the light-front:

$$\Phi_\Gamma(q^+ \equiv x) = \int dq^- \, \frac{d^2 q_\perp}{(2\pi)^2} \, \phi_\Gamma(p,q) \, . \tag{2.2}$$

For a quark-antiquark pair of masses m_1 and m_2, we obtain ($\beta \equiv g^2/\pi a_\perp^2$, P$\equiv$principal value) [1],

$$\begin{aligned}
\mu^2(x)\Phi(x) &\equiv \left[M^2 - \frac{m_1^2-\beta}{x} - \frac{m_2^2-\beta}{1-x}\right]\Phi(x) \\
&= \frac{1}{2x(1-x)}\left[\frac{m_1^2}{2x}\gamma^+ + x\gamma^- + m_1\right] \\
&\times \int_0^1 \frac{dy}{2\pi}\Big\{-\frac{g^2}{a_\perp^2}\mathrm{P}\Big[\frac{1}{(x-y)^2}\Big]\gamma^+\Phi(y)\gamma^+ \;+\; \frac{\kappa^2}{a_\perp^2}\Big[2\Phi(y) + \sum_n \gamma^n\Phi(y)\gamma^n\Big]\Big\} \\
&\times \left[\frac{\mu^2(x)}{2}\gamma^+ + \frac{m_2^2}{2(1-x)}\gamma^+ + (1-x)\gamma^- - m_2\right] .
\end{aligned} \tag{2.3}$$

2.1 Symmetry consequences

Eq.(2.3) is a 16-component matrix integral equation in Dirac space, and physical meson states have to be obtained by diagonalising it. Discrete symmetries of the action allow block-diagonalisation of Eq.(2.3) to four blocks of 4 components each. Writing meson wavefunctions in a basis that is the direct product of Clifford algebra bases in continuum and lattice directions,

$$\Phi = \sum_{C,L} \Phi_{C;L}\Gamma^{C;L} \; , \quad \Gamma^{C;L} \in \left\{1, \gamma^+, \gamma^-, \tfrac{1}{2}[\gamma^+,\gamma^-]\right\} \otimes \left\{1, \gamma^{n_1}, \gamma^{n_2}, \tfrac{1}{2}[\gamma^{n_1},\gamma^{n_2}]\right\} \; , \tag{2.4}$$

we have $\sum_n \gamma^n \Gamma^{C;L} \gamma^n \propto \Gamma^{C;L}$ for each value of L. Thus the lattice index "L" can be used as the block label, and the exact degeneracy of $L = n_1$ and $L = n_2$ blocks implies that only three of the four blocks are independent.

Without the transverse lattice dynamics, Eq.(2.3) reduces to the 't Hooft equation [2], and the limit $\kappa \to 0$ is smooth. The non-singular transverse contribution proportional to κ^2 represents a colour singlet quark-antiquark pair hopping from one light-front to the next. (Leaving out the γ-matrices, the form of this contribution is similar to the fermion-antifermion annihilation diagram for the massive Schwinger model [5].) The strong transverse gauge coupling limit produces a tight binding δ-function constraint during the hopping, so that the transverse interaction is a wavefunction at the origin effect and the meson orbital angular momentum L_z vanishes. With spin-half quarks, the meson helicities are restricted to $0, \pm 1$, and the allowed spin-parity quantum numbers for the mesons are $J^P = 0^\pm, 1^\pm$.

Under the exchange of the quark and the antiquark, the meson wavefunction transforms as $E_{q\bar{q}}\Phi(x; m_1, m_2) = C\Phi^T(1-x; m_2, m_1)C^{-1}$, where C is the charge conjugation operator. Eq.(2.3) is invariant under this exchange operation. Although parity is an exact symmetry of our formalism, it is not manifest because the light-front is not invariant under parity transformation. The $(x \leftrightarrow 1-x)$

part of $E_{q\bar{q}}$ can be associated with parity, however, and this parity symmetry is exact in every block labeled by L. As a result, the eight possible spin-parity quantum numbers are distributed in to the four blocks as two states of opposite parity in each block. In conventional notation, $\{\pi, a_1(0)\} \in \Phi_{C;n_1 n_2}$, the degenerate pair $\{\rho(n), a_1(n)\} \in \Phi_{C;n}$, and $\{\rho(0), \sigma\} \in \Phi_{C;1}$.

Restricted to the finite box $x \in [0,1]$, the spectrum of M^2 is purely discrete as in the 't Hooft model [2], and the meson states can be labeled by a radial excitation quantum number $n = 1,2,3\ldots$ in each block. Since the lowest state meson wavefunction $\Phi^{(n=1)}_{-;L}$ is symmetric, the exchange symmetry alternates with n, and the quark and the antiquark have opposite intrinsic parities, the meson states have parity $P = (-1)^n$ in each block.

2.2 Behaviour in the chiral limit

The singular part of the interaction kernel in Eq.(2.3) is the same as in the 't-Hooft model, and depends only on the components $\Phi_{-;L}$ for each value of L. The behaviour of the solutions in certain limiting situations, therefore, can be obtained using the same methods as for the 't Hooft model [2, 5–8].

The singular part can be separated using the projection, $\gamma^+\Phi\gamma^+ = \sum_L \Phi_{-;L}(2\Gamma^L \otimes \gamma^+)$,

$$M^2\Phi_{-;L}(x) = \left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x}\right]\Phi_{-;L}(x) + \beta\int_0^1 dy\ \mathrm{P}\left[\frac{\Phi_{-;L}(x)-\Phi_{-;L}(y)}{(x-y)^2}\right] + \chi_{-;L}(x)\ , \tag{2.5}$$

$$\chi_{-;L}(x) = \frac{\kappa^2}{2\pi a_\perp^2}\int_0^1 dy \cdot \begin{cases} 4\Phi_{+;1}(y) - \frac{2m_1m_2}{x(1-x)}\Phi_{-;1}(y)\ , \\ -2\Phi_{+;n}(y) - \frac{m_1m_2}{x(1-x)}\Phi_{-;n}(y) - (\frac{m_1}{x}+\frac{m_2}{1-x})\Phi_{1;n}(y) + (\frac{m_1}{x}-\frac{m_2}{1-x})\Phi_{+-;n}(y)\ , \\ 2(\frac{m_1}{x}-\frac{m_2}{1-x})\Phi_{1;n_1n_2}(y) - 2(\frac{m_1}{x}+\frac{m_2}{1-x})\Phi_{+-;n_1n_2}(y)\ . \end{cases} \tag{2.6}$$

For finite norm solutions, the integrals appearing in $\chi_{-;L}$ have to be finite. The solutions $\Phi_{-;L}$ therefore vanish at the boundaries as x^{β_1} and $(1-x)^{\beta_2}$, and the exponents can be determined using an ansatz $\Phi_{-;L} \sim x^{\beta_1}(1-x)^{\beta_2}$.

The equations simplify in the chiral limit. $\chi_{-;n_1n_2}$ vanishes when $m=0$, and consequently $\Phi^{(n)}_{-;n_1n_2}(m=0)$ coincide with the corresponding solutions of the 't Hooft equation. In particular, the lightest meson is massless and has the wavefunction $\Phi^{(n=1)}_{-;n_1n_2}(m=0) = 1$. This is the pseudoscalar Goldstone boson of the theory, with the decay constant

$$(f_\pi)_{4d}\ =\ \sqrt{2}(f_\pi)_{2d} \underset{m\to 0}{\longrightarrow} \frac{1}{a_\perp}\sqrt{\frac{2N}{\pi}}\ . \tag{2.7}$$

Asymptotically for large n, the masses and the wavefunctions behave as

$$n \gg 1:\ \ \Phi^{(n)}_{-;n_1n_2}(x) \simeq \sqrt{2}\sin(n\pi x)\ ,\ \ M_n^2 \simeq n\pi g^2/a_\perp^2\ . \tag{2.8}$$

With the physical values $N=3$ and $f_\pi \simeq 130$ MeV, we estimate the cut-off as $\pi/a_\perp \simeq 300$ MeV. Fitting the slope of the $\pi - \pi(1300) - \pi(1800)$ trajectory to its asymptotic behaviour, then gives the gauge coupling $g^2/4\pi \simeq 2.3$. With these parameters, the chiral condensate turns out to be $\langle\overline{\psi}\psi\rangle \simeq -(165\text{ MeV})^3$.

The masses of mesons in the other spin-parity blocks are shifted due to the non-singular part of the interaction kernel. The solutions have to be determined numerically, and the parameter κ can then be fixed by making the helicity$=0,\pm$ states for $J=1$ mesons as degenerate as possible.

3. Baryon states

Baryons are semi-classical solitons in the $N \to \infty$ limit. In this scenario, each valence quark can be considered to be moving in the common Hartree potential provided by the other $N-1$ valence quarks (sea quarks drop out in the $N \to \infty$ limit) [9]. This potential is static and of finite range, and carries colour opposite to that of the valence quark. The potential experienced by a valence quark bound to a heavy antiquark has the same features, although it may have a different spatial dependence, and several techniques used to study the heavy quark effective theory can be applied to the baryon case as well.

The baryon wavefunctions are completely antisymmetric in colour, and so fully symmetric in space, spin and flavour indices. Furthermore, in the Hartree approximation, the ground state baryons have all the valence quarks in the same lowest state of the potential. The total wavefunction is thus the product of identical single-particle wavefunctions; it is fully symmetric in space and satisfies $I = J$. The baryons are solutions of the bosonised effective action, with

$$Q(x^+) \equiv a_\perp^2 \sum_{x_\perp} \int dx^- \overline{\psi}(x,x_\perp)\gamma^+\psi(x,x_\perp) = \text{const.} \tag{3.1}$$

So in the single baryon sector, the $N \to \infty$ stationary point of the effective action has to satisfy $a_\perp^2 \sum_{x_\perp} \int dx^- \mathrm{tr}(\overline{\sigma}_{\alpha\beta}(x,x)\gamma^+_{\alpha\beta}) = N$. Extremisation of $V_{\text{eff}}(\sigma; J=0)$ leads to

$$\begin{aligned} 1 &= i\overline{\sigma}^T(x,y)(i\partial\!\!\!/ - m)\delta^{(2)}(x-y) - \frac{ig^2}{2}|x^- - y^-|\ \overline{\sigma}^T(x,y)\gamma^+\overline{\sigma}^T(y,x)\gamma^+ \\ &- i\delta^{(2)}(x-y)\sum_n \big[\overline{\sigma}^T(x,x)G(\overline{R})(1-i\gamma_n)\overline{\sigma}^T(x-n,x-n)(1+i\gamma_n) \\ &\qquad + \overline{\sigma}^T(x,x)(1+i\gamma_n)\overline{\sigma}^T(x+n,x+n)G(\overline{R})(1-i\gamma_n)\big] , \end{aligned} \tag{3.2}$$

where we have chosen units to set $a_\perp = 1$ for simplicity, and

$$G(R) = \frac{\kappa^2}{2(1+\sqrt{1+\kappa^2 R})} , \quad R = -(1-i\gamma_n)\sigma^T(x,x)(1+i\gamma_n)\sigma^T(x+n,x+n) . \tag{3.3}$$

The meson stationary point is translationally invariant,

$$\overline{\sigma}^T_{B=0}(x,y) = -i\int \frac{d^2p}{(2\pi)^2}\ \frac{1}{p\!\!\!/ - m - \Sigma_{B=0}(p) + i\varepsilon}\ e^{ip\cdot(x-y)}\delta_{x_\perp y_\perp} , \quad \Sigma_{B=0}(p) = -\frac{g^2\gamma^+}{2\pi p^+} . \tag{3.4}$$

With $\overline{\sigma}_{B=0}(x,x) \propto 1$ and $\overline{R}_{B=0} = 0$, the transverse lattice dynamics doesn't contribute to it. On the contrary, the baryon stationary point is not translationally invariant, and we decompose [10]

$$\overline{\sigma}_{B=1}(x,y) = \overline{\sigma}_{B=0}(x,y) + \delta_{x_\perp y_\perp} f^*(x^-)\gamma^+ f(y^-) , \quad \int dx^- |f(x^-)|^2 = \tfrac{1}{4} , \tag{3.5}$$

in the Hartree approximation. The stationary point equation then yields (in momentum space),

$$\begin{aligned} 0 &= \int \frac{d^2k}{(2\pi)^2}|\tilde{f}(k)|^2\gamma^+\gamma^-(k\!\!\!/ - \widetilde{m} + \frac{g^2\gamma^+}{\pi k^+}) \\ &+ 4g^2\gamma^+\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}\mathrm{P}\Big[\frac{1}{(p^+)^2}\Big]\tilde{f}^*(q)\tilde{f}(k)\tilde{f}^*(k+p)\tilde{f}(q+p) . \end{aligned} \tag{3.6}$$

Here the transverse lattice dynamics contributes only through the wavefunction at the origin effect, and renormalises the quark mass as $\widetilde{m} = m + 8G(0)\langle\overline{\psi}\psi\rangle/N$. Extracting the γ^+-component,

$$\int \frac{dk^+}{4\pi^2 k^+} |\tilde{f}(k^+)|^2 + \int \frac{dp^+}{2\pi}\frac{dq^+}{2\pi}\frac{dk^+}{2\pi} \mathrm{P}\Big[\frac{1}{(p^+)^2}\Big] \tilde{f}^*(q^+)\tilde{f}(k^+)\tilde{f}^*(k^+ + p^+)\tilde{f}(q^+ + p^+) = 0 \ . \quad (3.7)$$

To obtain the valence quark density in the baryon, $4|f(k^+)|^2$ with $k^+ \geq 0$, this nonlinear integral equation has to be solved numerically, as in the case of the 't Hooft model [10].

Alternatively, the baryon number constraint can be incorporated in the functional integral using a constant temporal Abelian background field, i.e. the chemical potential μ (see e.g. [11]),

$$\delta(Q - NB) = \int [D\mu] \ \exp\big[i(Q - NB)\mu\big] \ . \quad (3.8)$$

Functional integration at a fixed chemical potential shifts the self-energy of the quark propagator,

$$S(p) = \frac{i}{\not{p} - m - \Sigma_{\mu\neq 0}(p) + i\varepsilon}\delta_{x_\perp y_\perp} \ , \quad \Sigma_{\mu\neq 0}(p) = -\left(\frac{g^2}{2\pi p^+} + \mu\right)\gamma^+ \ . \quad (3.9)$$

Indeed, this structure justifies the decomposition in Eq.(3.5).

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